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RADICAL AND NONUNITAL BANACH ALGEBRA WITH CENTRAL IDEMPOTENTS

BY

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1. Introduction. It is known from Simmons, 1963, that the radical R of a unital Banach algebra A is a closed two sided ideal of A. This leads to the result that A/R is a semisimple Banach algebra. If however, A is not unital, then it is not possible to get an idea of regular (invertible) element, in the usual sense. But, in this case, the general definition of regular elements as given by Von Neumann, 1936 can be used. If the algebra contains central idempotents, then invertible elements can be defined as in Bhattacharya and Maity, 1992. In fact, in this paper, it has been shown how the study of radical can be carried out with Banach algebra having finite number of central idempotents.

2. Invertible elements and e–radical.

2.1. Some known definitions and results.

Definition 2.1.1. Let A be a nonunital Banach algebra. An element $x \in A$ is called left invertible if there exists a unique $y \in A$ such that $x = yxx$. In this case, y is called right invertible. If x is both left and right invertible, with y as its inverse, then (x, y) is called an invertible pair. Of course, in such case, $y = xyy$.

Definition 2.1.2. An element e of an algebra A is called a central idempotent if $e^2 = e$ and $ex = xe$, $\forall x \in A$.

Result 2.1.1. Let A be a nonunital algebra with E as the set of its central idempotents. Then $x \in A$ is left invertible iff there exists a unique $y \in A$ such that $yx = e$, $ex = x$, for some $e \in E$. In this case, x is called

e–left invertible in A and y is called e–right invertible in A. x is called e–invertible if it is both e–left invertible and e–right invertible.

Result 2.1.2. Let A be a nonunital algebra whose idempotents are central. Then the following are equivalent:

a) (x, y) is an invertible pair.
b) $x = yxx$, $y = xyy$.
c) $x = xyx,\ y = yxy$.
d) $xy = yx = e,\ ex = x,\ ey = y$ for some idempotent e of A, where $x, y \in A$.

2.2. Some New Definitions.

Definition 2.2.1. Let A be a nonunital algebra with E as its set of central idempotents. An ideal M of A is called e–proper, if it does not contain the central idempotent $e \in E$.

Definition 2.2.2. An e–maximal left ideal is defined as an e–proper left ideal which is not contained in any other e–proper left ideal.

It is denoted by $(MLI)_e$; $(MRI)_e$ are defined likewise.

Definition 2.2.3. An e–radical R_e is defined as the intersection of all e–maximal left ideals. It is written as $R_e = \cap(MLI)_e$.

3. Radical and nonunital algebra with finite number of central idempotents.

Theorem 3.1. *the e–radical R_e of a nonunital algebra with central idempotents is a two sided ideal.*

Proof. Let A be the algebra with E as the set of its central idempotents. We first prove the following Lemmas:

Lemma 1. *If $r \in R_e$, then each $e - r$ is e–left invertible.*

If possible, let $r \in R_e$ but $e - r$ be not e–left invertible. Then $L = A(e-r) = \{xe-xr,\ x\in A\}$ is a proper e–left ideal which contains $e(e-r) = e - er$. We now imbed L in an e–maximal left ideal M; then M contains $e - er$. But $r \in M$, so $er \in M$. Hence, $(e - er) + er = e \in M$. This contradicts the fact that M is e–proper. So $e - r$ is e–left invertible.

Lemma 2. *If $r \in R_e$, then each $e - r$ is e–invertible.*

By Lemma 1, there exists a unique $s \in A$ such that $s(e-r) = e$, $e(e-r) = e-r$. So, $se = e-(-sr)$. As $-sr \in R_e$, so, it follows by Lemma 1, that se is e–left invertible. Again, $s(e-r) = r \Longrightarrow s(e-r)e = e$. So, $se(e-r) = e$, also $e(e-r) = e-r$. Hence, se is right invertible. So, se is e–invertible and hence $e-r$ is e–invertible.

Lemma 3. *If $r \in R_e$, then $er = r$.*

Lemma 4. $R_e = \cap(MLI)_e = [r \in A \mid er = r$ *and* $e-xr$ *is* e*–invertible* $\forall x \in A]$.

In fact, if $r \in R_e$, then as $xr \in R_e$ so $e-xr$ is e–invertible and $er = r$. Conversely, let $r \in A$ such that $e-xr$ is e–invertible for every $x \in A$. If possible, let $r \notin R_e$ then r is not in some e–maximal left ideal M of A. Now, $M + Ar = \{m + xr,\ m \in M,\ x \in A\}$ is a left ideal which contains both M and er. So, $M + Ar$ cannot be e–proper, as M is e–maximal. So, $m+xr = e$, for some $x \in A$. For this x, $m = e-xr$ and hence by assumption, m is e–invertible. This is again a contradiction, as $m \in M$ where M is an e–proper ideal. So, $r \in R_e$.

Lemma 5. $\cap(MRI)_e = [r \in A \mid er = r$ *and* $e - rx$ *is* e*–invertible* $\forall x \in A]$.

Lemma 6. $\cap(MLI)_e = \cap(MRI)_e = R_e$.

First we show that $\cap(MLI)_e \subseteq \cap(MRI)_e$. Let $r \in \cap(MLI)_e$ then $e - xr$ is e–invertible for every $x \in A$. Hence, there exists a unique $s \in A$ such that $s(e-xr) = (e-xr)s = e$, $e(e-xr) = e-xr$, $es = s$. Now $(e-rx)(e+rsx) = e+e(rsx)-rxe-rx(rsx) = e-(rx)e+r(es-xrs)x =$ $= e-(rx)e+rex$ (since $(e-xr)s = e$) $e-(rx)e+(rx)e = e$.

Similarly, it may be shown that $(e+rsx)(e-rx) = e$.

Also $e(e+rsx) = e+ersx = e+resx = e+rsx$ (since $es = s$) lastly, $er = r$ implies $e(e-rx) = e-rx$. Hence $(e-rx)$ is e–invertible and $((e-rx), (e+rsx))$ is an invertible pair. So, $r \in \cap(MRI)_e$. So, $\cap(MLI)_e \subseteq \cap(MRI)_e$. Similarly, it may be proved that $\cap(MRI)_e \subseteq \cap(MLI)_e$. So, $\cap(MLI)_e =$ $= \cap(MRI)_e = \cap(MI)_e = R_e$, $(MI)_e$ being an e–maximal both sided ideal.

This proves Lemma 6 and Theorem 3.1 follows directly from this Lemma.

We now make the following definitions:

Definition 3.1. A maximal ideal M of an algebra with central idempotents is defined as the union of all the e–maximal ideals of A, $e \in E$, where E is the finite set of all the idempotents of A. It is denoted by $M = \cup_e (MI)_e$, $e \in E$.

Definition 3.2. An ideal M of A is said to be proper, if it does not contain any $e \in E$.

Definition 3.3. The radical R of A is defined as the union of all e–radical R_e of A, which does not contain the set of idempotents E. It is expressed as $R = \bigcup_{e \in E} R_e \setminus E$.

The following theorem is now immediate:

Theorem 3.2. *The radical R of an algebra A with finite set of central idempotents E, is a proper two sided ideal.*

4. Radical and nonunital Banach algebra with central idempotents. A central idempotent e if a Banach algebra A is assumed to satisfy an additional property $\|e\| \geq 1$.

4.1. Some known results:

Proposition 4.1.1. *Let A be a Banach algebra whose idempotents are central. Let $H_e = \{x \in A \mid$ for each x, there exists a unique y such that $xy = yx = e$, $ex = x$, $ey = y$, e being a central idempotent$\}$.*

Let $r \in A$ such that $\|r - e\| < 1$, $er = r$. Then r is invertible with (r, s) as an invertible pair, where

$$s = \sum_{n=1}^{\infty} (e - r)^n.$$

Proposition 4.1.2. *The set H_e defined as in Proposition* 4.1.1 *is open.*

Theorem 4.1.1. *The radical R of a Banach algebra A with finite set of central idempotents is a proper closed two sided ideal.*

Proof. Let L be any e–maximal ideal of A. We show that L is closed. If possible, let L be not closed. Then $L \subset \overline{L}$ where $\overline{L}$ is the closure of L. By joint continuity of algebraic operations, it follows that $\overline{L}$ is also a closed two sided ideal of A. As L is e–proper, so $L \subset S_e$ where $S_e = A \setminus H_e$ is a closed set, as H_e is open by Proposition 4.1.2. Hence $\overline{L} \subset \overline{S}_e = S_e$. Thus $\overline{L}$ is an e–proper ideal of A. Now $L \subset \overline{L}$ contradicts the maximality of L. So, every e–maximal ideal of A is closed. So, $R_e = \bigcap_e (MI)_e$ is closed. As E is finite, so $R = \bigcup_e R_e \setminus E$, $e \in E$, is also closed. This proves the theorem.

Theorem 4.1.2. *Let A be a Banach algebra with finite number of central idempotents. Then A/R is a semisimple Banach algebra with central idempotents, where R is the radical of A.*

Proof. It follows in the standard way by using Theorem 4.1.1.

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