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DIFFERENTIAL OPERATORS AND CONVEXITY ON VECTOR BUNDLES ENDOWED WITH (h, v) - METRICS

BY

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Abstract. The paper introduces and describes naturally the properties of the gradient, divergence, Hessian, Laplacian on a vector bundle endowed with a (h, v)-metric and with a suitable connection. Results on convexity and explicit formulas for special (h, v) - metrics and for particular manifolds (generalized Lagrange, Lagrange and Finsler spaces) are obtained.

1. Preliminaries. Let $\xi = (E, p, M)$ be a vector bundle of local coordinates $(x^i, y^a), i = \overline{1, n}, a = \overline{1, m}$. Let $N = \{N_i^a\}$ be a nonlinear connection on ξ and TE locally spanned by the adapted basis $(\delta_i, \dot\partial_a), i = \overline{1, n}, a = \overline{1, m}$ with [7],

$$\begin{cases} \delta_i = \partial_i - N_i^a \dot\partial_a, & (\partial_i = \frac{\partial}{\partial x^i}) \\ \dot\partial_a = \frac{\partial}{\partial y^a}, \end{cases}$$

and let $\{dx^i, \delta y^a\}$ be the corresponding dual basis. Let also

$$G = g_{ij} dx^i \otimes dx^j + h_{ab} \delta y^a \otimes \delta y^b$$

be a (h, v) - metric on E.

Let $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ be a d-connection [7], having the coefficients $\{L_{jk}^i, \widetilde{L}_{bk}^a, \widetilde{C}_{ja}^i, C_{bc}^a\}$. Let $\mathcal{T}$ be its torsion in adapted coordinates, with the coefficients

$$\mathcal{T}(\delta_j, \delta_i) = T_{ij}^k \delta_k + R_{ij}^a \dot\partial_a$$

$$\mathcal{T}(\dot\partial_a, \delta_j) = \tilde{C}_{ja}^i \delta_i + P_{ja}^b \dot\partial_b, \qquad \mathcal{T}(\dot\partial_b, \dot\partial_a) = S_{ab}^c \dot\partial_c$$

characterized by the relations

$$T^i_{jk} = L^i_{[jk]} = L^i_{jk} - L^i_{kj}, R^a_{jk} = \partial_{[k}N^a_{j]}, P^b_{ja} = \partial_a N^b_j - L^b_{aj}, S^a_{bc} = C^a_{[bc]}.$$

For a metrical, h- and v-symmetrical connection we have

$$L^i_{jk} = \frac{1}{2}g^{is}(\delta_{\{j}g_{sy\}} - \delta_s g_{jk}), C^a_{bc} = \frac{1}{2}h^{ad}(\delta_{\{b}h_{dc\}} - \dot\partial_d h_{bc}), \tag{1}$$

where we used the notations $\tau_{\{ij\}} = \tau_{ij} + \tau_{ji}, \tau_{[ij]} = \tau_{ij} - \tau_{ji}$.

Let ∇ be a (h,v)-metrical, (h,v)-symmetrical d-connection; then we have [4,7]

$$\begin{cases} \widetilde{L}^a_{bk} = N^a_k + \frac{1}{2}h^{ac}\delta_k h_{cb} + O^{ca}_{bd}Y^d_{ck} \\ \widetilde{C}^i_{ja} = \frac{1}{2}g^{ik}\dot\partial_a g_{kj} + O^{mi}_{jk}X^k_{ma} \end{cases} \tag{2}$$

with X^k_{ma}, Y^d_{ck} arbitrary d-tensor fields and

$$\begin{cases} O^{ih}_{jk} = \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^{ih}) \\ O^{ab}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - h_{cd}h^{ab}) \end{cases}$$

the Obata operators.

Proposition 1.1. *The (h,v) - metrical, (h,v) - symmetrical d- connection ∇ given by (1),(2) is completely symmetrical iff*

a) $R^a_{ij} = 0$,

b) $\dot\partial_a g_{ij} = 0$,

c) $\delta_j h_{ab} = 0, i,j = \overline{1,n}, a,b = \overline{1,m}$ *or, equivalently, iff [7]*

a) The distribution N is integrable,

b) The h-metric g_{ij} is Riemannian, $g_{ij} = g_{ij}(x)$,

c) The v-metric h_{ab} satisfies the relations

$$\delta_k h_{bc} = h_{\{bd}\dot\partial_{c\}}N^d_k, b,c = \overline{1,m}, k = \overline{1,n}.$$

This is the unique case in which the Riemannian connection of E is a d-connection.

Proof. Let us denote

$$\begin{cases} (OX)^i_{jc} = O^{mi}_{jk}X^k_{mc} = O^{im}_{kj}X^k_{mc} \\ (OY)^a_{bk} = O^{ca}_{bd}Y^d_{ck} = O^{ac}_{db}Y^d_{ck}. \end{cases}$$

The most general metrical, (h,v)-symmetrical connection is totally symmetrical iff

$$R^a_{ij}=0,(OY)^a_{bk}=\widetilde{Y}^a_{bk},(OX)^i_{jc}=\widetilde{X}^i_{jc},$$

where

$$\begin{cases}\widetilde{Y}^a_{bk}=-\frac{1}{2}h^{ac}(\delta_k h_{bc}-h_{d\{c}\dot{\partial}_{b\}}N^d_k)\\ \widetilde{X}^i_{jc}=-\frac{1}{2}g^{ik}\dot{\partial}_c g_{jk}\end{cases}$$

Let $U^{ca}_{bd}=\frac{1}{2}(\delta^c_b\delta^a_d+h_{bd}h^{ca})$. Then, since O^{ca}_{bd} and U^{ca}_{bd} are supplementary projectors on $\mathcal{T}^{01}_{11}$ and since $\widetilde{Y}=U\widetilde{Y}$ it follows $OY=U\widetilde{Y}$, and hence $OY=\widetilde{Y}=0$. Similarly, one gets $OX=\widetilde{X}=0$. Thus $\widetilde{Y}=0$ and $\widetilde{X}=0$, and these are equivalent to the last conditions b) and c), respectively.

Corollary 1.2. *The mixed coefficients of the metrical symmetrical d-connection are*

$$\begin{cases}\widetilde{L}^a_{bk}=\dot{\partial}_b N^a_k\\ \widetilde{C}^i_{jc}=\frac{1}{2}g^{ik}\dot{\partial}_c g_{kj}\end{cases}$$

We shall introduce on the total space (E,G) of ξ, the gradient, divergence, Hessian and Laplacian operators, establishing basic properties and developing the geometry-analysis created by these operators.

2. Differential operators. Let G be a (h,v)-metric on E of components $\{\underline{G_{\alpha\beta}}=\{g_{ij},h_{ab}\}$,
$\alpha,\beta=\overline{1,m+n}$.

Gradient. The natural splitting $TE=N\oplus VE$ produced by N determines the two partial gradient operators [1]

$$\begin{cases}\mathrm{grad}_h(f)=g^{ij}(\delta_i f)\delta_j\\ \mathrm{grad}_v(f)=h^{ab}(\dot{\partial}_a f)\dot{\partial}_b, f\in\mathcal{F}(E)\end{cases}$$

and hence, we state

Definition 2.2. *Let $f\in\mathcal{F}(E)$. The vector field*

$$\mathrm{grad}f=\mathrm{grad}_h(f)+\mathrm{grad}_v(f)\in\mathcal{X}(E),(\mathrm{grad}f)^\alpha=G^{\alpha\beta}\partial_\beta f$$

is called the global gradient *of f with respect to G.*

Proposition 2.2. *The gradient satisfies the relations*
a)$\mathrm{grad}(cf_1+f_2)=c\mathrm{grad}\ f_1+\mathrm{grad}\ f_2$,
b)$\mathrm{grad}(f_1f_2)=f_1\mathrm{grad}\ f_2+f_2\mathrm{grad}\ f_1$,
c)$\mathrm{grad}(f_1/f_2)=(f_2\mathrm{grad}\ f_1-f_1\mathrm{grad}\ f_2)/f_2^2$,
d)$G(\mathrm{grad}\ f,X)=X(f),\forall f\in\mathcal{F}(E),\forall X\in\mathcal{X}(E)$.

Remark. The gradient vector field is used in the extrema theory, since $\text{grad} f(x)$ gives the direction and the sense of steepest increase of the function f at the point x.

Divergence. In order to introduce an object similar to the classical divergence [9], we use the adapted volume element

$$dV = \sqrt{gh}dx^1\wedge\ldots\wedge dx^n\wedge\delta y^1\wedge\ldots\wedge\delta y^m = \sqrt{gh}dx^1\wedge\ldots\wedge dx^n\wedge dy^1\wedge\ldots\wedge dy^m,$$

where g and h are the determinants of matrices (g_{ij}) and (h_{ab}) respectively.

Definition 2.3. *Let X be a vector field in $\mathcal{X}(E)$. The scalar field div X defined by [1]*

$$(\text{div } X)dV = L_X dV,$$

where L_X is the Lie derivative in the direction of the field X, is called the divergence of the vector field X.

Let ∇ be a metrical connection on E, i.e., $\nabla_X G = 0, \ \ \forall X \in \mathcal{X}(M)$.

Proposition 2.4. *a) The divergence of a vector field X has the equivalent expressions*

$$\text{div } X = -\text{Trace } A_X, \text{where } A_X = L_X - \nabla_X, \forall X \in \mathcal{X}(E), \text{and}$$

$$\text{div } X = \text{Trace}(Y \to (\nabla_Y X + \mathcal{T}(X,Y))).$$

b) The local expression of the divergence is

$$\text{div } X = X^i_{|i} + X^a|_a + X^j(\mathcal{T}^i_{ij} - P_j) + X^a(C_a + S^b_{ba}),$$

$$\forall X = X^i\delta_i + X^a\dot{\partial}_a \in \mathcal{X}(E),$$

where $_{|i}, |_a$ are the h- and v-covariant derivations associated to ∇, and

$$P_i = P^a_{ia} = \dot{\partial}_a N^a_i - L^a_{ai}, C_a = \widetilde{C}^k_{ka}.$$

Proof. a) $\nabla_X G = 0$ infers $\nabla_X dV = 0$ whence the method in [9] produces the two equivalent expressions; also we use the relation $A_X(Y) = -\nabla_Y X - \mathcal{T}(X,Y)$. b) Considering the partial divergences

$$\begin{cases} \text{div}_h X = \text{Trace}(Y \to (\nabla_Y hX + \mathcal{T}(hX,Y))) \\ \text{div}_v X = \text{Trace}(Y \to (\nabla_Y vX + \mathcal{T}(vX,Y))), \end{cases}$$

we obtain

$$\mathrm{div}_h X = dx^i[(hX)_{|i} + X^j T^k_{ij}\partial_k + R^a_{ij}\dot{\partial}_a] +$$

$$+\delta y^a\left[(hX)|_a + X^j(\widetilde{C}^k_{ja}\delta_k - R^b_{ja}|_b)\right] =$$

$$= X^j_{|j} + (T^i_{ij} - P^a_{ja})X^j$$

$$\mathrm{div}_v X = dx^i\left[(vX)_{|i} + X^a\widetilde{C}^k_{ia}\delta_k + P^b_{ia}\dot{\partial}_b\right] + \delta y^a\left[(vX)|_a + \dot{X}^b S^c_{ab}\dot{\partial}_c\right] =$$

$$= \dot{X}^a|_a + \dot{X}^a(\widetilde{C}^i_{ia} + S^b_{ba});$$

Hence

$$X = \mathrm{div}_h X + \mathrm{div}_v \mathrm{X}.$$

Corollary 2.5. *If ∇ is h- and v - symmetrical (the case considered in [1]), then div $X = \mathrm{div}_h X + \mathrm{div}_v X$, where*

$$\begin{cases} \mathrm{div}_h X = X^i_{|i} - X^j P_j, \\ \mathrm{div}_v X = \dot{X}^a|_a + \dot{X}^a\widetilde{C}_a \end{cases}$$

and $P_j = P^a_{ja}, \widetilde{C}_a = \widetilde{C}^j_{ja}$.

Proposition 2.6. *The divergence is also given by*

$$\mathrm{div}\ X = \frac{1}{\sqrt{gh}}\left[\delta_k(\sqrt{gh}X^k) + \dot{\partial}_a(\sqrt{gh}\dot{X}^a)\right],$$

where $g = \det(g_{ij}), h = \det(h_{ab}), X = X^i\delta_i + \dot{X}^a\dot{\partial}_a \in \mathcal{X}(E)$.

Proof. Direct computation [9,5].

Proposition 2.7. *The following relations hold true*

a) $\mathrm{div}(cX + Y) = c\ \mathrm{div}\ X + \mathrm{div}\ Y$,

b) $\mathrm{div}(fX) = f\mathrm{div}\ X + G(\mathrm{grad}\ f, X), \forall X, Y \in \mathcal{X}(E), \forall f \in \mathcal{F}(E)$.

Remark. The divergence of a vector field defines the speed of contraction – dilatation of volumes with respect to the flow generated by that vector field.

Hessian. Let ∇ be an arbitrary metrical connection and $|_i, |_a$ the covariant derivatives associated to ∇.

Definition 2.8. *Let $f \in \mathcal{F}(E)$. The tensor field*

$$H_\nabla(f) = \text{Hess } f \in \mathcal{T}_2^0(E), H_\nabla = H_{hh} + H_{hv} + H_{vh} + H_{vv},$$

$$\begin{cases} (H_{hh})(f)_{ij} = (\delta_i f)_{|j} + (T^K_{ki} - P_i)\delta_j f \\ (H_{hv})(f)_{ia} = (\delta_i f)|_a + (T^K_{ki} - P_i)\dot\partial_a f \\ (H_{vh})(f)_{ai} = (\dot\partial_a f)_{|i} + (\widetilde{C}_a + S^b_{ba})\delta_i f \\ (H_{vv})(f)_{ab} = (\dot\partial_a f)|_b + (\widetilde{C}_a + S^b_{ba})\dot\partial_b f \end{cases}$$

is called the Hessian *of f with respect to the metrical d-connection ∇. If ∇ is the unique metrical and torsionless (Riemannian) connection on E, then*

$$H(f) = H_\nabla(f) = (\delta_i f)_{|j} dx^i \otimes dx^j + (\delta_i f)|_a dx^i \otimes \delta y^a +$$

$$+(\dot\partial_a f)_{|i} \delta y^a \otimes dx^i + (\dot\partial_a f)|_b \delta y^a \otimes \delta y^b$$

is called (simply) the Hessian *of f.*

Remark. The Hessian tensor field is used in the critical points theory, extrema theory and in the convexity theory of functions.

Laplacian. Let G be a (h, v)-metric on E and ∇ be a d-connection on$E$.

Definition 2.9. *The scalar field*

$$\Delta f = \text{div(grad } f),$$

is called the Laplacian *of the function f* with respect to the (h, v)-metric G and connection ∇.

Remark. For a metrical d-connection ∇, we can write

$$\Delta f = \text{ div(grad } f) = X^\alpha_{|\alpha} + X^\alpha \mathcal{T}^\gamma_{\gamma\alpha}, X^\alpha = G^{\alpha\beta}\partial_\beta f,$$

i.e.,

$$\Delta f = G^{\alpha\beta} f_{|\alpha|\beta} + \partial_\beta f \mathcal{T}^\gamma_{\gamma\alpha}.$$

Using the Hessian,

$$(H(f))_{\alpha\beta} = f_{|\alpha|\beta} + \partial_\beta f \mathcal{T}^\gamma_{\gamma\alpha}$$

we obtain $\Delta f = \text{Trace } H(f)$.

Assume that ∇ is completely torsionless and that it satisfies the condition $\nabla_X dV = 0$, i.e.,

$$\begin{cases} \delta_k \mathcal{G} = 2\mathcal{G}(L^j_{jk} + \widetilde{L}^a_{ak}) \\ \dot\partial_a \mathcal{G} = 2\mathcal{G}(\widetilde{C}^j_{ja} + C^b_{ba}), \ \mathcal{G} = \det\mathcal{G} = gh \end{cases} \tag{3}$$

Lemma 2.10. *a) If ∇ is metrical, then (3) are automatically fulffiled. b) The conditions (3) are equivalent to*

$$\begin{cases} \delta_k \ln\sqrt{\mathcal{G}} - L^j_{jk} - \widetilde{L}^a_{ak} = 0 \\ \dot\partial_a \ln\sqrt{\mathcal{G}} - \widetilde{C}^j_{ja} - C^b_{ba} = 0. \end{cases}$$

Proof. a) The relation $\nabla_X G = 0$ infers

$$\begin{cases} g_{ij|k} = 0 \Leftrightarrow \delta_k g_{ij} - g_{\{il}L^l_{j\}k} = 0 \\ h_{ab|k} = 0 \Leftrightarrow \delta_k h_{ab} - h_{\{ac}\widetilde{L}^c_{b\}k} = 0 \end{cases}$$

whence $\delta_k G_{\alpha\beta} - G_{\alpha\gamma\Gamma^\gamma_{\beta}k} = 0, \alpha,\beta,\gamma = \overline{1, m+n}$.
Similarly, we have $\dot\partial_a G_{\alpha\beta} - G^\gamma_{\alpha\gamma\Gamma^\gamma_{\beta}a+n} = 0$,

$$G = G_{\alpha\beta} dX^\alpha \otimes dX^\beta, \{dX^\alpha\} \equiv \{dx^i, \delta y^a\}, \alpha = \overline{1, m+n}.$$

Contracting with $\mathcal{G}G^{\alpha\beta}$, $\mathcal{G} = \det\mathrm{G}$, and using the outcoming relation

$$\partial_\theta G_{\alpha\beta} - G_{\alpha\gamma\Gamma^\gamma_{\beta}\theta} = 0, \partial_\theta = \{\delta_i, \partial_a\}, \theta = \overline{1, m+n},$$

we obtain, by means of $\partial_\theta \mathcal{G} = (\partial_\theta G_{\alpha\beta})\mathcal{G}G^{\alpha\beta}, (g = \det G)$, the final relation

$$\partial_\theta \mathcal{G} - 2\Gamma^\gamma_{\gamma\theta}\mathcal{G} = 0,$$

which represents in fact the condensed form of (3).

Remark. In this case, $\nabla_{\partial_\theta} dV = \left[(\partial_\theta \mathcal{G})/2\mathcal{G} - \Gamma^\gamma_{\gamma\theta}\right] dV$. Hence $\nabla_{\partial_\theta} dV = 0$ and the divergence can be expressed in terms similar to [9].

Proposition 2.11. *Under the conditions above, the Laplacian has the expression*

$$\Delta f = \Delta_h(f) + \Delta_v(f),$$

where

$$\begin{cases} \Delta_h(f) = g^{ij}(H_{hh}(f))_{ij} = g^{ij}(\delta_i f)_{|j} \\ \Delta_v(f) = h^{ab}(H_{vv}(f))_{ab} = h^{ab}(\dot\partial_a f)|_b. \end{cases}$$

Notation	Δ	grad	div	∂_α	$\mathcal{G}^{\alpha\beta}$	$\partial_\beta f$	$\mathcal{G}$
Replacement I	Δ_h	grad_h	div_h	∂_i	g^{ij}	$\partial_j f$	g
Replacement II	Δ_v	grad_v	div_v	$\dot\partial_a$	h^{ab}	$\dot\partial_a f$	h

Proposition 2.12. *Let the complete torsionless connection ∇ satisfy (3). Then we have the following relations*

a) $f_1\Delta f_2 - f_2\Delta f_1 = \text{div}(f_{[1}\text{grad } f_{2]})$.

b) $\Delta(f_1 f_2) = f_1\Delta f_2 + 2G(\text{grad } f_1, \text{grad } f_2)$.

c) The Laplacian admits the expression

$$\Delta f = \frac{1}{\sqrt{\mathcal{G}}}\partial_\alpha\left(\sqrt{\mathcal{G}}G^{\alpha\beta}\partial_\beta f\right), f \in \mathcal{F}(E), \mathcal{G} = \text{detG } = gh.$$

Remark. The results above hold also true under the replacement of the symbols from the first line in the subsequent table, with any of the two lines below

Proof. The techniques from [6,13] lead to the conclusion.

Remark. The Laplacian is used in the spectral theories on manifolds.

3. Convexity. Let $G = (g_{ij}, h_{ab})$ be a (h, v)-metric on ξ and ∇ the associated Riemannian d-connection.

Definition 3.1. *A* geodesic *in ξ [2,3,8] is a curve $C : I \subset \boldsymbol{R} \to E, C(t) = (x^i(t), y^a(t))$, which has vanishing covariant h- and v-accelerations,*

$$\frac{\nabla V^i}{dt} = 0, \frac{\nabla V^a}{dt} = 0, \quad i = \overline{1,n}, a = \overline{1,m}, \text{where}$$

$$V^i = \frac{dx^i}{dt}, V^a = \frac{dy^a}{dt} + N^a_j\frac{dx^j}{dt}$$

The relations (10) can be rewritten [2]

$$\frac{dV^i}{dt} + L^i_{jk}V^jV^k + \widetilde{C}^i_{ja}V^jV^a = 0.$$

$$\frac{dV^c}{dt} + \widetilde{L}^a_{bk}V^bV^k + C^a_{bc}V^aV^c = 0$$

Remarks. The h-paths [7] are particular cases of geodesics, since in the defining relations

$$\frac{\nabla V^i}{dt} = 0, \frac{\delta y^a}{dt} = 0$$

the second one represents in fact $V^a = 0$, and consequently $\frac{\nabla V^a}{dt} = 0$.

The v-paths are also geodesics, since the first relation in

$$\frac{dx^i}{dt} = 0, \frac{\nabla V^a}{dt} = 0$$

represents $x^i = c^i$, and then automatically $\frac{\nabla V^i}{dt} = 0$.

Let hereafter (E, G) be a Riemannian manifold, assumed *complete* - in order to be sure that for any pair of points $p, q \in E$, there exists at least one geodesic arc $C_{pq} : [0, 1] \to E$ joining p and q, i.e., $C_{pq}(0) = p, C_{pq}(1) = q$.

Definition 3.2. *A subset D of E is said to be* totally convex *if D contains each geodesic C_{pq} of E whose endpoints p and q are in D.*

Let D be a totally convex subset of E, and $\mathcal{M}$ the set of all geodesics from p to q, where $p, q \in D$.

Definition 3.3. *The function $f : D \to R$ is called* convex *if*

$$f(C_{pq}(t)) \leq (1-t)f(p) + tf(q), \forall p, q \in D, \forall C_{pq} \in \mathcal{M}, \forall t \in [0, 1].$$

If–f is convex, *then f is called* concave.

This definition of convexity is strongly dependent on the Riemannian connection, since it involves geodesics. However, the basic theory of usual convex functions on R^n is transfered to convex functions in the present work.

Theorem 3.4. *The function $f : D \to R$ is convex iff $\forall p, q \in D, \forall C_{pq} \in \mathcal{M}$ the functions $\varphi_{pq} = f \circ C_{pq}$ are convex on [0, 1].*

Theorem 3.5. *The function $f : D \to R$ is convex iff whatever $p \in D$, there exists $\omega_p \in T_p^* D$ such that $\forall q \in D, \forall C_{pq} \in \mathcal{M}$, we have*

$$f(q) - f(p) \geq \omega_p(\dot{C}_{pq}(0)).$$

Let D be open in E. For a function $f : D \to R$ of class C^1 we have $\omega = \mathrm{d}f$. For a function $f : D \to R$ of class C^2 we find the following

Theorem 3.6. *Let D be an open totally convex set and $f : D \to R$ be a function of class C^2. The function f is convex iff the function*

$$\varphi_{pq}(t) = f(C_{pq}(t)), t \in [0,1]$$

satisfies

$$\frac{d^2\varphi_{pq}}{dt^2} \geq 0, \forall t \in [0,1], \forall p, q \in D, \forall C_{pq} \in \mathcal{M}.$$

Theorem 3.7. *Let D be on open totally convex set and $f : D \to R$ be a function of class C^2. The function f is convex iff $H(f)$ is positive semidefinite on D.*

Proof. $\frac{d^2\varphi_{pq}}{dt^2} = (H(f) \circ C_{pq})(\dot{C}_{pq}, \dot{C}_{pq})$.

Theorem 3.8. *For $\xi = (TM, p, M)$-the tangent bundle of the Riemannian manifold $(M, g_{ij} = h_{ij} = \gamma_{ij})$, the canonical nonlinear connection $N_i^a = \gamma_{i0}^a$, and ∇ being the N-lift of the Levi–Civita connection given by γ_{jk}^i, the energy $\varepsilon = \frac{1}{2}h_{ab}y^a y^b$ of the Liouville field $\mathcal{C} = y^a|_a$ is a convex function on $E = TM$.*

Example. Let (M, g_{ij}) be a Riemannian manifold. We consider the primal convex programming problem

$$\max f_0(x) \text{ subject to } f_a(x) \leq 0, a = \overline{1,n}, x \in M.$$

The feasible region of this program will be denoted by F and its interior by $\check{F}$.

Assumptions: 1) $\check{F}$ is non-empty, 2) $\check{F}$ is bounded, 3) the functions $-f_0$ and f_a are C^2 convex functions on $\check{F}$. The associated dual problem is

$$\min f_0(x) - \sum_{a=1}^m y^a f_a(x) \text{ subject to } \sum_{a=1}^m y^a df_a(x) = df_0(x), y^a \geq 0.$$

Obviously, the function

$$\Phi(x,y) = f_0(x) - \sum_{a=1}^m y^a f_a(x)$$

is defined on the vector bundle $M \times \boldsymbol{R}^m$. Moreover, its restriction to $\check{F} \times \boldsymbol{R}_+^m$ is shown to be concave for the Riemannian metric $G = (g_{ij}, \delta_{ab})$ and its associated Riemannian connection.

Remarks. 1) Generalizations of convex functions which are found in the literature [10,12] (quasiconvex, pseudoconvex, etc.) can be transfered on the Riemannian manifold (E, G).

2) The convexity on E implies the h-convexity and v-convexity.

4. Particular cases.

A. Differential operators on vector bundles with special (h,v)–metrics.

Let $G=(g_{ij},h_{ab})$ be a (h,v)-metric on ξ. Its v-energy (the energy of the Liouville field $\mathcal{C}=y^a\dot\partial_a$) is $\varepsilon=\frac{1}{2}h_{ab}y^ay^b$ and we have:

$$2\ \mathrm{grad}_h\varepsilon=g^{ij}\delta_j(h_{00})\delta_i=g^{ij}(\delta_jh_{ab})y^ay^b\delta_i+$$

$$+2g^{ij}h_{ao}(-N_j^a)\delta_i=g^{ij}\left[(\delta_jh_{ab})y^ay^b-2h_{ao}N_j^a\right]\delta_i$$

$$2\ \mathrm{grad}\ {}_v\varepsilon=h^{ab}\left[(\dot\partial_bh_{cd})y^cy^d+2y_b\right]\dot\partial_a,$$

where the null index denotes contraction by y^a.

Particular cases

1° Case $h_{ab}(x,y)=e^{2\sigma(x,y)}\gamma_{ab}(x)$, with $\{\gamma_{ab}\}$ Riemannian metric on $M,\sigma\in\mathcal{F}(M)$. The v-energy is given by $\varepsilon=e^{2\sigma}\gamma_{oo}$ .Then

$$\begin{cases}\mathrm{grad}_h\varepsilon=g^{ij}(2\sigma_jh_{oo}+(\partial_j\gamma_{ab})e^{2\sigma}y^ay^b2e^{2\sigma}\gamma_{ao}N_j^a)\delta_i\\ \mathrm{grad}_v\varepsilon=h^{ab}(2\dot\sigma_bh_{oo}+2y^a)\dot\partial_a,\end{cases}$$

where $\sigma_j=\delta_j\sigma,\dot\sigma_a=\dot\partial_a\sigma$.

2° For $h_{ab}=\gamma_{ab}(x)+u(x,y)y_ay_b,y_a=\gamma_{ac}y^c,u\in\mathcal{F}(E)$, we have

$$\varepsilon=\gamma_{oo}(1+u\gamma_{oo})$$

$$\begin{cases}\mathrm{grad}_h\varepsilon=g^{ij}\left\{(1+2u\gamma_{00})\left[(\partial_j\gamma_{ab})y^ay^b-2N_j^a\gamma_{ao}\right]+\gamma_{oo}^2u_j\right\}\delta_i\\ \mathrm{grad}_v\varepsilon=h^{ab}[(1+2u\gamma_{oo})\gamma_{bo}+\gamma_{oo}^2\dot u_b]\dot\partial_a\end{cases}$$

where $u_j=\delta_ju,\dot u_a=\dot\partial_au$.

Extensively studied subcases in which the above results are considerably simplified are, e.g.

1.1) $\sigma(x,y)=\alpha_ix^i,\alpha_i>0$[2]; in this case $\sigma_j=\alpha_j,\dot\sigma_a=0$.

1.2) The Riemannian case, $\sigma=\sigma(x)$, producing $\sigma_j=\partial_j\sigma,\dot\sigma_a=0$.

1.3) The R.K.Tavakol case [8, 3]$\sigma_j=0$.

2.1) The non-dispersive case in relativistic optics $u=\frac{1}{k},k>0$ producing $u_k=0,\dot u_a=0$.

2.2) The strongly dispersive case, $u=u(y)$, in which $u_k=-N_k^a\dot u_a$

Remarks.

a) For computing the Hessian of a function $f\in\mathcal{F}(E)$, are necessary:

— a metrical d-connection (or at least one satisfying $\nabla_X dV = 0, \forall X \in \mathcal{X}(M)$), which we choose to be h- and v-symmetrical, of coefficients (1,2);

— the non-linear connection; in $1^o, 2^o$, it can be provided for $m = n$ by $N_i^a = \gamma_{io}^a$ (γ_{bc}^a being the Levi–Civita connection of a metric $\gamma_{ab}(x)$ on M). Then Hess f can be computed by formulas 2.8;

b) The classical Laplacian Δf needs the Riemannian connection of $\mathcal{G}$. In this case Δf is provided by 2.9.

B. Differential operators on generalised Lagrange, Lagrange and Finsler spaces

If $\xi = (TM, p, M), (m = n)$, for a generalized Lagrange metric $\{g_{ij}(x, y)\}$ [7] we can consider on $E = TM$, the metric $G = (g_{ij}, g_{ij})$.
In this case, considerable reductions are obtained by taking the d-connection to be normal, i.e., $\{L_{jk}^i, C_{bc}^a\}$, which are given in (1), determine the mixed coefficients

$$\begin{cases} \widetilde{L}_{bk}^a = \delta_i^a \delta_b^j L_{jk}^i \\ \widetilde{C}_{ja}^i = \delta_b^i \delta_j^c C_{ca}^b \end{cases}$$

also, we have $h_{ab} = \delta_a^i \delta_b^j g_{ij}$.

Particular cases

3^o The cases 1° and 2°, for $m = n, g_{ij} = h_{ij}$ and canonical d-connection [7].

4^o For a Lagrange space $(M, L(x, y))$ with a p-positively homogeneous Lagrangian, $(p \geq 2), L \in \mathcal{F}(TM)$ being $\mathcal{C}^\infty$ on $TM\backslash\{0\}$ and $\mathcal{C}^0$ on TM. In this case, L produces the canonical nonlinear connection [7]

$$N_j^i = \dot{\partial}_j G^i = \dot{\partial}_j \gamma_{oo}^i = L_{jo}^i, \qquad G^i = \frac{1}{4} g^{ik} \left((\dot{\partial}_k \partial_j L) y^j - \partial_k L \right),$$

and also $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L$. Then $(M, F = \sqrt{[p]L})$ is a Finsler, deflection-free space ($y_{|i}^i = 0$). We consider the energy $\varepsilon = g_{oo}$ and the fields $S = y^i \delta_i, \mathcal{C} = y^a|_a$. It is easy to show that $\mathrm{div}_v S = \mathrm{div}_h \mathcal{C} = 0$ [1] and

$$\begin{cases} \mathrm{div}_h S = y_j^j - y^j P_j = L_{oK}^k - N_k^k - y^k (\dot{\partial}_a N_k^a - L_{ak}^a) = L_{ok}^k + L_{ko}^k - N_k^k - \\ \qquad y^k \dot{\partial}_a N_k^a = 2L_{ko}^k - y^k \dot{\partial}_k N_a^a = 0 \\ \mathrm{div}_v \mathcal{C} = y^b|_b + y^b C_b = \delta_b^b + C_{ob}^b + C_{bo}^b = n + 2\frac{p-2}{2} = n(p-1). \end{cases}$$

Since

$$\varepsilon = g_{oo} = \frac{1}{2} (\dot{\partial}_i \dot{\partial}_j L) y^i y^j = \frac{p-1}{2} (\dot{\partial}_j L) y^j = p \frac{(p-1)}{2} L,$$

using $(\dot\partial_a\varepsilon)|_b=\frac{p^2}{2}g_{ab},\dot\partial_k\varepsilon=py_k$, we infer

$$\Delta_h\varepsilon=0,\Delta_v\varepsilon=n(p-1)p.$$

5^o For the generalized Lagrange case (subcase of 2°) satisfying

$$N_i^a=\gamma_{oi}^a,g_{ij}=e^{2\sigma}\gamma_{ij}(x),\begin{cases}L_{jk}^i=\delta_{\{j}^i\sigma_{k\}}-\gamma_{jk}\gamma^{is}\sigma_s\\ C_{bc}^a=\delta_{\{b}^a\dot\sigma_{c\}}-\gamma_{bc}\gamma^{ad}\dot\sigma_d\end{cases}$$

we have

Proposition 4.1. *The partial divergences of the fields S and $\mathcal{C}$ have the expressions*

$\mathrm{div}_hS=2n\sigma_o$, $\mathrm{div}_vS=0$, $\mathrm{div}_h\mathcal{C}=0$, $\mathrm{div}_v\mathcal{C}=n(1+2\dot\sigma_o)$

Proof. Direct computation provides

$$\mathrm{div}_hS=y^i_{|i}-y^jP_j=\delta^i_j(y^j_{|i})-y^j\delta^a_b(P^b_{ja})=\delta^i_j\delta_iy^j+L^j_{si}y^s-$$

$$-y^j\delta^a_b(\dot\partial_aN^b_j-L^b_{aj})=\delta^i_jN^j_i+L^i_{oi}-(\gamma^a_{ao}-L^a_{ao})=$$

$$=\gamma^i_{io}-\gamma^a_{ao}+2y^s(\delta^i_{\{i}\sigma_{s\}}-\gamma^{ij}\gamma_{si}\sigma_j)=2n\sigma_o.$$

Also, by means of the auxiliary results

$$g^a|_a=\delta^c_b(y^b|_a)=\delta^a_b\delta^b_a+y^sC^b_{sa}=n+y^s(\delta^b_{\{s}\dot\sigma_{b\}}-\gamma_{sb}\gamma^{bt}\dot\sigma_t)=$$

$$=n+y^b\dot\sigma_b+\delta^b_by^s\dot\sigma_s-y^s\dot\sigma_s=n+n\dot\sigma=n(1+\dot\sigma_o)$$

$$\widetilde{C}^i_{ia}=\delta^i_j\widetilde{C}^j_{ia}=\delta^i_j(\sigma^j_{\{i}\dot\sigma_{a\}}-\gamma^{js}\gamma_{ia}\dot\sigma_s)=n\dot\sigma_a+\dot\sigma_a-\dot\sigma_a=n\dot\sigma_a$$

$$y^a\widetilde{C}_a=n\dot\sigma_o$$

we obtain $\mathrm{div}_v\mathcal{C}=y^a|_a+y^a\widetilde{C}_a$. The last result inferes the following

Corollary 4.2. *The relation $div_hS=0$ holds true iff the d-vector field S is orthogonal to* $\mathrm{grad}_h\sigma$.

Remark. Analysing the sign of div $S=\mathrm{div}_hS+\mathrm{div}_vS=2n\sigma_0$ we can decide if the flow generated by the vector field S decreases, conserves, or increases the E-space volume. Similar results can be obtained for the vector field $\mathcal{C}$.

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