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DUALITY FOR STATE–CONSTRAINED OPTIMAL CONTROL PROBLEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

BY

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Abstract. We show how the adjoint state associated with a linear quadratic optimal control governed by an elliptic partial differential equation can be viewed as a dual variable associated to a suitable convex perturbation function. This observation along with a recent duality result in convex analysis leads to a general and simple way of deriving optimality conditions in convex optimal control problem with state constraints. We generalize this optimality condition to a class of non convex problems.

Key words. Optimal control, partial differential equations, convex duality, optimality conditions.

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1. Introduction. Many people working in the field of convex analysis suspect the fact that the adjoint state in a convex problem of optimal control can be viewed as a dual variable associated to a convex duality scheme. This approach has been fruitfully used for example in [13], [12], [4], [1], [2], [3], [34] in order to derive optimality conditions for some state– constrained control problems. In section 2 of this note we present a quite general and simple method in order to perturb a convex optimal control problem and to produce a dual variable which specializes into the classical adjoint state in the basic case (see [32]). This method is quite similar to the one introduced in [22] is a slightly different and less general situation. The perturbation method leads to Theorem 2.1 in which are given optimality conditions for abstract linear–quadratic optimal control with constraints on the state. As an application, we use our method in subsection 2.2 to tackle in a simple way the problems studied by F. A b e r g e l and R. T e m a m in [4]. In this case our framework leads to a slightly different dual problem than the

one given in [4]. In particular we do not require the use of convex functions of measures. In subsection 2.3, we generalize in a very simple way some recent results of M. B e r g o u n i o u x in [14] on optimality conditions for optimal control problems with state constraints. In order to get optimality conditions, the main assumption is a qualification one. Hereafter we use an Attouch–Brezis type condition which is more general than the interior type conditions previously used in the quoted papers. A related approach to the problem of finding optimality conditions is developped in a lecture note [37] by J.-P. R a y m o n d where the adjoint state is considered as a Lagrange multiplier associated to the constraint defined by the state equation. In the same spirit we generalize Theorem 2.1 in section 3 to a class of non convex abstract optimal control problems.

In the sequel we shall denote by X a normed vector space whose closed unit ball is denoted by B_X and whose dual space is denoted by X^*. Given a subset S of X, we denote by i_S its indicator function defined by

$$i_S(z) = \begin{cases} 0 & \text{if } z \in S \\ +\infty & \text{if } z \in Z \setminus S. \end{cases}$$

The closed ball with center x and radius r is denoted by $B(x,r)$. We denote by $\Gamma_0(X)$ (resp. $\Gamma_0(X^*)$) the set of convex proper (not identically $+\infty$) functions on X (resp. X^*) which are $\sigma(X,X^*)$ (resp. $\sigma(X^*,X)$) lower semicontinuous. Given a function $f\colon X \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ its domain is the set dom f of those $x \in X$ such that $f(x) < +\infty$.

Let us briefly recall some basic feature on convex duality (see [24], [40], [31] for further details). Given Banach spaces $X, W, Y = W^*$ and given $F \in \Gamma_0(X \times W)$ the primal problem associated to the perturbation function F consists in

$$\inf_{x \in X} F(x,0).$$

The dual problem consists in

$$\inf_{y \in Y} F^*(0,y) = \inf_{y \in Y} \psi(y),$$

where $\psi(y) = \varphi^*(y)$ and $\varphi(w) = \inf_{x \in X} F(x,w)$. The convex–concave Lagrangian associated to the perturbation function is the function $L\colon X \times Y \longrightarrow \overline{\boldsymbol{R}}$ defined for all $(x,y) \in X \times Y$ by

$$L(x,y) = \inf_{w \in W} (F(x,w) - \langle w, y \rangle).$$

It is important to observe that

$$F^*(0,y) = \sup_{x\in X}(-L(x,y)). \tag{1}$$

Assuming that the marginal function φ is lower semicontinuous at 0 with $\varphi(0)$ finite, one has

$$\inf_{x\in X} F(x,0) = -\inf_{y\in Y} F^*(0,y). \tag{2}$$

Moreover a pair $(\overline{x},\overline{y}) \in X \times Y$ solves

$$F(\overline{x},0) = \inf_{x\in X} F(x,0) \text{ and } F^*(0,\overline{y}) = \inf_{y\in Y} F^*(0,y)$$

$$\text{with } F(\overline{x},0) = -F^*(0,\overline{y})$$

if and only if $(\overline{x},\overline{y})$ is a saddle–point of the Lagrangian that is

$$L(y,\overline{x}) \le L(\overline{x},\overline{y}) \le L(x,\overline{y}) \quad \text{for all} \quad (x,y) \in X \times Y.$$

This is equivalent to

$$L'_x((\overline{x},\overline{y});\quad x-\overline{x}) \ge 0 \text{ and } L_y((\overline{x},\overline{y});\quad y-\overline{y}) \le 0 \tag{3}$$

for all $(x,y) \in X \times Y$ where $L(\overline{x},y)$ and $L(x,\overline{y})$ are finite with

$$L'_x((\overline{x},\overline{y});\quad x-\overline{x}) = \lim_{t\downarrow 0} \frac{L(\overline{x}+t(x-\overline{x}),\overline{y}) - L(\overline{x},\overline{y})}{t}$$

and

$$L_y((\overline{x},\overline{y});\quad y-\overline{y}) = \lim_{t\downarrow 0} \frac{L(\overline{x},\overline{y}+T(y-\overline{y})) - L(\overline{x},\overline{y})}{t}.$$

Observe also that the dual problem admits solutions whenever the function φ is continuous at 0 which is equivalent to the fact that it is bounded above in a neighborhood of 0. Let us mention a result which can be deduced from Proposition 3 of [41]. This result, which was independently proved in [8] and [17], (see also Corollary 1 in [38]) deals with stability of the convex duality scheme in the non reflexive setting. It is not required as in [40], Theorem 18,(c) that the space X^* must be a Banach space for some topology compatible with the duality (X, X^*).

Theorem 1.1. *Assume that X and W are Banach spaces, that $F \in \Gamma_0(X \times W)$, that $\varphi(0)$ is finite and that*

$$\boldsymbol{R}_+(dom\ \varphi) = Z \tag{4}$$

where Z is a closed vector space. Then φ is lower semicontinuous at 0 and

$$\inf_{x\in X} F(x,0) = -\min_{y\in Y} F^*(0,y).$$

2. The convex case.

2.1. Abstract results. In this section we follow the notations of [32]. Let $V, (H, \langle\cdot,\cdot\rangle)$ be Hilbert spaces such that V is continuously and densely embedded in H, let

$$a(\cdot,\cdot)\colon V \times V \longrightarrow \boldsymbol{R}$$

be a continuous and elliptic bilinear form on V and let $A\colon V \to V^*$ be the isomorphism defined by $A(y) = f$ if and only if $a(y,\cdot) = f(\cdot)$ on V. Let $\mathcal{U}$ be the Hilbert space of controls, let $\mathcal{H}$ be a Hilbert space, let $B \in L(\mathcal{U}, V^*)$ and let $C \in L(V, \mathcal{H})$. Let us begin by a basic example.

Example 2.1. Let $\mathcal{U}_{\rm ad}$ be a closed convex subset of $\mathcal{U}$. The optimal control problem we are dealing with is

$$\min_{v\in\mathcal{U}_{\rm ad}} \frac{1}{2}\|CA^{-1}(B(v)+f) - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} \tag{5}$$

where $f \in V^*$ is given, $N \in L(\mathcal{U},\mathcal{U})$ is a symmetric definite positive operator and $z_d \in \mathcal{H}$ is given. Let us set

$$\mathcal{C} = \{((y,v),g) \in (V \times \mathcal{U}_{\rm ad}) \times V^*\colon Ay = f + Bv + g\}.$$

Then problem (5) is equivalent to

$$\min_{(y,v)\in V\times\mathcal{U}} F((y,v),0)$$

where

$$F((y,v),g) = \frac{1}{2}\|C_y - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} + i_{\mathcal{C}}((y,v),g). \tag{6}$$

The associated Lagrangian $L\colon (V\times\mathcal{U})\times V \longrightarrow \boldsymbol{R}$ is defined for all $((y,v),p) \in (V\times\mathcal{U})\times V$ by

$$L((y,v),p) = \frac{1}{2}\|C_y - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} - \langle Ay - f - Bv, p\rangle + i_{\mathcal{U}_{\rm ad}}(v)$$

where we also denote by $\langle\cdot,\cdot\rangle$ the duality between V and V^*. Let $v_0 \in \mathcal{U}_{\rm ad}$. There exists $\rho > 0$ such that $\sup_{g\in B_{V^*}} \|A^{-1}(f + Bv_0 + g)\|_V \leq \rho$. This yields

$$\sup_{g\in B_{V^*}} \varphi(g) \leq \frac{1}{2}(\|C\|_\rho + \|z_d\|)^2 + \frac{1}{2}\|N\|\ \|v_0\|^2$$

where

$$\varphi(g) = \inf_{(y,v)\in V\times\mathcal{U}} F((y,v),g)$$

is the value function associated to F. Thus φ is continuous at 0 which entails that the dual problem admits a solution along with the primal one. We derive from (3) that $(\overline{y},\overline{v}) \in V\times\mathcal{U}_{\rm ad}$ is the primal solution and $\overline{p}\in V$ is the dual solution if and only if

$$L'_{(y,v)}(((\overline{y},\overline{v}),\overline{p});\ \ (y,v)-(\overline{y},\overline{v})) \geq 0 \ \text{ for all } \quad (y,v)\in V\times\mathcal{U}_{\rm ad},$$

and

$$L'_p(((\overline{y},\overline{v}),\overline{p});\ \ p-\overline{p}) \leq 0 \ \text{ for all } \quad p\in V.$$

A straightforward computation leads to

$$\langle C\overline{y} - z_d, C(y-\overline{y})\rangle_{\mathcal{H}} - \langle A(y-\overline{y}),\overline{p}\rangle + \langle N\overline{v}, v-\overline{v}\rangle_{\mathcal{U}} + \langle B(v-\overline{v}),\overline{p}\rangle \geq 0$$

$$\text{for all } \quad (y,v)\in V\times\mathcal{U}_{\rm ad},$$

and

$$\langle A(\overline{y}) - f - B\overline{v}, p-\overline{p}\rangle \geq 0 \ \text{ for all } \quad p\in V$$

which can be clearly decoupled in

$$\langle C(\overline{y}) - z_d, C(y-\overline{y})\rangle_{\mathcal{H}} - \langle A(y-\overline{y}),\overline{p}\rangle \geq 0 \ \text{ for all } \quad y\in V,$$

$$\langle N\overline{v}, v-\overline{v}\rangle_{\mathcal{U}} + \langle B(v-\overline{v}), \overline{p}\rangle \geq 0 \quad \text{for all} \quad v \in \mathcal{U}_{\text{ad}}$$

and

$$\langle A(\overline{y}) - f - B\overline{v}, p-\overline{p}\rangle \geq 0 \quad \text{for all} \quad p \in V.$$

Thus introducing the canonical isomorphisms $\Lambda_{\mathcal{H}}$ from $\mathcal{H}$ into $\mathcal{H}^*$, $\Lambda_{\mathcal{U}}$ from $\mathcal{U}$ into $\mathcal{U}^*$, and the adjoint operators $A^* \in L(V, V^*), B^* \in L(V,\mathcal{U}^*)$ and $C^* \in L(\mathcal{H}^*, V^*)$, we recover the classical necessary and sufficient condition of optimality (see [32], p. 52)

$$A\overline{y} = f + B\overline{v},$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle \Lambda_{\mathcal{U}}^{-1}B^*\overline{p} + N\overline{v}, v-\overline{v}\rangle_{\mathcal{U}} \geq 0 \quad \text{for all} \quad v \in \mathcal{U}_{\text{ad}}.$$

Remark 1. Let us compare our duality method and the one of V. Barbu and T. Precupanu (see [13] p. 220). This method amounts to use, in the case $\mathcal{H} = V, C = I_V, f \equiv 0$, the following perturbation function

$$\widetilde{F}{:}\mathcal{U} \times V \longrightarrow \boldsymbol{R}$$

defined for all $(v, y) \in \mathcal{U} \times V$ by

$$\widetilde{F}(v,y) = \frac{1}{2}\|A^{-1}(Bv) + y - z_d\|_H^2 + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} + i_{\mathcal{U}_{\text{ad}}}(v).$$

The perturbation variable lies in V and the dual one lies in V^*. The associated Lagrangian $\widetilde{L}{:}\mathcal{U} \times V^* \longrightarrow \overline{\boldsymbol{R}}$ is defined for all $(v, q) \in \mathcal{U} \times V^*$ by

$$\widetilde{L}(v,q) = \begin{cases} +\infty & \text{if } v \notin \mathcal{U}_{\text{ad}} \\ \frac{1}{2}\langle Nv, v\rangle + \langle q, A^{-1}(B(v)) - z_d\rangle_H - \frac{1}{2}\|q\|_H^2 - i_H(q) & \text{otherwise.} \end{cases}$$

The dual problem is

$$\inf_{q\in H} \widetilde{\psi}(q) \tag{7}$$

where

$$\widetilde{\psi}(q) = \theta^*(-B^*(A^*)^{-1}(q)) + \frac{1}{2}\|q\|_H^2 + \langle q, z_d\rangle_H$$

with $\theta \in \Gamma_0(\mathcal{U})$ defined for all $v \in \mathcal{U}$ by $\theta(v) = \frac{1}{2}\langle Nv, v\rangle + i_{\mathcal{U}_{\rm ad}}(v)$. The perturbation introduced in (6) leads to the dual problem

$$\inf_{\{p\in V : A^*(p)\in H\}} \psi(p) \tag{8}$$

where

$$\psi(q) = \theta^*(-B^*(p)) + \frac{1}{2}\|A^*(p)\|_H^2 + \langle A^*(p), z_d\rangle_H.$$

Observe that for all $p \in V$ one has

$$\psi(p) = \widetilde{\psi}(A^*(p))$$

and that the dual variable attached to (8) is exactly the adjoint state.

The duality scheme attached to the perturbation function defined in (6) allows also to treat the case of optimal problems with state constraints. Let $Z \subset H$ be a Banach space continuously and densely embedded in H. Observe that $Z \subset V^*$ with continuous embedding. For all $g \in Z$ and $v \in V$ the action of g on v is equal to $\langle g, v\rangle$ and

$$|\langle g, v\rangle| \leq \|g\|_H\|v\|_H \leq c\|g\|_Z\|v\|_V,$$

yielding $\|z\|_{V^*} \leq c\|z\|_Z$. We consider a closed convex subset $K \subset V$ such that

$$A(K) \cap (f + B(\mathcal{U}_{\rm ad})) \neq \emptyset$$

and the following optimal control problem

$$\min_{\{v\in\mathcal{U}, B(v)\in A(K)-f\}} \frac{1}{2}\|CA^{-1}(B(v)+f) - z_d\|_{\mathcal{H}}^2 + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}}. \tag{9}$$

Problem (9) is equivalent to

$$\min_{(y,v)\in V\times\mathcal{U}_{\rm ad}} F((y,v),0) \tag{10}$$

where $F\colon (V \times \mathcal{U}) \times Z \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ is defined by

$$F((y,v),g) = \frac{1}{2}\|Cy - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} + i_K(y) + i_{\mathcal{C}}((y,v),g), \tag{11}$$

with

$$\mathcal{C} = \{((y,v),g) \in (V \times \mathcal{U}_{\rm ad}) \times Z\colon Ay = f + Bv + g\},$$

$f \in V^*$ given. The set $\mathcal{C}$ is a closed convex subset of $V \times \mathcal{U} \times Z$. We shall need a qualification assumption. Let us assume that

$$Z \subset \boldsymbol{R}_+ \left(-f - B(\mathcal{U}_{\rm ad}) + A(K)\right). \tag{S_Z}$$

Theorem 2.1. *Assume that (S_Z) is in force. Then a pair $(\overline{y}, \overline{v}) \in K \times \mathcal{U}_{\rm ad}$ is optimal for the problem (10) if and only if there exists $\overline{q} \in Z^*$ and $\overline{p} \in V$ such that*

$$A(\overline{y}) = f + B\overline{v},$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle A(y - \overline{y}), \overline{p}\rangle_{\mathcal{H}} - \langle A(y - \overline{y}) - B(v - \overline{v}), \overline{q}\rangle + \langle N\overline{v}, v - \overline{v}\rangle_{\mathcal{U}} \geq 0$$

$$\textit{for all} \quad (y,v) \in \mathcal{D}$$

where

$$\begin{aligned}\mathcal{D} &= \{(y,v) \in K \times \mathcal{U}_{\rm ad}\colon A(y) - f - B(v) \in Z\}\\ &= \{(y,v) \in K \times \mathcal{U}_{\rm ad}\colon A(y - \overline{y}) - B(v - \overline{v}) \in Z\}.\end{aligned}$$

Moreover if $Z = H$ and $B \in L(\mathcal{U}, H)$ the preceding condition becomes decoupled in

$$A(\overline{y}) = f + B\overline{v},$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle A(y - \overline{y}), \overline{p} - \overline{q}\rangle \geq 0 \quad \textit{for all} \quad y \in (\overline{y} + A^{-1}(H)) \cap K$$

and

$$\langle N\overline{v} + B^*\overline{q}, v - \overline{v}\rangle \geq 0 \quad \textit{for all} \quad v \in \mathcal{U}_{\rm ad}$$

with $\overline{q} \in H$.

Proof. Observe that $F((y,v),0)$ is convex proper lower semicontinuous and coercive whenever F is defined in (11). Moreover using assumption (S_Z), we obtain that for all $g \in Z$ there exists $v \in \mathcal{U}_{\rm ad}$ and $y \in K$ such that

$$\lambda^{-1}g = -f - B(v) - A(y).$$

Thus $\lambda^{-1}g \in \mathrm{dom}\varphi$ hence

$$Z = \boldsymbol{R}_+(\mathrm{dom}_\varphi).$$

Applying Theorem 1.1., we get that the dual problem admits a solution and that

$$\inf_{(y,v)\in K\times\mathcal{U}_{\mathrm{ad}}} F((y,v),0) = -\min_{q\in Z^*} F^*((0,0),q).$$

The associated Lagrangian is easily shown to be equal to

$$L((y,v),q) = \frac{1}{2}\|Cy - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} + i_K(y) -$$

$$-\langle Ay - f - Bv, q\rangle + i_{\mathcal{F}}(y,v)$$

for all $((y,v),q) \in (V\times\mathcal{U})\times Z^*$ where $\langle\cdot,\cdot\rangle$ denotes also the duality (Z,Z^*) and

$$\begin{aligned}\mathcal{F} &= \{(y,v)\in V\times\mathcal{U}_{\mathrm{ad}}\colon A(y) - f - B(v)\in Z\}\\ &= \{(y,v)\in V\times\mathcal{U}_{\mathrm{ad}}\colon A(y-\overline{y}) - B(v-\overline{v})\in Z\}.\end{aligned}$$

Applying (3), we find that $((\overline{y},\overline{v}),\overline{q})\in\mathcal{D}\times Z^*$ is optimal if and only if

$$\langle C(\overline{y}) - z_d, C(y-\overline{y})\rangle_{\mathcal{H}} - \langle A(y-\overline{y}) - B(v-\overline{v}),\overline{q}\rangle + \langle N\overline{v}, v-\overline{v}\rangle_{\mathcal{U}} \geq 0$$

$$\text{for all}\quad (y,v)\in\mathcal{D}$$

and

$$\langle A(\overline{y}) - f - B\overline{v}, q-\overline{q}\rangle \geq 0 \ \text{ for all }\quad q\in Z^*.$$

Introducing the unique solution $\overline{p}\in V$ of

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

the preceding conditions turn to

$$\langle A(y-\overline{y}),\overline{p}\rangle_{\mathcal{H}} - \langle A(y-\overline{y}) - B(v-\overline{v}),\overline{q}\rangle + \langle N\overline{v}, v-\overline{v}\rangle_{\mathcal{U}} \geq 0$$

$$\text{for all} \quad (y, v) \in \mathcal{D}$$

and

$$A(\overline{y}) - f - B\overline{v} = 0.$$

Assuming that $Z = H$ and $B \in L(\mathcal{U}, H)$ one obtains the announced decoupled conditions

$$A(\overline{y}) - f - B\overline{v} = 0,$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle A(y - \overline{y}), \overline{p} - \overline{q}\rangle \geq 0 \ \text{ for all } \quad y \in (\overline{y} + A^{-1}(H)) \cap K$$

and

$$\langle N\overline{v} + B^*\overline{q}, v - \overline{v}\rangle \geq 0 \ \text{ for all } \quad v \in \mathcal{U}_{\text{ad}}.$$

■

The requirement for $K \subset V$ to be closed is sometimes too strong in practical situations such as pointwise constraints like in (13) below. It can be weakened in the following way.

Theorem 2.2. *Let us set*

$$W = \{y \in V : A(y) \in H\}$$

and let us assume that the convex set K is contained in W. Let us also assume that the set

$$\{(y, A(y)) : y \in K\} \ \textit{is closed in} \ \ V \times H. \tag{12}$$

Let us assume that $f \in H, B \in L(\mathcal{U}, H)$ and that (S_Z) is in force. Then a pair $(\overline{y}, \overline{v}) \in K \times \mathcal{U}_{\text{ad}}$ is optimal for the problem (10) if and only if there exists $\overline{q} \in Z^*$ and $\overline{p} \in V$ such that

$$A(\overline{y}) = f + B\overline{v},$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle A(y - \overline{y}), \overline{p}\rangle_{\mathcal{H}} - \langle A(y - \overline{y}) - B(v - \overline{v}), \overline{q}\rangle + \langle N\overline{v}, v - \overline{v}\rangle_{\mathcal{U}} \geq 0$$

$$for\ all \quad (y,v) \in \mathcal{D}$$

where

$$\begin{aligned}\mathcal{D} &= \{(y,v) \in K \times \mathcal{U}_{\rm ad}\colon A(y) - f - B(v) \in Z\}\\ &= \{(y,v) \in K \times \mathcal{U}_{\rm ad}\colon A(y-\overline{y}) - B(v-\overline{v}) \in Z\}.\end{aligned}$$

Moreover if $Z = H$ the preceding condition becomes decoupled in

$$A(\overline{y}) = f + B\overline{v},$$

$$A^*\overline{p} = C^*\Lambda_{\mathcal{H}}(C\overline{y} - z_d),$$

$$\langle A(y-\overline{y}), \overline{p} - \overline{q}\rangle \geq 0 \quad for\ all \quad y \in K$$

and

$$\langle N\overline{v} + B^*\overline{q}, v - \overline{v}\rangle \geq 0 \quad for\ all \quad v \in \mathcal{U}_{\rm ad}$$

with $\overline{q} \in H$.

Proof. The optimal control we are dealing with takes the form

$$\min_{(y,v)\in V\times\mathcal{U}_{\rm ad}} F((y,v),0)$$

where $F\colon (V \times \mathcal{U}) \times Z \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ is defined by

$$F((y,v),g) = \frac{1}{2}\|C_y - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} + i_{\mathcal{C}}((y,v),g),$$

with

$$\mathcal{C} = \{((y,v),g) \in (K \times \mathcal{U}_{\rm ad}) \times Z\colon Ay = f + Bv + g\}.$$

From (12), the set $\mathcal{C}$ is a closed convex subset of $V \times \mathcal{U} \times Z$ which allows us to conclude the proof as in Theorem 2.1. ∎

Remark 2. Assumption (S_Z) holds true whenever the following one

$$\text{there exist } \ r > 0 \text{ and } \ \rho > 0 \text{ with } \ rB_Z \subset -f - B(\mathcal{U}_{\rm ad} \cap \rho B_{\mathcal{U}}) + A(K) \ \ (\widetilde{S}_Z)$$

used in [14] is satisfied. Observe that (S_Z) is sometimes easier to check than $(\widetilde{S}_Z)$ as shown by the following example. Let $\Omega \subset \boldsymbol{R}^d$ be a bounded open set with a smooth boundary Γsuch that Ω is locally on one side of Γ, let $V = H^1(\Omega)$, or $V = H^1_0(\Omega)$ and let $f \equiv 0$. Let us set

$$\mathcal{U}_{\text{ad}} = \{h \in L^2(\Omega) \colon h \geq 0\},$$

$$K = \{y \in V \colon \int_\Omega y(x)dx = 1\}$$

or assuming $d \leq 3$ which entails $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$

$$K = \{y \in V \cap H^2(\Omega) \colon y(\omega) = 1\} \tag{13}$$

for some $\omega \in \Omega$ and let us set

$$a(y,z) = \int_\Omega \langle Dy, Dz\rangle dx + \int_\Omega yz dx \quad \text{for all} \quad y, z \in V.$$

Given $h \in H = L^2(\Omega)$, we can write

$$h = -1 + g - h^-$$

with $g = 1 + h^+$. Let $z \in V$ be the solution of $A(z) = g$. From the maximum principle for the Dirichlet problem (see [19], Théorème IX.27) or for the Neumann problem ([19], Proposition IX.30) we obtain that $z \geq 1$ on Ω thus $\lambda \int_\Omega z(x)dx = 1$ or $\lambda z(\omega) = 1$ for some $\lambda > 0$ yielding

$$H \subset \boldsymbol{R}_+(A(K) - \mathcal{U}_{\text{ad}}).$$

2.2. Applications to a nonqualified problem. Let us apply the method developped in subsection 2.1 to the problems treated by F. Abergel and R. Teman in [4].

Example 2.2. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded open set with a smooth boundary $\Gamma = \partial\Omega$ such that Ω is locally on one side of Γ. Let $V = H_0^1(\Omega), H = L^2(\Omega), \mathcal{U} = H, \mathcal{U}_{\text{ad}} = L^2(\Omega), \alpha > 0$ and

$$K = \{y \in V \colon |y| \leq \alpha \ a.e.\}.$$

Given $z_d \in H$, let us consider the problem

$$\min_{v \in H} J(v) \tag{14}$$

where

$$J(v) = \frac{1}{2}\|y - z_d\|_H^2 + \frac{1}{2}\|v\|_H^2$$

and

$$\begin{cases} -\Delta y + y = v & \text{on } \Omega \\ |y| \leq \alpha & \text{on } \Omega, y = 0 \text{ on } \Gamma. \end{cases}$$

Equivalently we have to slove

$$\min_{(y,v)\in V\times H} F((y,v),0)$$

where

$$\begin{aligned} &F((y,v),g) = \tfrac{1}{2}\|y - z_d\|_H^2 + \tfrac{1}{2}\|v\|_H^2 + i_K(y) + i_{\mathcal{C}}((y,v),g), \\ &\mathcal{C} = \{((y,v),g) \in (V \times H) \times H : -\Delta y + y = v + g\}. \end{aligned} \tag{15}$$

Problem (14) can be written

$$\min_{v\in W\cap K} \left(\frac{1}{2}\|y - z_d\|_H^2 + \frac{1}{2}\| - \Delta y + y\|_H^2\right) \tag{16}$$

where

$$W = \{y \in V : -\Delta y + y \in H\}.$$

It is clear that problem (16) admits a unique solution $\overline{y} \in W \cap K$ characterized by

$$\langle \overline{y} - z_d, y - \overline{y}\rangle + \langle \overline{p}, -\Delta(y - \overline{y}) + y - \overline{y}\rangle \geq 0 \text{ for all } y \in K \cap W$$

where $\overline{p} = -\Delta\overline{y} + \overline{y}$. The use of duality theory will provide some regularity property on $\overline{p}$ that could not be derived from the preceding direct approach.

Observe that $(\widetilde{S}_H)$ and then (S_H) is obviously satisfied since given $g \in H$ and $v = -g$ one has $y \equiv 0$ hence $y \in K$. The associate Lagrangian is equal to

$$L((y,v),q) = \frac{1}{2}\|y - z_d\|_H^2 + \frac{1}{2}\|v\|_H^2 + i_K(y) - \langle q, -\Delta y + y - v\rangle + i_W(y)$$

for all $((y,v),q) \in (V \times H) \times H$. Observe that for all $y \in K \cap W$ and for all $v \in H$ one has

$$\langle q, -\Delta y + y\rangle \leq F^*((0,0),q) + \frac{1}{2}\|y - z_d\|_H^2 + \frac{1}{2}\|v\|_H^2 + \|v\|_H\|q\|_H. \tag{17}$$

Let $((\overline{y},\overline{v}),\overline{q}) \in (V \times H) \times H$ be a saddle-point of the Lagrangian L. As a consequence of (S_H), we get

$$F^*((0,0),\overline{q}) = -F((\overline{y},\overline{v}),0).$$

Thus we derive that $F^*((0,0),\overline{q}) < +\infty$ from what we deduce via (17) that

$$-\Delta\overline{q} + \overline{q} \in \mathcal{M}(\Omega)$$

the set of bounded Radon measures on Ω. Indeed given $\varphi \in \mathcal{D}(\Omega)$ with $\|\varphi\|_\infty \leq \alpha$ one gets

$$\langle \overline{q}, -\Delta\varphi - \varphi\rangle \leq F^*((0,0),\overline{q}) + \frac{1}{2}\|\varphi - z_d\|_H^2 + \frac{1}{2}\|\overline{v}\|_H^2 + \|\overline{v}\|_H\|\overline{q}\|_H.$$

Moreover one has $\overline{q} = 0$ on $\Gamma = \partial\Omega$ in the sense that

$$\int_\Omega \psi d(-\Delta\overline{q} + \overline{q}) = \int_\Omega (-\Delta\psi + \psi)\overline{q}dx$$

for all $\psi \in \mathcal{C}^\infty(\overline{\Omega})$ with $\psi = 0$ on Γ. Indeed the distribution

$$T(\varphi) = \int_\Omega (-\Delta(\varphi\psi) + (\varphi\psi))\overline{q}dx = \langle -\Delta\overline{q} + \overline{q}, \varphi\psi\rangle$$

is a bounded Radon measure since we obtain from (17) that

$$\sup_{\|\varphi\|_\infty \leq \alpha\|\psi\|_\infty^{-1}} T(\varphi) < +\infty,$$

yielding the announced fact by allowing $\varphi \equiv 1$. Observing that $\overline{y} \in H^2(\Omega)$, an analogous computation leads to the fact that the distribution

$$T(\varphi) = \int_\Omega (-\Delta(\varphi\overline{y}) + (\varphi\overline{y}))\overline{q}dx$$

is also a bounded Radon measure denoted by $(-\Delta\overline{q} + \overline{q}) \cdot \overline{y}$. Relying on (3) and using the generalized Green's formula proved in [4], one readily obtains

the following necessary and sufficient for $((\overline{y},\overline{v}),\overline{q}) \in K \times H \times H$ to be optimal

$$\overline{q} = -\overline{v} = \Delta\overline{y} - \overline{y}, \text{ and } -\Delta\overline{q} + \overline{q} \in \mathcal{M}(\Omega), \overline{q} = 0 \text{ on } \Gamma$$

and

$$\int_\Omega (\overline{y} - z_d)(y - \overline{y})dx - \int_\Omega (-\Delta\overline{q} + \overline{q}) \cdot (y - \overline{y}) \geq 0 \text{ for all } y \in K.$$

In what precede we did not need to compute the dual problem associated to the perturbation defined in (15). A simple calculation would show that

$$\mathrm{dom}(F^*((0,0),\cdot)) = BL_0(\Omega) = \{q \in H\colon -\Delta q + q \in \mathcal{M}(\Omega) \text{ and } q|\Gamma = 0\}$$

and that for all $q \in BL_0(\Omega)$ one has

$$F^*((0,0),q) = \frac{1}{2}\|q\|_H^2 + \int_\Omega \Psi(-\Delta q + q + z_d) - \frac{1}{2}\|z_d\|_H^2$$

where $\psi(\cdot)$ is the convex function of measure (see [21]) associated to the convex function $\psi(s) = \frac{s^2}{2}$ if $|s| \leq \alpha, \psi(t) = \alpha(|s| - \frac{\alpha}{2})$ if $|s| \geq \alpha$. It is of interest to observe that this dual problem turns to be the relaxed dual problem introduced in [4]. At last, let us mention that it is possible, following the same lines, to treat the case of the unilateral constraint $y \leq \alpha$ as in [4].

2.3. Application to state–constrained problems governed by elliptic equations. The method used in section 2.1 allows also to generalize the results of [14] without using penalization techniques. For the sake of brievity we give only an example of application to a distributed parameter problem with Neumann boundary condition. An interested reader could easily write the results corresponding to the case of Dirichlet boundary condition and so on. Let $\Omega \subset \boldsymbol{R}^d$ be a bounded open subset with a smooth boundary Γ such that Ω is locally on the same side of Γ. Let $M(x) = (a_{ij}(x))_{(i,j)\in[1,d]^2}$ be a matrix depending on $x \in \overline{\Omega}$ with smooth coefficients, let $a_0(x)$ be a smooth function on $\overline{\Omega}$. We assume that there exists $c > 0$ such that $\langle M(x)\xi, \xi\rangle \geq c|\xi|^2$ for all $x \in \overline{\Omega}$ and $\xi \in \boldsymbol{R}^d$ and that a_0 is such that the bilinear form $a(\cdot,\cdot)\colon V \times V \longrightarrow \boldsymbol{R}$ defined on $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$ by

$$a(y,z)=\int_\Omega(\langle M(x)Dy(x),Dz(x)\rangle+a_0(x)y(x)z(x))dx$$

is elliptic, that is $a_0\geq 0$ if $V=H_0^1(\Omega)$ or $\inf_{\overline{\Omega}}a_0>0$ if $V=H^1(\Omega)$. We set

$$\Lambda y=-\mathrm{div}(MDy)+a_0y=-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial y}{\partial x_j}\right)+a_0y,$$

and for all $x\in\Gamma$

$$\frac{\partial y}{\partial_{\nu_\Lambda}}=\sum_{i,j=1}^{d}a_{ij}\frac{\partial y}{\partial x_j}\nu_i$$

where ν is the outward normal vector to Γ at x. We define analogously Λ^* which is associated to the transposed matrix $M^*(x)$ of $M(x)$.

Example 2.3. Let us set $H=L^2(\Omega)$ and $V=H^1(\Omega)$ and let $f\in H$. Assume for simplicity that $\mathcal{U}=L^2(\Omega),B=Id_{\mathcal{U}},\mathcal{H}=L^2(\Omega),C=Id_{\mathcal{H}}$. Let $\mathcal{U}_{\mathrm{ad}}\subset L^2(\Omega)$, let $K\subset H^1(\Omega)$ be closed convex subsets. Then the control problem

$$\min\frac{1}{2}\|y-z_d\|_H^2+\frac{1}{2}\langle Nv,v\rangle_H$$

where $y\in K$ and $v\in\mathcal{U}_{\mathrm{ad}}$ are coupled by

$$\begin{cases}\Lambda y=f+v & \text{on }\Omega\\ \frac{\partial\overline{y}}{\partial\nu_\Lambda}=0 & \text{on }\Gamma\end{cases}$$

is equivalent to

$$\min_{(y,v)\in V\times H}F((y,v),0)$$

where

$$F((y,v),g)=\tfrac{1}{2}\|y-z_d\|_H^2+\tfrac{1}{2}\langle Nv,v\rangle_H+i_K(y)+i_{\mathcal{C}}((y,v),g),$$
$$\mathcal{C}=\{((y,v),g)\in(V\times\mathcal{U}_{\mathrm{ad}})\times H\colon Ay=f+v+g\}.$$

We posit assumption (S_Z) namely

$$Z \subset \boldsymbol{R}_+ \left(-f - \mathcal{U}_{\rm ad} + A(K)\right). \qquad (S_Z)$$

Let $Z = \mathcal{C}_0(\Omega)$ be the set of continuous functions on Ω that vanish on Γ. Assume that (S_Z) holds. Then, according to Theorem 2.1, a pair $(\overline{y}, \overline{v}) \in K \times \mathcal{U}_{\rm ad}$ is optimal if and only if there exists a bounded Radon measure $\overline{\mu} \in \mathcal{M}(\Omega) = Z^*$ such that

$$\langle A(y - \overline{y}), \overline{p}\rangle_H - \langle A(y - \overline{y}) - (v - \overline{v}), \overline{\mu}\rangle + \langle N\overline{v}, v - \overline{v}\rangle_H \geq 0$$
$$\text{for all } (y, v) \in \mathcal{D}$$

where

$$\begin{aligned} \mathcal{D} &= \{(y, v) \in K \times \mathcal{U}_{\rm ad} \colon A(y) - f - v \in Z\} \\ &= \{(y, v) \in K \times \mathcal{U}_{\rm ad} \colon A(y - \overline{y}) - (v - \overline{v}) \in Z\} \end{aligned}$$

and $\overline{p} \in H^1(\Omega)$ is the unique solution of

$$(18) \qquad \begin{cases} L^*(\overline{p}) = \overline{y} - z_d & \text{on } \Omega \\ \frac{\partial \overline{p}}{\partial \nu_{\Lambda^*}} = 0 & \text{on } \Gamma. \end{cases}$$

If $Z = L^2(\Omega)$ then we obtain from Theorem 2.1 that a pair $(\overline{y}, \overline{v}) \in K \times \mathcal{U}_{\rm ad}$ is optimal if and only if there exists $\overline{q} \in L^2(\Omega)$ such that

$$\langle A(y - \overline{y}), \overline{p} - \overline{q}\rangle \geq 0 \text{ for all } y \in (\overline{y} + A^{-1}(H)) \cap K$$

and

$$\langle N\overline{v} + \overline{q}, v - \overline{v}\rangle \geq 0 \text{ for all } v \in \mathcal{U}_{\rm ad}$$

where $\overline{p} \in H^1(\Omega)$ is the unique solution of (18).

The following example shows the versatility of duality methods in optimal control. It deals with the case of boundary control which does not enter in the general framework described in section 2.1.

Example 2.4. Let us consider again the bilinear form $a(\cdot, \cdot)$ used in example 2.3. Let us set

$$W = \{y \in L^2(\Omega) \colon \Lambda y \in L^2(\Omega)\}.$$

Given $y \in W$ the restriction $y|\Gamma = \gamma_0(y)$ makes sense with $y|\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ see [33]. Let us consider the function

$$F\colon (W \times L^2(\Gamma)) \times (L^2(\Omega) \times L^2(\Gamma)) \longrightarrow \boldsymbol{R} \cup \{+\infty\}$$

defined by

$$F((y,v),(g,h)) = \frac{1}{2}\|y - z_d\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(\Gamma)} + i_k(y) + i_{\mathcal{C}}((y,v),(g,h)),$$

where

$$\mathcal{C} = \{((y,v),(g,h)) \in (W \times \mathcal{U}_{\text{ad}}) \times (L^2(\Omega) \times L^2(\Gamma))\colon \Lambda y = \\ = f + g \text{ on } \Omega, y = v + h \text{ on } \Gamma\}$$

where $N > 0$ and where K is a closed convex subset of W. Recall (see [32] p. 83) that given $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ one says that

$$\begin{cases} \Lambda y = f & \text{on } \Omega \\ y = g & \text{on } \Gamma \end{cases}$$

if and only if $y \in L^2(\Omega)$ and

$$\int_\Omega y\Lambda^*\psi dx = \int_\Omega f\psi dx - \int_\Gamma g\frac{\partial\psi}{\partial\nu_{\Lambda^*}}d\Gamma \text{ for all } \psi \in H^2(\Omega) \cap H^1_0(\Omega).$$

The problem is to minimize

$$\frac{1}{2}\|y - z_d\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(\Gamma)} + i_K(y)$$

where $y \in K$ and $v \in \mathcal{U}_{\text{ad}}$ are coupled by the relation

$$\begin{cases} \Lambda y = f & \text{on } \Omega \\ y = v & \text{on } \Gamma. \end{cases}$$

Let Z_Ω and Z_Γ be Banach spaces continuously and densely embedded in $L^2(\Omega)$ and $L^2(\Gamma)$ respectively and let us introduce the operator $T\colon W \longrightarrow L^2(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ defined for all $w \in W$ by

$$T(w) = (\Lambda w - f, \gamma_0(y)).$$

We assume that the following qualification assumption is in force

$$Z_\Omega \times Z_\Gamma \subset \boldsymbol{R}_+ \left(T(K) - (\{0\} \times \mathcal{U}_{\rm ad})\right). \tag{19}$$

Observe that assumption (19) is satisfied whenever there exist $r > 0, \rho \geq 0$ such that for all $(g,h) \in rB_{Z_\Omega} \times rB_{Z_\Gamma}$,

$$h \in -\mathcal{U}_{\rm ad} \cap \rho B_{L^2(\Gamma)} + \gamma_0(\Lambda^{-1}(f+g) \cap K) \tag{20}$$

which is the assumption made in [14]. The associated Lagrangian

$$L\colon (W \times L^2(\Gamma)) \times (Z_\Omega^* \times Z_\Gamma^*) \longrightarrow \overline{\boldsymbol{R}}$$

is defined for all $((v,y),(q,r)) \in (W \times L^2(\Gamma)) \times (L^2(\Omega) \times L^2(\Gamma))$ by

$$\begin{aligned} L((y,v),(q,r)) = \tfrac{1}{2}\|y - z_d\|^2_{L^2(\Omega)} + \tfrac{N}{2}\|v\|^2_{L^2(\Gamma)} + i_K(y) - \\ -\langle q, \Lambda y - f\rangle - \langle r, \gamma_0(y) - v\rangle + i_{\mathcal{F}}(y,v). \end{aligned}$$

where

$$\mathcal{F} = \{(y,v) \in V \times \mathcal{U}_{\rm ad}\colon (\Lambda y - f, \gamma_0 y - v) \in Z_\Omega \times Z_\Gamma\}.$$

Assumption (19) ensures via Theorem 1.1 that F is proper and that the value function

$$\varphi(g,h) = \inf_{(y,v)\in W\times L^2(\Gamma)} F((y,v),(g,h))$$

is lower semicontinuous at 0 yielding the existence of a solution $(\overline{q},\overline{r}) \in Z_\Omega^* \times Z_\Gamma^*$ for the dual problem. Applying (3) we get that $((\overline{y},\overline{v}),(\overline{q},\overline{r})) \in (K \times \mathcal{U}_{\rm ad}) \times (Z_\Omega^* \times Z_\Gamma^*)$ is optimal if and only if

$$\begin{aligned} &\langle \overline{y} - z_d, y - \overline{y}\rangle + \langle N\overline{v}, v - \overline{v}\rangle - \langle \overline{q}, \Lambda(y - \overline{y})\rangle - \\ &-\langle \overline{r}, \gamma_0(y - \overline{y}) - (v - \overline{v})\rangle \geq 0 \text{ for all } (y,v) \in \mathcal{D}, \\ &\langle q - \overline{q}, \Lambda y - f\rangle \geq 0 \text{ and } \langle r - \overline{r}, \gamma_0(\overline{y}) - \overline{v}\rangle \geq 0 \text{ for all } (q,r) \in Z_\Omega^* \times Z_\Gamma^* \end{aligned}$$

where

$$\mathcal{D} = \{(y,v) \in K \times \mathcal{U}_{\rm ad}\colon (\Lambda y - f, \gamma_0 y - v) \in Z_\Omega \times Z_\Gamma\}.$$

Introducing the unique solution $\overline{p} \in H^2(\Omega) \cap H_0^1(\Omega)$ of $\Lambda^*(\overline{p}) = \overline{y} - z_d$ and using the fact that

$$\langle \Lambda^*(\overline{p}), y-\overline{y}\rangle = \langle \overline{p}, \Lambda(y-\overline{y})\rangle - \left\langle \frac{\partial \overline{p}}{\partial \nu_{\lambda^*}}, \gamma_0(y-\overline{y})\right\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}}$$

we get

$$\begin{gathered}\langle N\overline{v}, v-\overline{v}\rangle + \langle \overline{p}-\overline{q}, \Lambda(y-\overline{y})\rangle - \langle \overline{r}, \gamma_0(y-\overline{y}) - (v-\overline{v})\rangle - \\ - \left\langle \tfrac{\partial \overline{p}}{\partial \nu_{\Lambda^*}}, \gamma_0(y-\overline{y})\right\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} \geq 0 \text{ for all } (y,v)\in \mathcal{D}, \\ \Lambda y = f \text{ and } \gamma_0(\overline{y}) = \overline{v}.\end{gathered}$$

In the case when $(Z_\Omega, Z_\Gamma) = (L^2(\Omega), L^2(\Gamma))$, the necessary and sufficient optimality condition turns to the existence of $(\overline{q}, \overline{r}) \in L^2(\Omega)\times L^2(\Gamma)$ such that

$$\begin{cases} \Lambda \overline{y} = f & \text{on } \Omega \\ \overline{y} = \overline{v} & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} \Lambda^* \overline{p} = \overline{y} - z_d & \text{on } \Omega \\ \overline{p} = 0 & \text{on } \Gamma, \end{cases}$$

$$\langle N\overline{v} + \overline{r}, v-\overline{v}\rangle \geq 0 \text{ for all } v \in \mathcal{U}_{\text{ad}},$$

$$\langle \overline{p}-\overline{q}, \Lambda(y-\overline{y})\rangle - \left\langle \frac{\partial \overline{p}}{\partial \nu_{\Lambda^*}} + \overline{r}, \gamma_0(y-\overline{y})\right\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} \geq 0 \text{ for all } y \in K.$$

3. Extension to a class of non convex problems. In the sequel a multifunction G from a set X into a set Y is identified with its graph which is a subset of $X\times Y$. For any $x\in X, G(x)$ is the possibly empty subset of Y defined by $G(x) = \{y\in Y\colon (x,y)\in G\}$. We denote by dom G the set of $x\in X$ such that $G(x)\neq \emptyset$ and by $G^{-1}\subset Y\times X$ the multifunction given by $G^{-1} = \{(y,x)\colon (x,y)\in G\}$.

We say that a multifunction $G\subset X\times Y$ where X and Y are normed vector spaces is metrically regular near $(a,b)\in G$ if there exist a neighborhood N of (a,b) and a constant $k\geq 0$ such that for all $(x,y)\in N$

$$d(x, G^{-1}(y)) \leq k d(y, G(x)),$$

where $d(z,S) = \inf_{x\in S} \|x - z\|$ denotes the distance function to the set S. Metric regularity near (a,b) is equivalent to openness at a linear rate near $(a,b) \in F$: there exist a neighborhood N of $(a,b), k \in \boldsymbol{R}_+$ and $r \in \boldsymbol{P}:=]0,+\infty[$ such that for each $(x,y) \in N \cap F$ and for each $t \in [0,r]$

$$y + tB_Y \subset F(x + tkB_X).$$

The contigent cone $T_S(\overline{x})$ to a closed set S of a normed vector space X at $\overline{x} \in S$ is the set of those $u \in X$ such that

$$\liminf_{t\downarrow 0} t^{-1} d(\overline{x} + tu, S) = 0.$$

A multifunction $\Phi \subset X \times Y$ between two normed vector spaces whose graph is a cone is called a process. The norm of a process Φ is defined ([6] p. 57) by

$$\|\Phi\| = \sup_{u\in \mathrm{dom}\Phi} \|u\|^{-1} d(0, \Phi(u)).$$

We denote by $R(\Phi)$ the subset $\bigcup_{x\in X} \Phi(x) \subset Y$ and by $N(\Phi)$ the subset $N(\Phi) = \Phi^{-1}(0)$. Given a subset A of a normed vector space X we denote by A^- its negative polar cone namely

$$A^- = \{p \in X^* \colon \langle p, x\rangle \leq 0 \text{ for all } x \in A\}$$

where X^* denotes the topological dual of X. The adjoint (see [6]) of a process $\Phi \subset X \times Y$ is the convex process $\Phi^* \subset Y^* \times X^*$ defined by

$$\Phi^* = \{(q,p) \in Y^* \times X^* \colon \langle p, x\rangle - \langle q, y\rangle \leq 0 \text{ for all } (x,y) \in \Phi\}.$$

Given a process $\Psi \subset X^* \times Y^*$ its adjoint $\Psi^* \subset Y \times X$ is defined by

$$\Psi^* = \{(y,x) \in Y \times X \colon -\langle q, y\rangle + \langle p, x\rangle \leq 0 \text{ for all } (p,q) \in \Psi\}.$$

The following lemma is close to Theorem 2.4.4 in [6]. Nevertheless no reflexivity assumption is needed in our lemma.

Lemma 3.1. *Let X, Y be Banach spaces and let $\Phi \subset X \times Y$ be a process. Assume that $R(\Phi) = Y$ and that $\|\Phi^{-1}\| < +\infty$. Then $R(\Phi^*)$ is w^* –closed.*

Proof. From the Banach–Krein–Šmulyan Theorem (see [28], Corollary 2, p. 154) it suffices to show that $R(\Phi^*) \cap rB_X$ is w^*–closed for all

$r \geq 0$. Let $r \geq 0$, let $(I, \leq)$ be a directed set and let $(p_i)_{i\in I}$ be a net in $R(\Phi^*) \cap rB_X$ converging to $p \in X^*$ with respect to the w^* topology. There exists $(q_i)_{i\in I} \subset Y^*$ such that $(q_i, p_i) \in \Phi^*$. From our assumptions, there exists $c \geq 0$ such that for all $v \in B_Y$ there exists $u \in cB_X$ with $(u, -v) \in \Phi$. We have for all $i \in I$

$$\langle p_i, u\rangle \leq \langle q_i, -v\rangle$$

yielding

$$\|q_i\| \leq cr.$$

It follows that there exists a subnet $(q_j)_{j\in J}$ which converges to some $q \in Y^*$ for the w^*–topology. Hence $(q, p) \in \Phi^*$ thus $p \in R(\Phi^*) \cap rB_X$. ∎

We say that a function defined on a neighborhood of an element x of a normed space X is Hadamard differentiable at x whenever the family $(t^{-1}(f(x + t(\cdot) - f(x)))$ converges uniformly on compact subset of X as t decrases to 0 to an element of X^* denoted $Df(x)$. We say that a mapping g defined on a neighborhood of an element x of X with values in a normed space Y is strictly differentiable at x if there exists a linear continuous mapping $\varphi \in L(X, Y)$ such that for all $\varepsilon > 0$ there exists $\eta > 0$ such that $f - \varphi$ is Lipschitzian with constant ε on $B(x, \eta)$; we then set $Dg(x) = \varphi$.

Theorem 3.2. *Let X, Y be Banach spaces, let $f\colon U \longrightarrow \boldsymbol{R}$ be a function defined on an open subset $U \subset X$, let $g\colon U \longrightarrow Y$ be a continuous mapping and let $\mathcal{K} \subset U$ be a closed convex subset. Assume that $\overline{x} \in g^{-1}(0) \cap \mathcal{K}$ is a local minimum for f on $g^{-1}(0) \cap \mathcal{K}$. Assume that f is Hadamard differentiable at $\overline{x}$ and that g is strictly differentiable at $\overline{x}$ with*

$$Dg(\overline{x})(\boldsymbol{R}_+(\mathcal{K} - \overline{x})) = Y. \tag{21}$$

Then there exists $q \in Y^*$ such that

$$\langle Df(\overline{x}) + q \circ Dg(\overline{x}), x - \overline{x}\rangle \geq 0 \quad \textit{for all} \quad x \in \mathcal{K}.$$

Proof. Let us set $C = \mathrm{cl}(\boldsymbol{R}_+(\mathcal{K} - \overline{x})) = T_K(\overline{x})$ and let us consider $u \in C$ with $Dg(\overline{x})u = 0$. There exist sequences (t_n) of positive numbers and (u_n) in X such that (t_n) tends to $0, (u_n)$ converges to u and $\overline{x} + t_n u_n \in \mathcal{K}$ thus $g(\overline{x} + t_n u_n) = o(t_n)$. From assumption (21), we can apply Corollary 4.1 of [18]. It follows that the multifunction $G = g|\mathcal{K}$ is metrically regular near $(\overline{x}, 0) \in G$. Hence there exists $k \geq 0$ such that for all $n \in \boldsymbol{N}$ large enough we have

$$d(\overline{x} + t_n u_n, G^{-1}(0)) \leq k\|o(t_n)\|.$$

Thus we get $u \in T_{G^{-1}(0)}(\overline{x})$. Using the fact that $\overline{x}$ is a local minimum for f on $G^{-1}(0)$ one readily obtains

$$\langle Df(\overline{x}), u\rangle \geq 0.$$

Introducing the closed convex process

$$\Phi = \{(u, Dg(\overline{x})u) \colon u \in C\}$$

we derive that

$$-Df(\overline{x}) \in (N(\Phi))^{-}.$$

Analogously to Proposition 2.5.6 of [6] one easily obtains that $(N(\Phi))^{-} = R(\Phi^*)^{--}$. Moreover we derive from (21) and from the Ursescu–Robinson's Theorem (see [38]) that the closed convex multifunction Φ is open at a linear rate near $(0,0)$ which entails $\|\Phi^{-1}\| < +\infty$. Thus we can apply Lemma 3.1 yielding

$$-Df(\overline{x}) \in R(\Phi^*),$$

so that there exists $q \in Y^*$ with

$$(q, -Df(\overline{x})) \in \Phi^*$$

thus

$$\langle -Df(\overline{x}), u\rangle - \langle q, Dg(\overline{x})u\rangle \leq 0 \text{ for all } u \in C,$$

hence the result observing that $C \supset \boldsymbol{R}_+(\mathcal{K} - \overline{x})$. ∎

Remark 3. The case of the minimization of f on $\mathcal{K} \cap g^{-1}(\mathcal{C})$ where $\mathcal{C}$ is a closed convex subset of Y can be tackled from the preceding theorem. Assume that $\overline{x} \in \mathcal{K} \cap g^{-1}(\mathcal{C})$ is a local minimum for f on $\mathcal{K} \cap g^{-1}(\mathcal{C})$ and that

$$\boldsymbol{R}_+(Dg(\overline{x})(\mathcal{K} - \overline{x}) - (\mathcal{C} - g(\overline{x}))) = Y. \tag{22}$$

Hence we can apply Theorem 3.2 to the function $\widetilde{f}(x,y) = f(x)$, to the mapping $\widetilde{g}(x,y) = g(x) - y$ and to the closed convex set $\mathcal{K} \times \mathcal{C}$ yielding the existence of $q \in Y^*$ such that

$$\langle Df(\overline{x}) + q \circ Dg(\overline{x}), x - \overline{x}\rangle - \langle q, y - g(\overline{x})\rangle \geq 0 \text{ for all } (x, y) \in \mathcal{K} \times \mathcal{C}.$$

Thus we recover, under a weaker assumption on f, the first order part of the dual necessary condition given by R. Cominetti in [20]. There exists $q \in N_{\mathcal{C}}(g(\overline{x}))$, the normal cone to the closed convex set $\mathcal{C}$ such that

$$\langle Df(\overline{x}) + q \circ Dg(\overline{x}), x - \overline{x}\rangle \geq 0 \text{ for all } x \in \mathcal{K}.$$

Example 3.1. We claim that Theorem 2.1 is a consequence of Theorem 3.2. Let us use the notations of Theorem 2.1 and let us assume that $f = 0$ for the sake of simplicity. Let us equip the vector space

$$X = \{(y, v) \in V \times \mathcal{U} \colon A(y) - B(v) \in Z\}$$

with the graph norm $\|(y, v)\| + \|A(y) - B(v)\|_Z$. Observe that X is a Banach space. Let us introduce the closed convex subset $\mathcal{K}$ of X defined by

$$\mathcal{K} = (K \times \mathcal{U}_{\text{ad}}) \cap X,$$

let us define $g \colon X \longrightarrow Z$ by

$$g(y, v) = A(y) - B(v) \text{ for all } (y, v) \in X.$$

and $f \colon X \longrightarrow \boldsymbol{R}$ by

$$f(y, v) = \frac{1}{2}\|Cy - z_d\|^2_{\mathcal{H}} + \frac{1}{2}\langle Nv, v\rangle_{\mathcal{U}} \text{ for all } (y, v) \in X.$$

Then the optimal control problem stated in (10) is equivalent to

$$\inf_{(y,v) \in g^{-1}(0) \cap \mathcal{K}} f(y, v).$$

Let $(\overline{y}, \overline{v}) \in K \times \mathcal{U}_{\text{ad}}$ br the optimal solution. Assume that (S_Z) holds namely

$$Z \subset \boldsymbol{R}_+ \left(-B(\mathcal{U}_{\text{ad}}) + A(K)\right).$$

It implies

$$Dg(\overline{y}, \overline{v}) \left(\boldsymbol{R}_+(\mathcal{K} - (\overline{y}, \overline{v})\right) = Z.$$

Thus we can apply Theorem 3.2 so that there exists $\overline{q} \in Z^*$ such that

$$\langle Df(\overline{y},\overline{v}),(y,v)-(\overline{y},\overline{v})\rangle+\langle\overline{q},A(y-\overline{y})-B(v-\overline{v})\rangle\geq 0 \text{ for all } (y,v)\in\mathcal{K},$$

which is exactly the conclusion of Theorem 2.1.

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