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ON AN INTEGRAL INEQUALITY INVOLVING FUNCTIONS AND THEIR DERIVATIVES

BY

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Abstract. The object of this note is to establish a new Opial–type integral inequality for many functions. This inequality generalizes some results of Pachpatte and Beesack.

1. Introduction. The Opial–type integral inequalities are among the most useful tools in the study of many quantitative and qualitative properties of solutions of differential and integral equations. Such integral inequalities involving functions and their derivatives have been established in the literature by many authors (See for example [2],[3] and the references given therein). In this paper, we obtain an integral inequality involving $n(n > 1)$ functions which reduces to a recent generalization (to three functions) of Opial's inequality given by Pachpatte [4].

2. Main result. The following simple results will be needed in the proof of our main result.

Lemma 2.1. *Let f be an absolutely continuous function on $[a, b]$, $-\infty < a < b < \infty$ and let $c \in [a, b]$. Suppose α and β are such that $a \leq \alpha \leq c \leq \beta \leq b$ and $f(\alpha) = 0 = f(\beta)$. If*

$$y(x) = \int_a^x |f'(t)|dt, \quad a \leq x \leq c$$

and

$$u(x) = \int_x^b |f'(t)|dt, \quad c \leq x \leq b \tag{2.1}$$

then

$$|f(x)| \leq y(x) + y(\alpha), \quad a \leq x \leq c$$

and

$$|f(x)| \leq u(x) + u(\alpha), \quad c \leq x \leq b \tag{2.2}$$

Proof. Suppose $a \leq x \leq c$. Since

$$\Big| \int_a^x f'(t)dt \Big| \leq \int_a^x \big|f'(t)\big|dt,$$

we have

$$|f(x) - f(a)| \leq y(x). \tag{2.3}$$

Hence

$$| - f(a)| \leq y(\alpha)$$

i.e.

$$|f(a)| \leq y(\alpha). \tag{2.4}$$

From (2.3), (2.4) and the triangle inequality, we have

$$|f(x)| \leq y(x) + y(\alpha), \quad a \leq x \leq c$$

which is (2.2).

The case when

$$c \leq x \leq b$$

is similarly prove. Our main result is the following :

Theorem 2.2. *Let $\{f_1, f_2, \ldots, f_n\}$ be $n(n \geq 1)$ absolutely continuous functions on $[a, b], -\infty < a < b < \infty$ and let $c \in [a, b]$ and $\alpha_i \in [a, c], \beta_i \in [c, b], i = 1, 2, \ldots, n$ be such that $f_i(\alpha_i) = f_i(\beta_i) = 0$. Suppose k is the smallest positive constant such that*

$$y_i(\alpha) \leq k(\alpha_i - a)y_i(c)$$

and

$$u_i(\beta_i) \leq k(b - \beta_i)y_i(c), \tag{2.6}$$

where

$$y_i(x) = \int_a^x \big|f_i'(t)\big|dt \tag{2.7}$$

and

$$u_i(x) = \int_x^b |f_i'(t)|dt. \tag{2.8}$$

Then

$$\begin{aligned} &\textstyle\int_a^b [|\prod_{i=1}^n f_i(x)|(\sum_{i=1}^n |f_i(x)|)+(\sum_{i=1}^n |f_i(x)|)(\prod_{i=1}^n |\sum_{\substack{j=1\\ j\neq i}}^n f_j(x)f_i'i(x)|)]dx \leq \\ &\leq (c-a)^n \textstyle\int_a^c [\sum_{i=1}^n (k(\alpha_i a)1)^{n+1}|f_i'(x)|^{n+1}]dx+ \\ &+(b-c)^n \textstyle\int_c^b [\sum_{i=1}^n (k(b_i-\beta_i)+1)^{n+1}|f_i'(x)|^{n+1}]dx. \end{aligned} \tag{2.9}$$

Equality holds if for $i = 1, \ldots, n$

$$f_i(x) = \begin{cases} M(x-a), & a \leq x \leq c \\ M(b-x), & c \leq x \leq b \end{cases}$$

where M *is a constant.*

Proof. From (2.7) and (2.8), we have for $i = 1, \ldots, n$

$$y_i'(x) = \big|f_i'(x)\big|, \quad a \leq x \leq c \tag{2.10}$$

and

$$u_i'(x) = -\big|f_i'(x)\big|, \quad c \leq x \leq b. \tag{2.11}$$

Also by Lemma 2.1,

$$\big|f_i(x)\big| \leq y_i(x) + y_i(\alpha_i), \qquad a \leq x \leq c \tag{2.12}$$

and

$$\big|f_i(x)\big| \leq u_i(x) + u_i(\alpha_i), \qquad c \leq x \leq b. \tag{2.13}$$

From (2.10), (2.12), (2.5) and the use of the following :
(a) Arith–Geometric Inequality :

$$\prod_{i=1}^n a_i \leq \left(\frac{1}{n}\sum_{i=1}^n a_i\right)^n;$$

(b) Jensen's Inequality :

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^m \leq \frac{1}{n}\sum_{i=1}^{n} a_i^m, \quad m \geq 1$$

and

(c) Holder's Inequality

where in (a) and (b) above the $a_i, i = 1, 2 \ldots, n$ are positive real numbers, we have

$$\begin{aligned}
&\textstyle\int_a^c \left[\left|\prod_{i=1}^n f_i(x)\right| \left(\sum_{i=1}^n \left|f_i'(x)\right|\right) + \right.\\
&\left. + \left(\sum_{i=1}^n \left|f_i(x)\right|\right) \left(\sum_{i=1}^n \left|f_i'(x) \prod_{j\neq 1}^n f_j(x)\right|\right)\right] dx \leq\\
&\leq \textstyle\int_a^c \left[\left\{\prod_{i=1}^n (y_i(x) + y_i(\alpha_i))\right\} \left(\sum_{i=1}^n y_i'(x)\right) + \right.\\
&\left. + \left\{\sum_{i=1}^n (y_i(x)+y_i(\alpha_i)) \cdot \sum_{i=1}^n y_i'(x) \prod_{j\neq 1}^n (y_j(x)+y_j(\alpha_j))\right\}\right] dx =\\
&= \textstyle\int_a^c \frac{d}{dx} \left[\left\{\prod_{i=1}^n (y_i(x) + y_i(\alpha_i))\right\} \left\{\sum_{i=1}^n (y_i(x) + y_i(\alpha_i))\right\}\right] dx =\\
&= \textstyle\left\{\prod_{i=1}^n (y_i(c) + y_i(\alpha_i))\right\} \left\{\sum_{i=1}^n (y_i(c) + y_i(\alpha_i))\right\} -\\
&- \textstyle\left\{\prod_{i=1}^n (y_i(\alpha_i)\right\} \left\{\sum_{i=1}^n y_i(\alpha_i)\right\} \leq\\
&\leq \textstyle\left\{\prod_{i=1}^n (k(\alpha_i - a) + 1) y_i(c)\right\} \left\{\sum_{i=1}^n (k(\alpha_i - a) + 1) y_i(c)\right\}\\
&\textstyle\left\{\sum_{i=1}^n (k(\alpha_i - a) + 1\right\}^{n+1} \left\{y_i(c)\right\}^{n+1}, \qquad n \geq 1\\
&= \textstyle\left\{\sum_{i=1}^n (k(\alpha_i - a) + 1\right\}^{n+1} \left\{\int_a^c \left|f_i'(x)\right| dx\right\}^{n+1}, \qquad n \geq 1\\
&\leq (c-a)^n \textstyle\sum_{i=1}^n \left\{k(\alpha_i - a) + 1\right\}^{n+1} \int_a^c \left|f_i'(x)\right|^{n+1} dx =\\
&= (c-a)^n \textstyle\int_a^c \left\{\sum_{i=1}^n \left[(k(\alpha_i - a) + 1)\left|f_i'(x)\right|\right]^{n+1}\right\} dx\\
&\textit{i.e. } \textstyle\int_a^c \left[\left|\prod_{i=1}^n f_i(x)\right| \left(\sum_{i=1}^n \left|f_i'(x)\right|\right) + \left(\sum_{i=1}^n \left|f_i(x)\right|\right) \times \right.\\
&\left. \times \left(\sum_{j=1}^n \left|\prod_{i\neq j}^n f_i(x) f_j'(x)\right|\right)\right] dx \leq\\
&\leq (c-a)^n \textstyle\int_a^c \left\{\sum_{i=1}^n \left[(k(\alpha_i - a) + 1)\left|f_1'(x)\right|\right]^{n+1}\right\} dx.
\end{aligned} \tag{2.14}$$

Similarly from (2.11), (2.13),(2.6) and the use of Arithmetic–Geometric, Jensen's and Hölder's inequalities, we obtain

$$\begin{aligned}
&\textstyle\int_a^c [|\prod_{i=1}^n f_i(x)|(\sum_{i=1}^n |f_i'(x)|) +\\
&+(\textstyle\sum_{i=1}^n |f_i(x)|) \cdot (\sum_{j=1}^n |\prod_{i\neq j}^n f_i(x) f_j(x)|)] dx \leq\\
&\leq (b-c)^n \textstyle\int_c^b \left\{\sum_{i=1}^n [(k(b - \beta_i) + 1)|f_i'(x)|]^{n+1}\right\} dx.
\end{aligned} \tag{2.15}$$

Adding (2.14) and (2.15) yields

$$
\begin{aligned}
&\int_a^b [|\prod_{i=1}^n f_i(x)|(\sum_{i=1}^n |f_i'(x)|)+ \\
&+(\sum_{i=1}^n |f_i(x)|)\cdot(\sum_{j=1}^n |\prod_{i\neq j}^n f_i(x)f_j'(x)|)]dx \leq \\
&\leq (c-a)^n \int_a^c \{\sum_{i=1}^n [(k(\alpha_i - a)+1)|f_i'(x)|]^{n+1}\}dx + \\
&+(b-c)^n \int_c^b \{\sum_{i=1}^n [(k(b-\beta_i)+1)|f_i'(x)|]^{n+1}\}dx.
\end{aligned} \tag{2.15'}
$$

That equality holds if for $i = 1, 2, \ldots, n$.

$$
f_i(x) = \begin{cases} M(x-a), & a \leq x \leq c \\ M(b-x), & c \leq x \leq b \end{cases}
$$

where M is a constant. As consequences of Theorem 2.2, we have the following results:

Corollary 2.3. *Let $f_1, \ldots, f_2$ be $n (n \geq 1)$ absolutely continuous functions, on $[a, b]$, $-\infty < a < b < \infty$, and let $c \in [a, b]$. Suppose $f_i(a) = 0 = f_i(b), \quad i = 1, \ldots, n$. Then*

$$
\begin{aligned}
&\int_a^b [\prod_{i=1}^n \left|f_i(x)\right|(\sum_{i=1}^n \left|f_i'(x)\right|)+ \\
&+(\sum_{i=1}^n \left|f_i(x)\right|)(\sum_{j=1}^n \left|\prod_{i\neq j}^n f_j(x)f_j'(x)\right|)]dx \leq \\
&\leq (c-a)^n \int_a^c \sum_{i=1}^n \left|f_i'(x)\right|^{n+1} dx + (b-c)^n \int_c^b \sum_{i=1}^n \left|f_i'(x)\right|^{n+1} dx
\end{aligned} \tag{2.16}
$$

In particular, if

$$
c = \frac{(a+b)}{2},
$$

we obtain

$$
\begin{aligned}
&-\int_a^b [\prod_{i=1}^n |f_i(x)|(\sum_{i=1}^n |f_i'(x)|)+ \\
&+(\sum_{i=1}^n |f_i(x)|)(\sum_{i=1}^n |\prod_{i\neq j}^n f_i(x)f_j'(x))|]dx \leq \\
&\leq (\tfrac{b-a}{2})^n \int_a^b \sum_{i=1}^n |f_i'(x)|^{n+1} dx.
\end{aligned} \tag{2.17}
$$

Proof. Inequality (2.16) follows from Theorem (2.2) by taking $\alpha_i = a$ and $\beta_i = b, \quad i = 1, \ldots, n$ while putting

$$
c = \frac{a+b}{2}
$$

gives

$$
c - a = b - c = \frac{b-a}{2}
$$

and hence inequality (2.16) follows.

Remark 1. If $n = 3$, we obtain from (2.17), the result of Pachpattee [2].

Corollary 2.4. *Let f be an absolutely continuous function on $[a,b], -\infty \le a < b \le \infty$ and let $c \in [a,b]$ and $\alpha \in [a,c],\ \beta \in [c,b]$ such that $f(\alpha) = 0 = f(\beta)$. For some constant $k, 0 < k < \infty$, anf for $n \in Z^+ (n \ge 1)$, we have*

$$\begin{aligned} &\textstyle\int_a^b |(f(x))^n f'(x)|dx \le \\ &\le \tfrac{1}{n+1}[(c-a)^n\{k(\alpha-a)+1\}^{n+1} \textstyle\int_a^c |f'(x)|^{n+1}dx + \\ &+(b-c)^n\{k(b-\beta)+1\}^{n+1} \textstyle\int_c^b |f'(x)|^{n+1}dx]. \end{aligned} \tag{2.18}$$

In particular

$$\int_a^b |(f(x))^n f'(x)|dx \le \frac{1}{n+1}\left(\frac{b-a}{2}\right)^n \int_a^b \left|f'(x)\right|^{n+1}dx. \tag{2.19}$$

Proof. If we set $f_i(x) = f(x),\ i = 1, \ldots, n$ in Theorem 2.2,

$$\int_a^b \left|(f(x))^n f'(x)\right|dx$$

we obtain (2.18). Further, if we put $\alpha = a,\ \beta = b$ and

$$c = \frac{a+b}{2}$$

in (2.18), we obtain (2.19).

Remark 2. When $n = 1$, inequality (2.19) yields Opial's inequality ([3]).

REFERENCES

1. H a r d y, G.H., L i t t l e w o o d, J.E. and P o l y a, G. – *Inequalities*, Cambridge University Press, 1952.
2. M i t r o n o v i c, D.S. (in cooperation with V a s i c, P.M.) – *Analytic Inequalities*, Berlin–Heidelberg–New York, 1970.
3. O p i a l, Z. – *Sur une inégalité*, Ann. Polon. Math., 8 (1960), pp.29–32.
4. P a c h p a t t e e, B.G. – *On an Integral Inequality Involving Functions and Their Derivatives*, Soochow Journal of Mathematics, Vol.13, No.2, (1987), 211–214.

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