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A GENERALIZATION OF THE LIMIT CLASS OF APPROXIMATION SPACES

BY

NICOLAE TIŢA

Introduction. Let $(X, \|\cdot\|_X)$ be a quasi–normed Abelian group and $(G_n)_{n=0}^{\infty}$ a sequence of subsets of X such that

$$G_0 = \{0\}; \tag{1}$$

$$G_n \subseteq G_{n+1},\ n = 0, 1, 2, \ldots; \tag{2}$$

$$G_n \pm G_m \subseteq G_{n+m},\ m, n = 0, 1, 2, \ldots. \tag{3}$$

Given any $f \in X$, put $E_n(f) := inf\{\|f - g\|_X : g \in G_{n-1}\}, n = 1, 2, \ldots$. In this way we associate to each $f \in X$ the sequence of non–negative and non–increasing numbers $(E_n(f))$ whose n–th coordinate is the error of approximation of f by elements of G_{n-1}.

The classical approximation space X_q^{α}, [4], [7], [8] consists of all $f \in X$ which have a finite quasi–norm

$$\|f\|_{X_q^{\alpha}} = \left[\sum_1^{\infty} (n^{\alpha} E_n(f))^q \cdot n^{-1}\right]^{1/q},$$

where $0 < \alpha, q < \infty$.

If E and F are two Banach spaces and $L(E, F)$ is the set of all linear and bounded operators $T : E \to F$, are known the Lorentz classes

$$L_{p,q}(E, F) := \left\{ T \in L(E, F) : \left(\sum_1^{\infty} (n^{1/p} a_n(T))^q n^{-1} \right)^{1/q} < \infty \right\},$$

where $0 < p, q < \infty$ and $a_n(T) = inf\{\|T - K\| : K \in L(E,F),\ \text{rank}\ K < n\}, n = 1, 2, \ldots$. The relations between $L_{p,q}(E,F)$ and X_q^α becomes clear when taking $X = L(E,F), G_n = \{K \in L(E,F) :\ \text{rank}\ K \leq n\}, n = 0, 1, 2, \ldots$. In this way we obtain $X_q^\alpha = L_{1/\alpha,q}(E,F)$.

In the papers [2], [3] are considered the limit classes X_q and respectively $L_{\infty,q}$. $\left(X_q = \left\{f \in X : \sum\limits_1^\infty \frac{E_n^q(f)}{n} < \infty,\ \ 0 < q < \infty\right\}\right)$.

Also are describe the behaviour of some "biliniar" operators under X_q–spaces.

In this paper, by means of the symmetric norming functions [5], [9], we consider a generalization of these spaces X_q and we describe the behaviour of some "biliniar" operators on such spaces. In the particular case we obtain, in a simpler way, the results from [2], [3].

The spaces X_ϕ. Let c_0 be the space of the real sequence converging to 0 and let $\overline{c}_0$ be the subclass of c_0 containing the sequences $x = (x_0, x_1, \ldots) \in c_0$ such that $x_1 \geq x_2 \geq \ldots \geq 0$. Denote by K the set of the sequence $x \in \overline{c}_0$ such that $x = \{x_0, \ldots, x_n, 0, 0, \ldots\}, n < \infty$, i.e., the sequence of finite rank n.

The symmetric norming function ϕ are defined in[5], [9], [10] as follows: $\phi : K \to R$ and

$$\phi(x) > 0 \text{ if } x \neq 0; \tag{1}$$

$$\phi(\alpha, x) = \alpha\phi(x) \text{ if } x \in K \text{ and } \alpha \geq 0; \tag{2}$$

$$\phi(x + y) \leq \phi(x) + \phi(y); \tag{3}$$

$$\phi(1, 0, 0, \ldots) = 1; \tag{4}$$

$$\text{if } \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i,\ k = 1, 2, \ldots,\ \text{then } \phi(x) \leq \phi(y). \tag{5}$$

For all symmetric norming function ϕ we can consider also the functions $\Phi_{(p)} : (x_i) \to (\Phi(\{x_i^p\}))^{\frac{1}{p}}, 1 \leq p < \infty$. These functions are also symmetric norming functions, [9].

Definition. *For all symmetric norming function ϕ we define the approximation space $X_{\phi_{(p)}}$ as follows :*

$$X_{\phi_{(p)}} = \{f \in X : \|f\|_{\phi_{(p)}} = \phi_{(p)}(\{E_n(f)\}) =$$

$$= \lim_{n\to\infty} \Phi_{(p)}(\{E_k(f)\}_{k=1}^{n}) < \infty\}$$

(1) If $X = L(E,F)$, then $X_\phi = L_\phi(E,F)$ [10]. Also the spaces X_q coincides whith $X_q^*(E,F) = \{f \in X : \Sigma E_{2^{n-1}}^q(f) < \infty\}$.

For to prove that $\|f\|_{\Phi_{(p)}}$ is a quasi–norm it is necessary to prove only the following :

Proposition. *For all symmetric norming function we have*

$$\|f+g\|_{\phi_{(p)}} \leq 2c\left(\|f\|_{\phi_{(p)}} + \|g\|_{\phi_{(p)}}\right), \quad f,g \in X,$$

where c is the quasi–norm constant on X.

Proof. Given $\epsilon > 0$ there exists f_0 and $g_0 \in G_{n-1}, n = 1,2,\ldots$ such that $E_n(f) + \epsilon/2 \geq \|f - f_0\|_X$ and $\|g - g_0\|_X \leq E_n(g) + \epsilon/2$. Then

$$E_{2n-1}(f+g) \leq \|f+g-(f_0+g_0)\|_X \leq c(\|f-f_0\|_X +$$

$$+\|f-g_0\|_X) \leq c(E_n(f) + E_n(g) + \epsilon)$$

Since ϵ is arbitrary it results, the relation :

$$E_{2n-1}(f+g) \leq c(E_n(f) + E_n(g)).$$

Now we can write :

$$\sum_1^k E_n(f+g) \leq \sum_1^k (E_{2n-1}(f+g) + E_{2n}(f+g)) \leq$$

$$\leq 2\left(\sum_1^k (E_{2n-1}(f+g)\right) \leq 2c\left(\sum_1^k (E_n(f) + E_n(g)\right), \quad k = 1,2,\ldots$$

By using the propertie (5) of the functions $\phi_{(p)}$ it results

$$\|f+g\|_{\phi_{(p)}} = \phi_{(p)}(\{E_n(f+g)\}) \leq 2c\phi_{(p)}(\{E_n(f) + E_n(g)\}) \leq$$

$$\leq 2c[\phi_{(p)}(\{E_n(f)\}) + \phi_{(p)}(\{E_n(g)\})] = 2c(\|f\|_{\phi_{(p)}} + \|g\|_{\phi_{(p)}}).$$

In the sequel we describe the behaviour of bilinear operators on some $X_{\phi_{(p)}}$–spaces. To this end we consider the functions $\overline{\phi}$ defined as follows $\overline{\phi} : (x_i) \to \phi\left(\left(\frac{x_i}{i}\right)\right)$, where ϕ is a symmetric norming function. The functions $\overline{\phi}$ are symmetric norming functions, since

$$\sum_1^k x_i \leq \sum_1^k y_i,\ k = 1, 2, \dots \text{ imply } \quad \sum_1^k \frac{x_i}{i} \leq \sum_1^k \frac{y_i}{i}$$

(we refere to [5]–and hence $\phi\left(\frac{x_i}{i}\right) \leq \phi\left(\frac{y_i}{i}\right)$. In this way the propertie (5) is verified and the other properties are obviously.

Take three quasi–normed Abelian groups X, Y, Z and let $(G_n)_0^\infty, (F_n)_0^\infty$ and $(H_n)_0^\infty$ be sequences of subsets in X, Y, Z satisfying the conditions (1) to (3) as above. Let $B : X \times Y \to Z$ be an operator. We assume that there exists $M < \infty$ such that for all $f, f_0, f_1 \in X$ and $g, g_0, g_1 \in Y$ the following holds:

(a) $B(f_0 \pm f_1, g) = B(f_0, g) \pm B(f_1, g)$
(b) $B(f, g_0 \pm g_1) = B(f, g_0) \pm B(f, g_1)$
(c) $\|B(f, g)\|_Z \leq M\|f\|_X\|g\|_Y, M < \infty$
(d) $B(G_m, F_n) \subset H_{mn},$ for $m, n = 0, 1, \dots$.

Theorem. *Let X, Y, Z and B as above. Then the operator $B : X_{\phi_{(p)}} \times Y_{\phi_{(p)}} \to Z_{\phi_{(p)}}$ is bounded for all functions ϕ.*

Proof. First we show that the exists a constant $c > 0$ (independent of f, g and n) such that

$$E_{n^2}(B(f, g)) \leq c(E_n(f)\|g\|_Y + E_n(g)\|f\|_X), \quad n = 1, 2, \dots .$$

For this we use a similar argument to one in [11] (we refere also to [3]).

Let c_1, c_3 be the constants of the quasi–triangle inequality in X and Z respectively. Given any $\epsilon > 0$, find $f_0 \in G_{n-1}$ and $g_0 \in E_{n-1}$ with $\|f - f_0\|_X \leq E_n(f) + \epsilon$ and $\|g - g_0\|_Y \leq E_n(g) + \epsilon$. By the above conditions we obtain $B(f_0, g_0) \in H_{n^2-1}$ and hence

$$\begin{aligned}
E_{n^2}(B(f, g)) &\leq \|B(f, g) - B(f_0, g_0)\|_Z \\
&= \|B(f - f_0, g) + B(f_0, g - g_0)\|_Z \leq \\
&\leq c_3M(\|f - f_0\|_X\|g\|_Y + \|f_0 - f + f\|_X\|g - g_0\|_Y) \leq \\
&\leq c_3M((E_n(f) + \epsilon)\|g\|_Y + c_1(E_n(f) + \epsilon + \|f\|_X)(E_n(g) + \epsilon)) \\
&\leq c_3M((E_n(f) + \epsilon)\|g\|_Y + c_1(2\|f\|_X + \epsilon)(E_n(g) + \epsilon)).
\end{aligned}$$

Passing to the limit when $\epsilon \to 0$ we obtain the relation with $c = Mc_1c_3$.

Now we obtain $\sum\limits_{1}^{s} \frac{E_n^p(B(f,g))}{n} \le c(p) \sum\limits_{1}^{s} \left[\frac{E_n^p(f)}{n} \|g\|_Y^p + \frac{E_n^p(g)}{n} \|f\|_X^p \right]$, $s = 1, 2, \ldots, \quad 1 \le p < \infty$.

For this we observe that for all $s \in N$ there exists $r \in N$ such that $r^2 \le s < (r+1)^2$.

Then we can write :

$$\sum_{1}^{s} \frac{E_n^p(B(f,g))}{n} \le \sum_{1}^{r} (2n+1) \frac{E_{n^2}^p(B(f,g))}{n^2} \le 3 \sum_{1}^{r} \frac{E_{n^2}^p(B(f,g))}{n} \le$$

$$\le c(p) \sum_{1}^{r} \left[\frac{E_n(f)^p}{n} \|g\|_Y^p + \frac{E_n(g)^p}{n} \|f\|_X^p \right] \le$$

$$\le c(p) \sum_{1}^{s} \left[\frac{E_n^p(f)}{n} \|g\|_Y^p + \frac{E_n^p(g)}{n} \|f\|_X^p \right], \quad s = 1, 2, \ldots$$

For to complete the proof we use the properties (5) of the functions ϕ and we obtain :

$$\Phi \left(\frac{E_n^p(B(f,g))}{n} \right) \le c(p) \Phi \left(\frac{E_n^p(f)}{n} \|g\|_Y^p + \frac{E_n^p(g)}{n} \|f\|_X^p \right)$$

and so

$$\|B(f,g)\|_{\overline{\phi}_{(p)}} \le \widetilde{c}(p) \ (\|f\|_{\overline{\phi}_{(p)}} \|g\|_Y + \|g\|_{\overline{\phi}_{(p)}} \|f\|_X) \le$$

$$\le M_1 \|f\|_{\overline{\phi}_{(p)}} \|g\|_{\overline{\phi}_{(p)}}, \ \widetilde{c}(p) = c(p)^{\frac{1}{p}}.$$

Hence B is bounded.

Remarks. (1) If E_i and $F_i (i = 1, 2)$ are normed spaces and $T_i : E_i \to F_i$ are linear and bounded operators, then $T_1 \otimes T_2 : E_1 \widehat{\otimes}_\tau E_2 \to F_1 \widehat{\otimes}_\tau F_2$ is a bilinear operator (and bounded if τ is a tensor norm [6], [11], i.e., $\|T_1 \otimes T_2\|_\tau \le \|T_1\| \cdot \|T_2\|$). From the above theorem it results that the classes of operators

$$L_{\overline{\phi}_{(p)}}(E, F) = \{T \in L(E, F) : \overline{\phi}_{(p)}(\{a_n(T)\}) < \infty\}$$

are tensor product stable. (If $T_i \in L_{\overline{\phi}_{(p)}}(E_i, F_i), \quad i = 1, 2$, then $T_1 \otimes T_2 \in L_{\overline{\phi}_{(p)}}(E_1 \widehat{\otimes}_\tau E_2, F_1 \widehat{\otimes}_\tau T_2)$ for all tensor norm).

(2) If ϕ is the function $\phi_\omega : \{E_n(f)\} \rightarrow \sum\limits_{n=1}^{\infty} \frac{E_n(f)}{n}$, [5], then $(\phi_\omega)_{(p)} : (E_n(f) \rightarrow \left(\sum\limits_{n=1}^{\infty} \frac{E_n^p(f)}{n}\right)^{\frac{1}{p}}$ is also a symmetric norming function $1 \leq p < \infty$, [9] and hence X_p [3] is a special case of $X_{\overline{\phi}_{(p)}}$.

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Department of Mathematics
University of Braşov
2200 Braşov
ROMANIA