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ON H–CLOSED SPACES AND GRILLS

BY

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Abstract. In this paper, an investigation of (quasi) H–closed spaces is undertaken in terms of grill as the basic appliance. This adds to the list of varieted techniques available so far for studying an H–closed space. In the first part of the deliberation, the theory of grills is developed to certain extent, which serves as the prerequisite to support for the sought – for characterizations and study of (quasi) H–closedness and allied notions in the latter part of the paper.

Keywords: Grill, (Quasi) H–closed, θ–closure, θ–adherence, grill, θ–linkage (quasi) H–closed.

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1. Introduction and preliminaries. The theory of H–closed spaces owes its initiation to the classical paper of A l e x a n d r o f f and U r y s o h n [1] in mid–twenties. The last seventy years have witnessed a great deal of activities concerning H–closed and quasi H–closed spaces from different orientations and via different approaches and appliances. Among the techniques adopted so far are by covers, filters and filterbases, nets, closed graphs, projections, cluster sets etc. In this regard, we refer to the papers [6,7,13] and especially to the work of B e r r i et al. [2] for a comprehensive description and a fairly good list of literature concerning H–closed and some allied types of spaces.

The concept of grill was introduced by C h o q u e t [5], and an extensive study of the same was made by T h r o n [12] and many others (e.g. see [4]). In certain situations, grills turn out to be a very useful device. Especially in the theory of proximity spaces and in discussing compactifications, the role of grills is undeniable.

The intent of this paper is to try for an investigation of (quasi) H–closed spaces through the concept of grills. We require, to this end, to elaborate certain concepts concerning grills, and we take up this general discussion on the theory of grills in the next section which is followed by our intended study of (quasi) H–closed spaces in the last section.

Throughout this paper, a space X or simply X will mean a topological space (X, τ), η_x will denote the open neighbourhood (henceforth nbd) system at the point $x \in X$, and $\mathcal{P}(X)$ shall stand for the power set of X. For a set A in X, let us denote the interior and the closure of A by $\operatorname{int} A$ and $\operatorname{cl} A$ respectively. The collection of all closed sets in (X, τ) will be denoted by σ.

A point x of a space X is called a θ–adherent (δ–adherent) point [13] of a subset A of X iff $\operatorname{cl} U \cap A \neq \emptyset$ (resp. $\operatorname{intcl} U \cap A \neq \emptyset$), for every open nbd U of x. The set of all θ–adherent (δ–adherent) points of A is called the θ–closure (resp. δ–closure) of A [13], to be denoted by θ–$\operatorname{cl} A$ (resp. δ–$\operatorname{cl} A$). A set A in X is called regular open (regular closed) iff $A = \operatorname{intcl} A$ (resp. $A = \operatorname{clint} A$). The δ–open sets in X form a topology on X, called the semi–regularization topology which is weaker than the topology on X, regular open sets on X being a base for it [3]. It is well–known [12] that for any open set U in X, $\operatorname{cl} U = \theta$–$\operatorname{cl} U = \delta$–$\operatorname{cl} U$.

By a grill on a space X, we mean a subfamily Ω of $\mathcal{P}(X)$ such that (i) $\emptyset \notin \Omega$, (ii) $(A \in \Omega,\ B \in \mathcal{P}(X), A \subseteq B \Longrightarrow B \in \Omega)$ and (iii) $(A, B \in \mathcal{P}(X),\ A \cup B \in \Omega \Longrightarrow A \in \Omega$ or $B \in \Omega)$. The collection $\Omega(c, x) = \{A \subseteq X : x \in \operatorname{cl} A\}$, where $x \in X$, is a grill on X, called the adherence grill [12] at $x \in X$. Moreover, $\{\Omega(c, x) : x \in X\}$ completely determines the Kuratowski closure operator c; for we have $c(A) = \{x \in X : A \in \Omega(c, x)\}$ [4]. For a grill Ω, we define $\sec \Omega = \{A \in \mathcal{P}(X) : A \cap G \neq \emptyset$, for each $G \in \Omega\}$. If Ω is a grill (filter) on a space X, then $\sec \Omega$ is a filter (grill) on X [12]. Further if $\mathcal{F}$ and Ω are respectively a filter and a grill on X with $\mathcal{F} \subseteq \Omega$, then there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \Omega$, [12]. A filter $\mathcal{F}$ is said to θ–adhere at $x \in X$ (θ–converges to $x \in X$) iff for each $F \in \mathcal{F}$ and each $U \in \eta_x$, $F \cap \operatorname{cl} U \neq \emptyset$ (resp. to each $F \in \mathcal{F}$, there corresponds a $U \in \eta_x$ such that $F \subseteq \operatorname{cl} U$) [13].

A topological space X is called almost regular [11] iff for every regular closed set A in X and for each $x \notin A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. We shall sometimes need the result that in an almost regular space X, θ–$\operatorname{cl} A$ is θ–closed (i.e. θ–$\operatorname{cl}(\theta - \operatorname{cl} A) = \theta$–$\operatorname{cl} A$), for each $A \subseteq X$.

A Hausdorff space is said to be an H–closed space [1] iff its homeomorphic image into any Hausdorff space is closed as a subspace.

The following theorem gives a short list of the well–known characterizations of H–closed spaces, that we shall require to use in the sequel.

Theorem 1.1. [8,10,13] *For a Hausdorff topological space X, the following are equivalent:*

(a) *X is H–closed.*

(b) *For every open cover of X, there is a finite proximate subcover, i.e., there is a finite subfamily whose union is dense in X.*

(c) *For each family of regular closed sets $\{F_\alpha\}$ such that $\bigcap\limits_\alpha F_\alpha = \emptyset$, there exists a finite subfamily $\{F_{\alpha i} : i = 1, 2, ..., n\}$ such that $\bigcap\limits_{i=1}^{n} \operatorname{int} F_{\alpha i} = \emptyset$.*

(d) *Every filterbase on X θ–adheres.*

(e) *Every open filterbase on X has non–void adherence.*

(f) *Every ultrafilterbase on X θ–converges in X.*

(g) *Every open ultrafilterbase on X converges in X.*

It is known that for any non–Hausdorff space X also, the conditions (b)–(g) of the above theorem remain equivalent. A space satisfying any one of these equivalent conditions is called a quasi H–closed space. Thus a quasi H–closed space X is an H–closed space iff X is Hausdorff. Consequently, any condition formulating a quasi H–closed space characterizes an H–closed space too, when in addition, the underlying space is assumed to be Hausdorff.

2. Grills and filters. In this section we intend to develop the theory of grills to the extent that we require to apply the concept to the study of (quasi) H–closed spaces to be taken up in the next section. Thus our endeavour here is to discuss the grill theory to some length and observe especially the interplay between grills and filters.

Definition 2.1. A grill Ω on a space X is said to

(a) adhere (θ–adhere) at $x \in X$ iff for each $U \in \eta_x$ and each $G \in \Omega$, $U \cap G \neq \emptyset$ (resp. $\operatorname{cl} U \cap G \neq \emptyset$);

(b) converge (θ–converge) to $x \in X$ iff for each $U \in \eta_x$, $G \subseteq U$ (resp. $G \subseteq \operatorname{cl} U$), for some $G \in \Omega$.

Remark 2.2. An immediate consequence of the definition of θ–convergence of a grill is that, a grill Ω on a space X, θ–converges to some $x \in X$ iff $\{\operatorname{cl} U : U \in \eta_x\} \subset \Omega$.

Theorem 2.3. *If a grill Ω on a space X θ–adheres at $x \in X$, then Ω θ–converges to x.*

Proof. The grill Ω on X θ–adheres at $x \in X \Longrightarrow$ For each $G \in \Omega$ and each $U \in \eta_x$, $\mathrm{cl}\, U \cap G \neq \emptyset \Longrightarrow \mathrm{cl}\, U \in \sec \Omega$, for each $U \in \eta_x \Longrightarrow X \backslash \mathrm{cl}\, U \notin \Omega$, for each $U \in \eta_x \Longrightarrow \mathrm{cl}\, U \in \Omega$, for each $U \in \eta_x$, [since $X = \mathrm{cl}\, U \cup (X \backslash \mathrm{cl}\, U) \in \Omega$]. Hence, by Remark 2.2, Ω θ–converges to x.

Remark 2.4. We note that unlike a filter, θ–adherence of a grill implies its θ–convergence. The crux of the theory of grills lies in the fact that θ–adherence is stictly stronger than θ–convergence. In fact the following simple example shows that a θ–convergent grill may not θ–adhere at any point of the space even if the space if finite. For further clarification, we refer to Example 3.15.

Example 2.5. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then (X, τ) is a topological space. Define $\Omega = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. It can be verified that Ω is a grill on X, which θ–converges to $c \in X$, while Ω does not θ–adhere at any of the three points of X.

Definition 2.6. A non–empty subcollection Ω of σ (τ) in a topological space (X, τ) is said to form a closed (resp. open) grill on X iff (a) $\emptyset \notin \Omega$; (b) $A \in \Omega$ and $C \in \sigma$ ($C \in \tau$) with $A \subseteq C \Longrightarrow C \in \Omega$; and (c) $A, B \in \sigma$ (resp. $A, B \in \tau$), with $A \cup B \in \Omega \Longrightarrow A \in \Omega$ or $B \in \Omega$.

Remark 2.7. It is quite evident from the above definition that if Ω is a grill on X then $\Omega \cap \tau$ and $\Omega \cap \sigma$ are open and closed grills respectively.

Theorem 2.8. *If Ω is a closed (open) grill on a space (X, τ) then* $\sec \Omega \cap \tau$ *(resp.* $\sec \Omega \cap \sigma$*) is an open (resp. closed) filter on X.*

Proof. Clearly $\sec \Omega \cap \tau$ ($\sec \Omega \cap \sigma$) is a non–empty subcollection of τ (resp. σ) such that $\emptyset \notin \sec \Omega \cap \tau$ ($\emptyset \notin \sec \Omega \cap \sigma$). If $A \in \sec \Omega \cap \tau$ ($A \in \sec \Omega \cap \sigma$) and $B \in \tau$ ($B \in \sigma$) with $A \subseteq B$, then it is clear that $B \in \sec \Omega \cap \tau$ ($B \in \sec \Omega \cap \sigma$). Now, $A, B \in \sec \Omega \cap \tau$ ($A, B \in \sec \Omega \cap \sigma$) $\Longrightarrow A, B \in \tau$ (σ) and $X \backslash A, X \backslash B \notin \Omega \Longrightarrow A, B \in \tau(\sigma)$ and $(X \backslash A) \cup (X \backslash B) = X \backslash (A \cap B) \notin \Omega$, since Ω is a closed (open) grill on $X \Longrightarrow A, B \in \tau$ (σ) and $A \cap B \in \sec \Omega \Longrightarrow A \cap B \in \sec \Omega \cap \tau$ ($A \cap B \in \sec \Omega \cap \sigma$). Thus, $\sec \Omega \cap \tau$ ($\sec \Omega \cap \sigma$) is an open (resp. closed) filter on X.

Theorem 2.9. *If Ω is an open (closed) filter on X then* $\sec \Omega \cap \sigma$ *(resp.* $\sec \Omega \cap \tau$*) is a closed (resp. open) grill on X.*

Proof. The proof, being similar to that of the last theorem, is omitted.

Theorem 2.10. *In a topological space X, every grill converges iff every open grill converges.*

Proof. We first assume that every grill on X converges in X and let Ω be any open grill on X. By Theorem 2.8, $\sec\Omega\cap\sigma$ is a closed filter on X and hence a filter–base on X. Let $\mathcal{F}$ be the filter generated by the filter–base $\sec\Omega\cap\sigma$. Then as $\sec\mathcal{F}$ is a grill on X, it converges to some $x\in X$. Hence for each $F\in\mathcal{F}$ and each $U\in\eta_x$, $U\cap F\neq\emptyset$. Now $\sec\Omega\cap\sigma$ being a base for $\mathcal{F}$, $U\cap R\neq\emptyset$ for each $U\in\eta_x$ and each $R\in\sec\Omega\cap\sigma$. We claim that $\eta_x\subseteq\Omega$. For otherwise, there is some $U\in\eta_x$ for which $X\backslash U\in\sec\Omega\cap\sigma$, a contradiction. Hence $\eta_x\subseteq\Omega$, proving that Ω converges to x.

Conversely, let us assume that every open grill on X converges in X and suppose Ω is any grill on X. Then $\Omega\cap\tau$ is an open grill on X, by Remark 2.7. By hypothesis, $\Omega\cap\tau$ converges to some $x\in X$, so $\eta_x\subseteq\Omega\cap\tau\subseteq\Omega$. Hence, Ω converges to x.

Theorem 2.11. *In a topological space, every open grill converges iff every closed filter adheres.*

Proof. Let Ω be any open grill on X in which every closed filter adheres. By Theorem 2.8, $\sec\Omega\cap\sigma$ is a closed filter on X, and hence it adheres at some $x\in X$. Then $P\cap U\neq\emptyset$, for each $P\in\sec\Omega\cap\sigma$ and each $U\in\eta_x$. Clearly, $U\in\Omega$, for each $U\in\eta_x$; for otherwise there would exist some $U\in\eta_x$ such that $X\setminus U\in\sec\Omega\cap\sigma$, which is a contradiction. Hence, Ω converges to x. Conversely, let $\mathcal{F}$ be a closed filter on X. Then $\sec\mathcal{F}\cap\tau$ is an open grill, by Theorem 2.9. By hypothesis, $\eta_x\subseteq\sec\mathcal{F}\cap\tau$, for some $x\in X$. Thus for each $U\in\eta_x$ and each $F\in\mathcal{F}$, $F\cap U\neq\emptyset$ so that $\mathcal{F}$ adheres at x.

Corollary 2.12. *In a topological space, every open grill converges iff every filter adheres.*

Proof. The corollary follows immediately from the fact that in a topological space every filter adheres iff every closed filter adheres.

It is well–known that a necessary and sufficient condition for a topological space X to be compact is that every filter adheres in X. So by Theorems 2.10 and 2.11 we obtain the following chracterizations of compact spaces:

Theorem 2.13. *In a topological space X, the following are equivalent:*

(i) *X is compact.*

(ii) *Every grill converges in X.*

(iii) *Every open grill converges in* X.

Theorem 2.14. *In a topological space, every open filter adheres iff every closed grill* θ*–converges.*

Proof. We assume tht every open filter on X adheres in X. Let Ω be any closed grill on X. Then by Theorem 2.8, $\sec\Omega \cap \tau$ is an open filter on X and hence adheres at some $x \in X$. Thus for each $U \in \eta_x$ and each $P \in \sec\Omega \cap \tau$,

$$P \cap U \neq \emptyset. \tag{1}$$

Supose, if possible, for some $U \in \eta_x$, $X\backslash \mathrm{cl}\, U \in \sec\Omega \cap \tau$ and then by (1), $(X\backslash \mathrm{cl}\, U) \cap U \neq \emptyset$, a contrdiction. Hence, $\mathrm{cl}\, U \in \Omega$ for each $U \in \eta_x$. By Remark 2.2, Ω θ–converges to x.

Conversely, for any open filter $\mathcal{F}$ on X, $\sec\mathcal{F} \cap \sigma$ is a closed grill on X and hence θ–converges in X to some $x \in X$. Then $\{\mathrm{cl}\, U : U \in \eta_x\} \subset \sec\mathcal{F} \cap \sigma$ (by Remark 2.2), which in turn gives $\mathrm{cl}\, U \cap F \neq \emptyset$, for each $F \in \mathcal{F}$ and each $U \in \eta_x$. Thus $\mathcal{F}$ θ–adheres at x. $\mathcal{F}$ being an open filter, it adheres at x, by Theorem 1.1.

3. (Quasi) H–closedness via grills. As already stated, our prime objective in this section is to study quasi H–closed spaces (and hence H–closedness for Hausdorff topological spaces) in terms of certain types of grills. In order to achieve our desired results we require to introduce a few definitions and some relevant theorems.

Theorem 3.1. *For each point* x *in* X*, the collection* $\Omega(\theta, x) = \{A \subseteq X : x \in \theta\text{–cl}\, A\}$ *is a grill on* X.

Proof. Straightforward.

Definition 3.2. The grill $\Omega(\theta, x)$ (of the above theorem), for $x \in X$, is called the θ–adherence grill on X at x.

These θ–adherence grills, as defined above, are very useful in characterizing θ–adherence and θ–convergence of a grill, as is apparent from what follows next.

Theorem 3.3. *A grill* Ω *on a space* X θ*–adheres at some point* $x \in X$ *iff* $\Omega \subseteq \Omega(\theta, x)$.

Proof. A grill Ω on X θ–adheres at $x \in X \Longleftrightarrow \mathrm{cl}\, U \cap G \neq \emptyset$, for each $U \in \eta_x$ and each $G \in \Omega \Longleftrightarrow x \in \theta - \mathrm{cl}\, G$, for each $G \in \Omega \Longleftrightarrow G \in \Omega(\theta, x)$, for each $G \in \Omega \Longleftrightarrow \Omega \subseteq \Omega(\theta, x)$.

Remark 3.4. For any space X, every θ–adherence grill at $x \in X$ evidently θ–adheres at x, thereby giving a justification for the terminology.

Theorem 3.5. *A grill Ω on a space X θ–converges to a point x in X iff* $\sec \Omega(\theta, x) \subseteq \Omega$.

Proof. First suppose tht the grill Ω on X θ–converges to $x \in X$. Then for any $U \in \eta_x$, there exists $G \in \Omega$ such that $G \subseteq \mathrm{cl}\, U$ and hence

$$\mathrm{cl}\, U \in \Omega. \tag{1}$$

Now, $B \in \sec \Omega(\theta, x) \Longrightarrow X \backslash B \notin \Omega(\theta, x) \Longrightarrow x \notin \theta - \mathrm{cl}\,(X \backslash B) \Longrightarrow \mathrm{cl}\, U \cap (X \backslash B) = \emptyset$, for some $U \in \eta_x \Longrightarrow \mathrm{cl}\, U \subseteq B$, where $U \in \eta_x \Longrightarrow B \in \Omega$ (by (1) and since Ω is a grill). Thus $\sec \Omega(\theta, x) \subseteq \Omega$.

Conversely, let $V \in \eta_x$. Since $x \in V \subseteq \theta - \mathrm{cl}\, V = \mathrm{cl}\, V$ (V being open), $x \notin \theta - \mathrm{cl}\,(X \backslash \theta - \mathrm{cl}\, V)$. Thus $(X \backslash \theta - \mathrm{cl}\, V) \notin \Omega(\theta, x)$, and consequently, $\theta - \mathrm{cl}\, V \in \sec \Omega(\theta, x) \subseteq \Omega$. As V is open, $\mathrm{cl}\, V = \theta - \mathrm{cl}\, V \in \Omega$; and this is true for each $V \in \eta_x$. Hence Ω θ–converges to x.

Definition 3.6. A grill Ω on a space X is said to be a

(a) θ–linked grill on X iff for any two members A, B of Ω, $\theta - \mathrm{cl}\, A \cap \cap \theta - \mathrm{cl}\, B \neq \emptyset$,

(b) θ–conjoint grill on X iff for every finite subfamily $\{A_1, A_2, ..., A_n\}$ of Ω, $\mathrm{int} \left[\bigcap_{i=1}^{n} \theta - \mathrm{cl}\, A_i \right] \neq \emptyset$.

Remark 3.7. Every θ–conjoint grill is evidently a θ–linked grill. But the converse is not, in general, true (see Example 3.15).

We are now in a position to use the appliances described so far to the investigtion of (quasi) H–closed space, some characterization of such a space being given as follows.

Theorem 3.8. *For a (Hausdorff) topological space X, the following are equivalent:*

(a) *X is quasi H–closed (resp H–closed).*

(b) *Every grill on X θ–converges in X.*
(c) *Every closed grill θ–converges in X.*
(c) *Every open θ–conjoint grill adheres in X.*

Proof. (a) $\Longrightarrow$ (b): Let Ω be any grill on a quasi H–closed space X. Then $\sec\Omega$ is a filter on X. Now $B \in \sec\Omega \Longrightarrow X\backslash B \notin \Omega \Longrightarrow B \in \Omega$ (since $X\backslash B \notin \Omega$ and $B \notin \Omega \Longrightarrow X = [(X\backslash B) \cup B] \notin \Omega$, which is not true). Thus $\sec\Omega \subseteq \Omega$. Consequently, there exists an ultrafilter $\mathcal{U}$ on X such that $\sec\Omega \subseteq \mathcal{U} \subseteq \Omega$. As X is quasi H–closed, in view of Theorem 1.1, the ultrafilter $\mathcal{U}$ θ–converges to some point x in X. Thus for each $U \in \eta_x$, $F \subseteq \mathrm{cl}\, U$ for some $F \in \mathcal{U}$. Consequently, $\mathrm{cl}\, U \in \mathcal{U} \subseteq \Omega$, i.e., $\mathrm{cl}\, U \in \Omega$, for each $U \in \eta_x$. Hence by Remark 2.2, Ω θ–converges to x in X.

(b) $\Longrightarrow$ (a): By virtue of Theorem 1.1 it is enough to show that every ultrafilter on X θ–converges in X, which is immediate from the fact that an ultrafilter on X is a grill on X as well.

(a) $\Longleftrightarrow$ (c): The proof follows from Theorem 2.14, as a (Hausdorff) space X is quasi H–closed (H–closed) iff every open filter adheres in X.

(a) $\Longrightarrow$ (d): Consider any open θ–conjoint grill Ω on a (quasi) H–closed space X. Then $\{\mathrm{cl}\, A : A \in \Omega\}$ is a collection of regular closed sets in X such that

$$\mathrm{int}\left[\bigcap_{i=1}^{n}\theta - \mathrm{cl}\, A_i\right] = \mathrm{int}\left[\bigcap_{i=1}^{n}\mathrm{cl}\, A_i\right] \text{ (as each } A_i \text{ is open)} \neq \emptyset,$$

for each finite sub–family $\{A_1, A_2, ..., A_n\}$ of Ω. So by Theorem 1.1, $\cap\{\mathrm{cl}\, A : A \in \Omega\} \neq \emptyset$. If $x \in \cap\{\mathrm{cl}\, A : A \in \Omega\}$, then Ω adheres at $x \in X$.

(d) $\Longrightarrow$ (a): Let every open θ–conjoint grill adhere in X. By virtue of Theorem 1.1, it is only to be shown that any open ultrafilter on X adheres in X. Let $\mathcal{U}$ be an open ultrafilter on X and hence an open grill on X. For any finite subfamily $\{A_1, A_2, ..., A_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} A_i \neq \emptyset$ and consequently $\mathrm{int}\left[\bigcap_{i=1}^{n} A_i\right] = \bigcap_{i=1}^{n} A_i \neq \emptyset$. Also, $\mathrm{int}\left[\bigcap_{i=1}^{n}\theta - A_i\right] \supseteq \left[\bigcap_{i=1}^{n} A_i\right] \neq \emptyset$ which proves that $\mathcal{U}$ is an open θ–conjoint grill on X and hence by hypothesis, $\mathcal{U}$ adheres in X.

We now obtain conditions slightly different from that of Theorem 3.8(d), which are respectively necesary and sufficient for a (Hausdorff) space to be quasi H–closed (resp. H–closed).

Theorem 3.9. *Let X be a (Hausdorff) topological space. Then the following implications hold:* (a)$\Longrightarrow$(b)$\Longrightarrow$(c), *where*

(a) *Every grill Ω on X with the property that $\bigcap_{i=1}^{n}\theta - \mathrm{cl}\,(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of Ω, θ-adheres in X,*

(b) *X is quasi H–closed (H–closed),*

(c) *every θ–conjoint grill θ–adheres in X.*

The conditions are equivalent if X is almost regular.

Proof. (a) $\Longrightarrow$ (b): Let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U}$ is a grill on X and also for each finite subcollection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$, $\cap\{\theta - \mathrm{cl}\,(U_i) : i = 1, ..., n\} \supset \cap\{U_i : i = 1, ..., n\} \neq \emptyset$, so that $\mathcal{U}$ is a grill with the given property. Hence by hypothesis, $\mathcal{U}$ θ–adheres. Consequently, by Theorem 1.1, the (Hausdorff) space X is quasi H–closed (resp. H–closed).

(b) $\Longrightarrow$ (c): Consider any θ–conjoint grill Ω on a quasi H–closed space X. Then $\{\theta - \mathrm{cl}\,(A) : A \in \Omega\}$ is a collection of δ–closed sets in X such that

$$\mathrm{int}\,[\cap\{\theta - \mathrm{cl}\,(A_i) : i = 1, ..., n\}] \neq \emptyset, \tag{1}$$

for each finite subfamily $\{A_1, A_2, ..., A_n\}$ of Ω.

Since $\theta - \mathrm{cl}\,(A)$ is δ–closed for each $A \in \Omega$,

$$\theta - \mathrm{cl}\,(A) = \cap\{B_\alpha : \alpha \in \Lambda_A\}, \tag{2}$$

where each B_α is regular closed in X, Λ_A being an indexed set.

Consider $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda_A,\ A \in \Omega\}$. Then $\mathcal{B}$ is a collection of regular closed sets in X. Now, let $\{B_{\alpha i} : i = 1, 2, ..., k\}$ be any finite subfamily of $\mathcal{B}$. To each $B_{\alpha i}$ $(i = 1, 2, ..., k)$ there corresponds some $A_{\alpha i} \in \Omega$ such that $\theta - \mathrm{cl}\,(A_{\alpha i}) \subseteq B_{\alpha i}$. Then $\cap\{\theta - \mathrm{cl}\,(A_{\alpha i}) : i = 1, ..., k\} \subseteq \{B_{\alpha i} : i = 1, ..., k\}$. Hence using (1), $\emptyset \neq \mathrm{int}\,[\cap\{\theta - \mathrm{cl}\,(A_{\alpha i}) : i = 1, ..., k\}] \subseteq \mathrm{int}\,[\cap\{B_{\alpha i} : i = 1, ..., k\}]$. So by theorem 1.1, $\cap\{B_\alpha : \alpha \in \bigcup_{A\in\Omega} \Lambda_A\} \neq \emptyset$. But by (2),

$$\cap\{B_\alpha : \alpha \in \cup\Lambda_A\} = \cap\{\theta - \mathrm{cl}\,(A) : A \in \Omega\}.$$

Hence there is $x \in X$ such that $x \in \cap\{\theta - \mathrm{cl}\,(A) : A \in \Omega\}$, i.e., $A \in \Omega(\theta, x)$ for each $A \in \Omega$. In other words Ω θ-adheres in X, by Theorem 3.3.

(c) $\Longrightarrow$ (a): Let $\mathcal{F}$ be a θ–conjoint grill on an almost regular quasi H–closed space. Then the collection $\mathcal{F}^* = \{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\}$ has the finite intersection property, and hence by Theorem 1.1, $\mathcal{F}^*$ θ–adheres, i.e.,

$\emptyset \neq \cap\{\theta - \mathrm{cl}\,(\theta - \mathrm{cl}\,F) : F \in \mathcal{F}\} = \cap\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\}$ (since X is almost regular). Hence $\mathcal{F}$ θ–adheres.

As a Urysohn H–closed space is known [9] to be almost regular, we at once get,

Corollary 3.10. *A Urysohn space X is H–closed iff any one of the following conditions holds:*

(a) *Every grill Ω on X with the property that $\bigcap_{i=1}^{n}\theta - \mathrm{cl}\,A_i \neq \emptyset$ for every finite subfamily $\{A_1, A_2, ..., A_n\}$ of Ω, θ–adheres in X.*

(b) *Every θ–conjoint grill θ–adheres.*

Another sufficient condition for (quasi) H–closedness which is equivalent to that of Theorem 3.9(a) is obtained from the following:

Theorem 3.11. *Every grill Ω on a space X with the property that $\cap\{\theta - \mathrm{cl}\,G_i : i = 1, ..., n\} \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of Ω, θ–adheres in X iff for every family $\mathcal{F}$ of subsets of X with the finite intersection property for the family*

$$\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\},\ \cap\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\} \neq \emptyset.$$

Proof. Let every grill with the given condition on a space X θ–adhere in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\}$ has finite intersection property. Let $\mathcal{U}$ be the collection of all those families Ω of subsets of X for which $\Omega^* = \{\theta - \mathrm{cl}\,G : G \in \Omega\}$ has finite intersection property and $\mathcal{F} \subset \Omega$. Then $\mathcal{F} \in \mathcal{U}$ and $\mathcal{U}$ is partially ordered by inclusion relation, in which every chain has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a grill with the given property. Hence $\cap\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\} \supset \cap\{\theta - \mathrm{cl}\,U : U \in \mathcal{U}^*\} \neq \emptyset$.

Conversely, let $\mathcal{F}$ be a grill on X with the given property. Then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\cap\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}_0\} \neq \emptyset$, and hence by hypothesis, $\cap\{\theta - \mathrm{cl}\,F : F \in \mathcal{F}\} \neq \emptyset$ so that $\mathcal{F}$ θ–adheres.

Definition 3.12. A (Hausdorff) space X is called θ–linkage quasi H–closed (resp. θ–linkage H–closed) iff every θ–linked grill θ–adheres in X.

Theorem 3.13. *Every θ–linkage (quasi)H–closed space is (quasi) H–closed.*

Proof. Follows from Remark 3.7 and Theorem 3.9.

Remark 3.14. The next example shows that the converse of the above Theorem is not true.

Example 3.15. Let us consider the space X of Example 7.2 of [4], where $X = \{x : x \text{ is a real number and } -2x < 2\}$ and d be the ordinary closure operator induced from the reals. An operator k on X is defined by specifying the adherence grills at each point $x \in X$ as follows:
$\Omega(k, x) = \Omega(d, x)$, for all $x \notin \{-1, 0, 1\}$
$\Omega(k, -1) = \Omega(d, -1) \cup \{A : A \cap (-\varepsilon, 0)$ is infinite for all $\varepsilon > 0\} \cup \{B : B \cap [(-2, -2 + \varepsilon) \cup (2 - \varepsilon, 2)]$ is infinite for all $\varepsilon > 0\}$
$\Omega(k, 0) = \Omega(d, 0) \cup \{A : A \cap (-1 - \varepsilon, -1 + \varepsilon)$ is infinite for all $\varepsilon > 0\} \cup \{B : B \cap (1 - \varepsilon, 1 + \varepsilon)$ is infinite for all $\varepsilon > 0\}$
$\Omega(k, 1) = \Omega(d, 1) \cup \{A : A \cap (0, \varepsilon)$ is infinite for all $\varepsilon > 0\} \cup \{B : B \cap [(-2, -2 + \varepsilon) \cup (2 - \varepsilon, 2)]$ is infinite for all $\varepsilon > 0\}$

It can be verified that k is a Kuratowski closure operator on X inducing a topology τ (say) on X and that (X, τ) is a T_1–compact topological space, and hence a quasi H–closure space.

Now, consider the family $\Omega = \{A : A \in \Omega(k, -1) \cup \Omega(k, 1)$ and A is infinite$\}$.

It is easy to see that Ω is a θ–linked grill on (X, τ).

We choose

$$A_1 = (-1 - 1/3, -1 + 1/3) \backslash \{-1\},$$
$$A_2 = (1 - 1/3), 1 + 1/3) \backslash \{1\} \text{ and}$$
$$A_3 = (2 - 1/3, 2).$$

Then clearly $A_1, A_2, A_3 \in \Omega$. We claim that A_1, A_2 and A_3 are open in (X, τ). In fact, if A_1 is not open, there exists $x \in A_1 \cap k(X \backslash A_1)$. But $x \in A_1 \Longrightarrow x \notin \{-1, 0, 1\}$; and $x \in k(X \backslash A_1) \Longrightarrow X \backslash A_1 \in \Omega(k, x) = \Omega(d, x)$ (as $x \notin \{-1, 0, 1\}) \Longrightarrow x \in d(X \backslash A_1)$, which is impossible. Thus A_1 is open in (X, τ). Similarly, we can show that A_2, A_3 are also open in (X, τ). Hence $k(A_i) = \theta - \operatorname{cl} A_i$, $i = 1, 2, 3$. Now, it is clear that $-1, 0 \in k(A_1)$, $1 \notin k(A_1)$; $0, 1 \in k(A_2)$, $-1 \notin k(A_2)$, and $-1, 1 \in k(A_3)$, $0 \notin k(A_3)$. Let, if possible, $x \in k(A_1) \cap k(A_2) \cap k(A_3) \backslash \{-1, 0, 1\}$. Then $x \in k(A_i)$, for $i = 1, 2, 3 \Longrightarrow A_i \in \Omega(k, x) = \Omega(d, x)$ (since, $x \notin \{-1, 0, 1\}$), for $i = 1, 2, 3 \Longrightarrow x \in d(A_i)$, for $i = 1, 2, 3 \Longrightarrow x \in [-1 - 1/3] \cap [1 - 1/3, 1 + 1/3] \cap [2 - 1/3, 2) = \emptyset$, which is absurd. Hence it now follows that $k(A_1) \cap k(A_2) \cap k(A_3) = \emptyset$, i.e., $\theta -$

$\mathrm{cl}\,(A_1) \cap \theta - \mathrm{cl}\,(A_2) \cap \theta - \mathrm{cl}\,(A_3) = \emptyset$, which shows that Ω does not θ–adhere in (X, τ). Hence (X, τ) is not θ–linkage quasi H–closed. Incidentally, we see that Ω is a θ–linked grill which is not θ–conjoint.

We now show that the two notions viz. quasi H–closedness and θ–linkage quasi H–closedness coincide for almost regular spaces.

Theorem 3.16. *In the class of almost regular topological spaces, the concepts of quasi H–closedness and θ–linkage quasi H–closedness become identical.*

Proof. In view of Theorem 3.13 it is enough to show that an almost regular quasi H–closed space is θ–linkage quasi H–closed. Let Ω be any θ–linked grill on an almost regular quasi H–closed space X such that Ω does not θ–adhere in X. Then for each $x \in X$, there exists $A_x \in \Omega$ such that $x \notin \theta - \mathrm{cl}\,(A_x)$. Since in an almost regular space X, $\theta - \mathrm{cl}\,(A_x)$ is θ–closed, it is δ–closed so that $\theta - \mathrm{cl}\,(A_x) = \cap\{K_\alpha(x) : \alpha \in \Lambda\}$, where each $K_\alpha(x)$ is regular closed. As $x \notin K_\alpha(x)$ for some $\alpha \in \Lambda$, there is an open nbd V_x of x such that $\mathrm{cl}\,(V_x) \cap K_\alpha(x) = \emptyset$. Then $\theta - \mathrm{cl}\,(V_x) \cap K_\alpha(x) = \emptyset$, i.e., $\theta - \mathrm{cl}\,(\theta - \mathrm{cl}\,V_x) \cap (\cap\{K_\alpha(x) : \alpha \in \Lambda\}) = \emptyset$. Thus $\theta - \mathrm{cl}\,(\theta - \mathrm{cl}\,V_x) \cap \theta - \mathrm{cl}\,(A_x) = \emptyset$. As Ω is a θ–linked grill and $A_x \in \Omega$, it then follows that $\theta - \mathrm{cl}\,V_x \notin \Omega$. By quasi H–closedness of X, the open cover $\{V_x : x \in X\}$ of X has a finite subfamily $\{V_{xi} : i = 1, 2, ..., n\}$ such that $X = \cup\{\mathrm{cl}\,(V_{xi} : i = 1, ..., n\}$. Since $A_{xi} \in \Omega$ and $\theta - \mathrm{cl}\,(V_{xi}) \notin \Omega$, for $i = 1, 2, ..., n$, we have $X = \cup\{\mathrm{cl}\,(V_{xi}) : i = 1, ..., n\} = \cup\{\theta - \mathrm{cl}\,(V_{xi}) : i = 1, ..., n\} \notin \Omega$, a contradiction (as $X \in \Omega$). Hence Ω must θ–adhere in X, proving that X is θ–linkage quasi H–closed.

Corollary 3.17. *A Urysohn space is H–closed iff it is θ–linkage H–closed.*

It is well–known that a regular quasi H–closed space is compact. Now we try to strengthen this result by defining a strictly weaker condition than regularity, what we call θ–regularity, defined in terms of grills.

Definition 3.18. A topological space (X, τ) is called θ–regular iff every grill on X, which θ–converges must converge (not necessarily to the same point).

Theorem 3.19. *A regular space (X, τ) is θ–regular.*

Proof. Let Ω be a grill on a regular space X θ–converging to a point $x \in X$. If $U \in \eta_x$, then there exists $V \in \eta_x$, such that $x \in \mathrm{cl}\,V \subseteq U$. Then

there is some $G \in \Omega$ such that $G \subseteq \mathrm{cl}\, V \subseteq U$. Hence Ω converges to x, proving that X is θ–regular.

Remark 3.20. The converse of the above theorem is false, which is easily seen from the following.

Example 3.21. Consider an infinite set X with the co–finite topology τ. Then (X, τ) is a compact non–regular space. Since (X, τ) is compact, every grill converges in X and hence every grill which θ–converges in X must converges in X. Thus X is θ–regular.

Theorem 3.22. *A θ–regular quasi H–closed space is compact.*

Proof. Follows at once from Theorems 2.13 and 3.8.

Subsets are quasi H–closed relative to a space X, first defined by V e – l i c k o [13], have got immense importance and thus been studied by several mathematicians from different standpoints. A subset A of a space X is called quasi H–closed relative to X or simply an H–set [13] iff for every cover $\mathcal{U}$ of A by open sets of X has a finite subcollection $\mathcal{U}_0$ such that $A \subseteq \cup \{\mathrm{cl}\, U : U \in \mathcal{U}_0\}$. Grills can also be used to the investigation of such concepts. We conclude by showing for instance, one such characterization of H–sets, many others could also be achieved via grills.

Theorem 3.23. *A subset A of a topological space X is an H–set iff each grill Ω on X such that for each regular closed set $U \supseteq A$, there is some $F \in \mathcal{F}$ with $F \subseteq U$, θ–converges in A.*

Proof. Let $A \subseteq X$ be an H–set and Ω be a grill on X such that Ω does not θ–converges to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in \eta_a$ such that $F \not\subset \mathrm{cl}\, U_a$ for all $F \in \Omega$. Now $\{U_a : a \in A\}$ is a cover of the H–set A by open sets of X. Then $A \subseteq \cup \{\mathrm{cl}\, U_{ai} : i = 1, ..., n\} = U$ (say), for some positive integer n. Then U $(\supseteq A)$ is regular closed. Since Ω is a grill, $F \not\subset U$, for all $F \in \Omega$.
Conversely, let A be not an H–set. Then for some cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by open sets of X,

$$\mathcal{F} = \left\{ A \setminus \bigcup_{\alpha \in \Lambda_0} \mathrm{cl}\, U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda \right\}$$

is a filterbase on A. Then $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on A. Then $\mathcal{F}^*$ is a grill on X and for any regular closed set $U \supseteq A$, there is $F \in \mathcal{F}^*$

with $F \subseteq U$ (as each $F \in \mathcal{F}$ is a subset of A). Now, for each $x \in X$, there must exist some $\beta \in \Lambda$ such that $x \in U_\beta$, as $\mathcal{U}$ is an open cover of A. Then for any $G \in \mathcal{F}^*$ as $G \cap (A \backslash \operatorname{cl} U_\beta) \neq \emptyset$, $G \not\subset \operatorname{cl} U_\beta$. Hence the grill $\mathcal{F}^*$ cannot θ–converge to any $x \in A$. This contradiction proves the desired result.

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