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TWO– AND THREE–DIMENSIONAL POINT GROUPS OF HYPER–TABLET P–SYMMETRIES AND THEIR GEOMETRICAL APPLICATION

BY

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Abstract. The paper contains the basic facts of the theory of P–symmetry and the results obtained by using two– and three–dimensional points groups of rosette, tablet and hyper-tablet P–symmetries for counting and modelling some categories of n–dimensional symmetry groups for $n \geq 4$.

1. Shubnikov teaching of antisymmetry [3] is used as the basic for various new generalizations of the classical theory of symmetry and their large application in discrete geometry [4,5]. The interpretation of antisymmetry as the two–colored symmetry brought the idea of Belov multi–colored symmetry, named in [4,5] the p–symmetry. As the other generalization of antisymmetry Zamorzaev antisymmetry of different patterns (multiple antisymmetry [3]) appeared, extending the n by assigning to the points of a transformal figure not only one, but several qualitatively different signs + or -. Diverse approaches to the colored antisymmetry introduced by Pawley, Neronova and Belov, and their further generalization–cryptosymmetry of Niggli and Wondratchek are the synthesis of the both [4,5].

The mentioned generalizations of antisymmetry and colored symmetry are included in P–symmetry, using arbitrary number of color p (not only $p = 2$, as antisymmetry), assigned to the points of a figure, and arbitrary group P of color–permutations (not only cyclic groups, as Belov p–symmetry).

The more detailed explanation of the ideas mentioned, the recent methods of applying two– and three–dimensional crystallographic groups of rosette, tablet and hyper–tablet P–symmetries to the study of multidimensional symmetry groups will be given in this communication.

2. The basic assumptions of P–symmetry, explained in details in [4,5], are the following: assigning to all the points of a figure at least one index $i = 1, 2, ..p$, its isometrical symmetry transforming each points with the index i into the point with the index k_i is called P–symmetry, where the permutation of indices

$$\varepsilon = \begin{pmatrix} 1 & 2 & \dots & p \\ k_1 & k_2 & \dots & k_p \end{pmatrix}$$

belongs to the already given permutation group P of these p indices. Each P–symmetry transformation g is a communicative product of a symmetry s and permutation of indices ε; the set of all the P– symmetries of the indexed figure is the P–symmetry groups G. If all permutations of indices belonging to P are exhausted in G, G is a group of complete P–symmetry. Each group G of complete P–symmetry can be derived from its generating (classical) symmetry group S by searching in S and P for the normal subgroups H and Q respectively, such that $S/H \simeq P/Q$, by multiplying the cosets corresponding in this isomorphism, and unifying the products obtained. If $Q = P$, $Q = e$, or $e \subset Q \subset P$, the complete P– symmetry group G is called the senior (then $S = H$ and $G = S \times P$), junior ($S/H \simeq P$ and $G \simeq S$), or middle, respectively. By the isomorphism of the factor–group $S/H \simeq P$ the homomorphism from S onto P with kernel H, i.e. the representation of the group P by permutations is naturally defined.

P–symmetry is the most extended genralization of the classical theory of symmetry, including all the generalizations of antisymmetry and colored symmetry, such that the change of properties is combined directly with isometrical transformations acting obly to the points, independently of the choice of the part of a figure. In the scheme of P–symmetries Shubnikov antisymmetry is the 2–symmetry with the group $P = \{(1, 2)\} \simeq C_2$, for Belov p–symmetry $P = \{(1, 2, ..., p)\} \simeq C_p$, Zamorzaev l–multiple symmetry is the (2,2,...,2)– symmetry where P is the direct product of l groups of order 2, for $(p2)$–symmetry $P = \{(1, 2, 3, ..., p), (2, p)(k, p-k+2)\} \simeq D_p(2p)$ where $D_p(2p)$ denotes the irregular dihedral permutation group, for Pawley $(p/)$–symmetry $P = \{(1, ..., p)(\overline{p}, ..., \overline{1}), (1, \overline{1})...(p, \overline{p})\} \simeq D_p$, for $(p, 2)$–symmetry $P = \{1, 2, ..., p)\} \times \{(+, -)\} \simeq C_p \times C_2$, and for the $(p/, 2)$–symmetry $P = \{1, ..., p)(\overline{p}, ..., \overline{1}), (1, \overline{1})...(p, \overline{p})\} \times \{+, -)\} \simeq D_p \times C_2$, etc.

The mentioned particular cases for P–symmetry can be simply geometrically visualized [4,5]. Hence, the permutations of indices in the case of 2–symmetry, (2,2)–, p and $(p/)$–symmetry are represented, respectively, by the permutations of the vertices of a segment, rectangle, oriented regular p–angle, regular p– angle and equiangular semiregular $2p$–angle, corresponding to their symmetry transformations. The $(p, 2)$– and $(p/, 2)$–symmetry

are represented by the permutations of the vertices of a regular prism with equally oriented p–angular basis and equiangular prism with the $2p$–gonal basis. Therefore, the permutation group P, characterizing each particular P–symmetry mentioned, is isomophic to the group C_2 in the case of 2– symmetry, C_{2v} in the case of (2,2)–symmetry, C_p in the case of p–symmetry, D_p in the case of $(p/)$– and $(p2)$–symmetry, C_{ph} in the case of $(p,2)$–symmetry and D_{ph} in the case of $(p/,2)$–symmetry, since the corresponding point groups given by Schönflies symbols are complete symmetry groups of the geometrical figures mentioned.

3. The sign + or -, assigned to the points of a figure, as well as the indices $i = 1, 2, ..., p$, possess the defined non–geometrical meaning, with regard to the space which the figure belongs. In the additional dimensions those signs and indices can be geometrically interpreted [4,5], making the possibility to tstudy multidimensional crystallographic symmetry groups, modelling by P–symmetry groups certain categories of "plane" subgroups of n–dimensional Edorov groups G_n, namely the categories G_{nm} (with the invariant m–dimensional plane) and $G_{nm...k}$ (with the invariant m–,...,k–dimensional planes properly inserted one into other).

Thus, for example, the zero–dimensional symmetry and antisymmetry groups of class G_0^1 (the generating group 1 and senior group $\underline{1}$) model one–dimensional point groups of class G_{10} (these denoted by 1 and m) if interpret the permutations of + and - signs assigned to an asymmetric point as the discrete dispositions of the points of a straight line over one or the other side of its invariant point. Further, the zero– dimensional symmetry and double antisymmetrygroups of class G_0^2 (consisting of the generating group 1, senior group of the first kind $\underline{1}$; senior of the second kind 1;; senior of the $(1,2)$ kind $\underline{1}'$ and senior of the third kind $\underline{1} \times 1'$) completely describe all various symmetry groups of class G_{210} $(1, 1m, m1, 2, mm)$. These groups leave invariant in the plane a point and two perpendicular lines which through the given invariant point. Here we interpret the fist + and - signs assigned to an asymmetric point as belonging to the upper or the lower semi–planes devided by the invariant line; and the second + and - signs we interpret as belonging to one or the second direction of the invariant line. Smilarly the zero–dimensional symmetry and three-fold antisymmetry groups of class G_0^3 completely model all various symmetry groups of class G_{3210} (finite bands), if we interpret the first + and - signs assigned to an antisymmetric point as belonging to one or the second normal direction of its first invariant plane; and the second + and - signs, assigned to an antisymmetric points, as belonging to the one or the second normal direction of its third invariant plane. Thus the zero–dimensional symmetry and three–fold antisymmetry

groups G_0^3 can model symmetry and two–fold antisymmetry groups of segments G_{10}^2 and symmetry and antisymmetry groups of finite edgings G_{210}^1 and symmetry groups of finite bands.

Table 1 compares the above four point groups (in the order of their enumeration). The groups compared have been written in the international symbols. The bar under the symbol of a symmetry element in column 1 indicates the replacement by the antisymmetry element of type 1. The prime above to the right of a symmetry element (or antisymmetry element of type 1) indicates the replacement by an antisymmetry element of type 2 (or type (1,2)) in the first column and by an antisymmetry element of type 1 in the second solumn respectively. The asterisk above to the left of a symmetry element (or antisymmetry element of type 1, type 2 or type (1,2)) idicates the replacement by an antisymmetry element or type 3 (or type (1.3), (2,3) or (1,2,3)) in the first column, by an antisymmetry element of type 2 (or type (1,2)) in the second column and by an element of type 1 in the third column respectively (cf.[1,3]).

Table 1 clearly illustrates that there is a simple isomorphism correspondence between all various zero–dimensional groups of three-fold antisymmetry G_0^3 and symmetry groups G_{3210} (finite bands). That is why the number of various zero–dimensional symmetry and three–fold antisymmetry groups G_0^3 fully coincides with the number of various symmetry groups of class G_{3210}, and the structure of a separate group of class G_0^3 determines the structure of the corresponding group of class G_{3210}.

Analogously, the symmetry groups of tables G_{320} can be modelled by the antisymmetry groups of rosettes G_{20}^1 [3-5]. Interpreting the l signs + or -, assigned to plane point, as the signs of coordonates in l additional dimensions, and developing this idea in multidimensional spaces, by the l–multiple antisymmetry groups of rosettes G_{20}^l can be modelled all the symmetry groups of the category $G_{(l+2)(l+1)l(l-1)...320}$. From this fact we conclude that in the $(l+2)$–dimensional space ($l = 1, 2, 3, 4$) there are 31 crystallographic symmetry group of tables G_{320}, 125 crystallographic symmetry group of hyper– tables G_{4320}, 671 crystallographic symmetry group of the category G_{54320}, and 4885 crystallographic symmetry group of the category G_{654320} (modelled by l–multiple antisymmetry groups of rosettes G_{20}^1, G_{20}^2, G_{20}^3 and G_{20}^4 respectivly [3]).

4. It should be noted that 10 symmetry groups of one–sided G_{20} rosettes can be described via zero–dimensional G_0 symmetry, G_0^1 antisymmetry, G_0^p and $G_0^{p/}$ p–and $(p/)$–symmetry, respectively, for $p = 2, 3, 4, 6$. In fact the extension of G_0 to antisymmetry, p– and $(p/)$–symmetry for $p = 2, 3, 4, 6$ gives generating group and nine senior groups $1 \times \underline{1}$, $1 \times 1^{(2)}$, $1 \times 1^{(3)}$, $1 \times 1^{(4)}$, $1 \times 1^{(6)}$, $1 \times 1^{(2\cdot 1')}$, $1 \times 1^{(3\cdot 1')}$, $1 \times 1^{(4\cdot 1')}$, $1 \times 1^{(6\cdot 1')}$, that is 10 P–symmetry groups (including $P = e$) in the class G_0^P. The pictorial geometrical schemes of these 10 P–groups on the plane are the following: an asymmetric point with index 1, two points with indices 1 and $\overline{1}$, symmetric to an axis, two points with indices 1, and 2, symmetric to a centre stands for 1–, (1/)– and 2–symmetry, respectively (Fig.1),

an oriented regular p–gon with vertices $1, 2, , ..., p$ stands for p–symmetry when $p > 2$ (Fig.2) and the $(p/)$–symmetry is modelled by a semiregular

$2p$–gon with vertices $1, 2, ..., p$ and $\overline{1}, \overline{2}, ...\overline{p}$, so that the permutation group $P = \{(1, 2, ..., p)(\overline{p}, ..., \overline{2}, \overline{1}), (1, \overline{1})(2, \overline{2})...(p, \overline{p})\}$ of their vertices describes its complete symmetry group when $p \geq 2$ (Fig.3).

Apparently, these scheme interpret P–groupsas one–sided rosette symmetry group G_{20} (cf.[1]).

Table 2 compares the above zero–dimensional P–symmetry groups to one–sided rosete symmetry groups in Shubnikov's symbols [3] (in the order of their enumeration).

Thus in this case, as it is illustrated in table 2, there is an isomorphic correspondence between zero–dimensional groups G_0^P (symmetry, antisymmetry, p– and $(p/)$–symmetry for $p = 2, 3, 4, 6$) and rosette symmetry groups G_{20}. There is the same correspondence (as between zero–dimensional groups G_0^P of the above particular P–symmetry cases and rosette symmetry groups G_{20} between symmetry, antisymmetry, p– and $(p/)$–symmetry groups for $p = 2, 3, 4, 6$, for example, between the class G_r^P and $(r + 2)$–dimensional symmetry groups of class $G_{(r+2)r}$ (cf.[1,4,5]). By this means, using the P–symmetry groups of rosette G_{20}^P and $G_{20}^{p/}$ it is proved that there are 263 crystallographic symmetry groups of the category G_{420} (including enantiomorphic groups) and 251 groups without enantiomorphism. In the same way, modelling them by p– and $(p/)$–symmetry groups of tables G_{320}^p and $G_{320}^{p/}$ where $p = 1, 2, 3, 4, 6$, we conclude that the category G_{5320} consists of 1274 crystallographic symmetry groups, and that the category G_{530}, modelled by G_{30}^p and $G_{30}^{p/}$, consists of 1208 crystallographic symmetry groups [5].

By the P–symmetry groups G_r^P of the 31 tablet P– symmetries in the geometrical classification [6] ($P \simeq G_{320}$), there are completely modelled all the symmetry groups of the category $G_{(r+3)(r+2)r}$. Using the P–symmetry groups of rosettes G_{20}^P, of the 31 tablet P–symmetries, we conclude that the category G_{5420} consists of 1274 different crystallographic symmetry groups. This is the independent proof of the previous result obtained for the category G_{54320}, since that categories coincide. Analogously, modelling the symmetry groups of category G_{654320} by P–symmetry groups of tablets G_{320}^P ($P \simeq G_{320}$), we conclude that it consists of 8806 crystallographic symmetry groups. From the three–dimensional point P–symmetry groups G_{30}^P

of tablet P–symmetries ($P \simeq G_{320}$) it results that there are 7579 crystallographic symmetry groups of the category G_{6530} [2].

By P–symmetry groups G_r of 125 hyper–tablet P–symmetries ($P \simeq G_{4320}$), all the symmetry groups of the category $G_{(r+4)(r+3)(r+2)r}$ are completely interpreted. By the same method, generalizing the symmetry groups of rosettes G_{20} by 125 hyper–tablet P–symmetries mentioned ($P \simeq G_{4320}$), we conclude that the category G_{65420} consists of 8806 crystallographic symmetry groups. The equality of the results obtained by independent approaches, proves the correctness of this results, since $G_{65320} = G_{65420}$ [5]. In the same way, the category G_{765320}, modelled by the symmetry groups of results generalized by 125 hyper–tablet P–symmetries, consists of 84313 7–dimensional crystallographic symmetry groups. Finally, using the three–dimensional point groups G_{30}^{P} ($P \simeq G_{4320}$) of hyper–tablet P–symmetry, modelling the symmetry groups of the category G_{76530}, we conclude that it consists of 73589 crystallographic symmetry groups.

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