ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIV, s.I.a, Matematică, 1998, f1

REGULAR AND SINGULAR ELEMENTS OF A COMPLETE METRIC PARAALGEBRA AND A METRIC PARA–SEMIALGEBRA

BY

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1. Introduction. Regular and singular elements of an unital Banach algebra have been studied by Rickart, C.E. [5], Simmons, G.F. [6] and many others. Similar elements have also been investigated in [1], for nonunital Banach algebra with central idempotents. The object of the present paper is to show that the aforesaid type of investigations can be done with unital Banach algebra, even without the homogeneity condition of the norm and the additive inverse of each element.

With this motivation, we have discussed regular and singular elements of a metric paraalgebra and a metric para–semialgebra where the latter possesses s–identity.

We have limited our discussions primarily to three articles – the first one deals with definitions and examples of different types of parametric algebras and parametric semialgebras; the second one deals with regular and singular elements of a metric paraalgebra and the third one deals with similar elements for a metric para–semialgebra with s–identity.

We have also made an additional article where we have found out some applications of the results obtained above. In fact, in this article, we have investigated the nature of radical of a metric paraalgebra and that of a metric para–semialgebra.

2. Metric paraalgebra, parametric algebra, metric para–semialgebra and parametric semialgebra.

Definition 1. Let X be a linear space over the field F. A function $p : X \to R^+ \cup \{0\}$ is called a total paranorm on X if (i) $p(x) = 0$ iff $x = \theta$, (ii) $p(x + y) \leq p(x) + p(y)$, (iii) $p(-x) = p(x)$ and (iv) if $|\lambda_n - \lambda| \to 0$,

$p(x_n - x) \to 0$ then $p(\lambda_n x_n - \lambda x) \to 0$, as $n \to \infty$, where $\{x_n\} \subset X$, $x, y \in X$, $\{\lambda_n\} \subset F$, $\lambda \in F$, θ is the null element of X.

Remark 1. Every total paranorm p on X gives rise to an invariant metric ρ on X given by $\rho(x, y) = p(x - y)$ such that X becomes a linear metric space. But if X is a linear space over F and is endowed with an invariant metric ρ, then $p(x) = \rho(x, \theta)$ may or may not be a total paranorm on X. In fact, $p(x) = \rho(x, \theta)$ always satisfies properties (i)–(iii), but property (iv) may or may not hold.

Example 1. $\ell(r) = \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^{r_k} < \infty\}$ where $\{r_k\}$ is a bounded sequence of real numbers, $0 < r_k \leq H$, $H = \sup_k r_k$.

$\ell(r)$ is a linear space over $\mathbb{R}$ under pointwise operations and $\rho(x, y) = = (\Sigma \mid x_k - y_k|^{r_k})^{1/M}$, where $M = \max(1, H)$ is an invariant metric on $\ell(r)$ and $p(x)$ satisfies properties (i)–(iv). So $\ell(r)$ is a linear metric space.

Example 2. $\ell_\infty(r) = \{x = x_k : \sup_k |x_k|^{r_k} < \infty\}$, $\{r_k\}$ being same as above.

$\ell_\infty(r)$ is a linear space under pointwise operation; an invariant metric on $\ell_\infty(r)$ is given by $\rho(x, y) = \sup_k |x_k - y_k|^{r_k/M}$; $p(x)$ satisfies properties (i)–(iii). But (iv) holds only when $\inf_k r_k > 0$. So, $\ell_\infty(r)$ is not a linear metric space unless $\inf_k r_k > 0$.

Definition 2. A semilinear space X over $\mathbb{R}^+ \cup \{0\}$ is an additively commutative semigroup with zero element θ, such that for all $\alpha, \beta \in R^+ \cup \{0\}$, and all $x, y \in X$, the following properties are fulfilled:

$$(\alpha + \beta)x = \alpha x + \beta x,\ \beta(x + y) = \beta x + \beta y,$$
$$1x = x,\ 0x = \theta = x0$$

A semilinear space is always considered over $\mathbb{R}^+ \cup \{0\}$.

Definition 3. A total semi paranorm on a semilinear space X is a function $p : X \to \mathbb{R}^+ \cup \{0\}$, such that (i)′ $p(x) = 0$ iff $x = \theta$, (ii)′ $p(x + y) \leq \leq p(x) + p(y)$, (iii)′ $|\lambda_n - \lambda| \to 0$ and $|p(x_n) - p(x)| \to 0 \Rightarrow p(\lambda_n x_n) - p(\lambda x) \to 0$, as $n \to \infty$.

Remark 2. A total semiparanorm on a semilinear space X can never determine a metric ρ on X given by $\rho(x, y) = p(x - y)$, as x does not possess

the additive inverse $-x$, where $x \in X$. However, if X is a semilinear space endowed with an invariant metric ρ, then $p(x) = \rho(x, \theta)$ may be a total semiparanorm on X.

Example 3. $\ell^+(r) = \{x = \{x_k\}, x_k > 0, \sum\limits_k (x_k)^{r_k} < \infty\}$.

$\ell^+(r)$ is a semilinear space under pointwise operations, $\rho(x, y) = \left(\sum\limits_k (x_k - y_k)^{r_k}\right)^{1/M}$ is an invariant metric on $\ell^+(r)$. Further, $p(x) = \rho(x, \theta)$ satisfies properties (i)′–(iii)′. Hence, $\ell^+(r)$ is also a metric semilinear space.

To show that $p(x) = \rho(x, \theta)$ may fail to be a paranorm, we consider the following example:

Example 4. $\ell_\infty^+(r) = \{x = \{x_k\};\ x_k > 0,\ \sup\limits_k (x_k)^{r_k/M} < \infty\}$.

$\ell_\infty^+(r)$ is a metric semilinear space under pointwise operations and metric $\rho(x, y) = \sup\limits_k (x_k - y_k)^{r_k/M}$, only when $\inf\limits_k r_k > 0$.

We now make the following definitions:

Definition 4. Let X be a linear space over F. Let X possess and invariant metric ρ. X is called a metric paralinear space over F, if $p(x) = \rho(x, \theta)$ is a total paranorm on X.

Definition 5. Let X be a semilinear space having an invariant metric ρ. X is called a metric para–semilinear space if $p(x)$ is a total paranorm.

Definition 6. Let X be an algebra over F. Let X possess an invariant metric p. X is called a metric paraalgebra over F, if X is a metric para–linear space having total paranorm $p(x) = \rho(x, \theta)$, where ρ satisfies the following additional property

$$\rho(xy, xz) \le \rho(x, \theta)\rho(y, z);\ \rho(xy, zy) \le \rho(x, z)\rho(y, \theta),\ \forall x, y, z \in X.$$

X is called unital, if X has an identity for multiplication I such that $p(I) = 1$.

Remark 4. For the metric paraalgebra $(X, p), p(xy) \le p(x)p(y)$, $\forall x, y \in X$.

Example 5. $\ell_\infty(r)$ is a metric paraalgebra under pointwise operations.

Definition 7. Let X be an algebra over F. Let X possess an invariant metric ρ. X is called a parametric algebra over F, if the paranorm $p(x)$ satisfies the following additional properties:

$\{x_n\} \subset X, x \in X, p(x_n - x) \to 0 \Longrightarrow p(x_n y - xy) \to 0$, as $n \to \infty$, for each $y \in X$.

$\{y_n\} \subset X, y \in X, p(y_n - y) \to 0 \Longrightarrow p(xy_n - xy) \to 0$, as $n \to \infty$, for each $x \in X$.

Remark 5. Every metric paraalgebra is a parametric algebra, but the converse is not true.

Example 5. $X = \{x = \{x_k\} : \sup_k |x_k| < \infty\}$

X is a parametric algebra over R under pointwise operations, where

$$\rho(x, y) = \sup_k \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

It is not a metric paraalgebra, as $p(x, y) \leq p(x)p(y)$ is not true.

Definition 8. A semialgebra X is a semilinear space over $R^+ \cup \{0\}$, where there is an additional composition $\circ : X \times X \to X$ such that

(i) $x \circ (y \circ z) = (x \circ y) \circ z$,
(ii) $x \circ (y + z) = x \circ y + x \circ z, (x + y) \circ z = x \circ z + y \circ z$,
(iii) $\theta x = \theta = x\theta$,
(iv) $\alpha(x \circ y) = (\alpha x) \circ y$

for all $x, y, z \in X$, $\alpha \in R^+$ and θ as the zero element of X.
A semialgebra X is called unital, if X has an identity for multiplication I.

Remark 6. A metric para–semialgebra and a parametric semialgebra may be similarly defined and examples may be similarly constructed.

3. Regular and singular elements of a complete unital metric paraalgebra. We simply mention those theorems whose proofs are similar in case of Banach algebra, otherwise we mention the crucial steps of the proofs, if necessary.

Theorem 1. *Let (X, ρ) be a complete metric paraalgebra with identity I. Then each $r \in X$ for which $\rho(r, I) < 1$, is regular and its inverse s is given by*

$$s = I + \sum_{n=1}^{\infty} (I - r)^n$$

Theorem 2. *Let G and S denote respectively the set of regular and singular elements of a complete metric paraalgebra (X, ρ). Then G is an open set and S is a closed set.*

Theorem 3. *The mapping $r \to r^{-1}$ of G into G is continuous.*

Definition 9. An element z of a metric paraalgebra (X, ρ) is called a topological divisor of zero, if there exists a bounded sequence $\{z_n\} \subset X$ such that $z_n \not\to 0$ as $n \to \infty$ and either $zz_n \to 0$ or $z_n z \to 0$, as $n \to \infty$.

Theorem 4. *Let Z denote the set of all topological divisors of zero of a metric paraalgebra (X, ρ), then $Z \subseteq S$ but bd S is a subset of Z.*

Proof. $Z \subset S$ is clear, because if $z \in Z$ but $z \notin S$, then z^{-1} exists and also, there exists $\{z_n\} \subset X$ such that $z_n \not\to 0$, whereas, for definiteness, $zz_n \to 0$. So, from continuity of multiplication, it follows that $z^{-1}(zz_n) = = z_n \to 0$, as $n \to \infty$, which is a contradiction.

Now, to prove the other part, we take $z \in \text{bd } S$. As S is closed, so there exists $\{r_n\} \subset G$ such that $p(r_n - z) \to 0$, as $n \to \infty$. Also, the sequence $\{p(r_n^{-1})\}$ is unbounded, for otherwise there exists $M > 0$ such that $p(r_n^{-1}) \leq M$ for all n and so we consider some n for which we have $p(z - r_n) < 1/M$. Thus, using the inequality $p(r_n^{-1}z - I) = p(r_n^{-1}(z - r_n)) \leq p(r_n^{-1})p(z - r_n)$, we get $p(r_n^{-1}z - I) < 1$. Hence, for this n, $r_n^{-1}z \in G$. So $z = r_n(r_n^{-1}z) \in G$, which is a contradiction. Therefore, $p(r_n^{-1}) \to \infty$ as $n \to \infty$. Let $z_n = r_n^{-1}/p(r_n^{-1})$. Then $z_n \not\to 0$ as $n \to \infty$. Now $p(zz_n) \leq p(I/p(r_n^{-1})) + p(z - r_n)$. Let $(1/(p(r_n^{-1})) = \alpha_n$. Then $\alpha_n \to 0$ as $n \to \infty$. As p is a total paranorm, so $p(\alpha_n I) \to p(0I)$, as $n \to \infty$. Thus $p(\alpha_n I) \to 0$. Hence $zz_n \to 0$ as $n \to \infty$. This completes the proof.

Remark. The proof is just a paranorm version of the corresponding result in normed case (see [6], p.307).

Theorem 5. *Let R be the radical of a complete metric paraalgebra (X, ρ) and let R be taken as the intersection of all maximal left ideals of X. Then X/R is a semisimple complete metric paraalgebra.*

4. Regular and singular elements of a metric para-semialgebra

Definition 10. Let X be a semialgebra. The multiplicative identity of X is called an s–identity, if for each $r \in X$, there exists a unique $r^0 \in X$ such that $r + r^0 = I$, where I is the multiplicative identity of X.

Example 6. Let $X = \{i, x, y, ...\}$ where $i = [1, 1, 1, ...]$, $x = [\{x_n\} : x_n \in R^+ \cup \{0\}]$, $y = [\{y_n\} : y_n \in R^+ \cup \{0\}]$, such that $0 < (x_n + y_n) \leq 1$, if none of x and y is i and $1 < (x_n + y_n) \leq 2$, if at least one of x and y is i.

X is a semialgebra under pointwise operations and i is its s–identity.

Theorem 6. *Let (X, ρ) be a complete metric para–semialgebra with I as its s–identity. Then each $r \in X$, for which $\rho(I, r) < 1$, is regular. Further, the inverse of r is given by $r' = I + \sum_{n=1}^{\infty} t^n$, where $r + t = I$.*

Proof. By given conditions, $p(t) = \rho(t, \theta) = \rho(r + t, r) = \rho(I, r) < 1$. As $p(t^n) \leq (p(t))^n < 1$, so partial sums of $\sum_{n=1}^{\infty} t^n$ form a Cauchy sequence in X. As X is complete, so $\sum_{n=1}^{\infty} t^n$ converges to some element of X. We denote it by $\sum_{n=1}^{\infty} t^n$. Now

$$\begin{aligned}\rho(rr', I) &= \rho\left(r + \sum_{n=1}^{\infty} rt^n, r + t\right) = \rho\left(\sum_{n=1}^{\infty} rt^n, t\right) \\ &= \left(\sum_{1}^{\infty} rt^n + \sum_{2}^{\infty} t^n, t + \sum_{2}^{\infty} t^n\right) = \left(\sum_{1}^{\infty} rt^n + \sum_{2}^{\infty} t^n, \sum_{1}^{\infty} t^n\right)\end{aligned}$$

Again, $I = r + t \Rightarrow t^n = rt^n + t^{n+1}$. So $\sum_{1}^{\infty} t^n = \sum_{1}^{\infty} rt^n + \sum_{1}^{\infty} t^{n+1} = \sum_{1}^{\infty} rt^n + \sum_{2}^{\infty} t^n$.

Hence, $\rho(rr', I) = \left(\sum_{1}^{\infty} t^n, \sum_{1}^{\infty} t^n\right) = 0$. So, $rr' = I$. Similarly, it may be shown that $r'r = I$. Thus r' is the inverse of r. This proves the theorem.

Theorem 7. *Let (X, ρ) be defined as in Theorem* 6. *Then the set G of regular elements is open and the set S of singular elements is closed.*

Proof. We only prove that G is open. Let $r \in G$ be arbitrary and let r' be the inverse of r. We show that an open ball of radius $1/p(r')$ with center r lies in G. Let $y \in X$ such that $\rho(y, r) < 1/p(r')$. We prove that $y \in G$. Now, $\rho(yr', I) = \rho(yr', rr') \leq \rho(y, r)\rho(r', \theta) < (1/p(r'))$ $(p(r') = 1)$. Hence $yr' \in G$. As, composed of two regular elements is a regular element. So, $(yr')r = y(r'r) = y \in G$. Hence G is open. Thus S is closed.

Theorem 8. *Let (X, ρ) be defined as in Theorem* 6. *Then the map $r \to r^{-1}$ of G into G is continuous.*

Proof. Let $r, r_0 \in G$ such that $\rho(r, r_0) < (1/2)p(r_0^{-1})$. Then

$$\rho(r_0^{-1}t, I) = \rho(r_0^{-1}r, r_0^{-1}r_0) \leq p(r_0^{-1})\rho(r, r_0) < 1/2.$$

So $r_0^{-1}r \in G$. Now,

$$r^{-1}r_0 = (r_0^{-1}r)^{-1} = I + \sum_1^\infty t^n, \text{ where } r_0^{-1}r + t = I.$$

Hence, $p(t) = \rho(t, \theta) = \rho(r_0^{-1}r + t, r_0^{-1}r) = \rho(I, r_0^{-1}r) < 1/2$. Now,

$$\begin{aligned}
\rho(r^{-1}, r_0^{-1}) &= \rho(r^{-1}r_0r_0^{-1}, Ir_0^{-1}) \leq p(r_0^{-1})\rho(r^{-1}r_0, I) \\
&= p(r_0^{-1})\rho\left(I + \sum_1^\infty t^n, I\right) = p(r_0^{-1})\rho\left(\sum_1^\infty t^n, \theta\right) \\
&= p(r_0^{-1})p\left(\sum_1^\infty t^n\right) \leq p(r_0^{-1})\sum_1^\infty p(t^n) \leq \\
&\leq p(r_0^{-1})\left(\sum_1^\infty (p(t))\right)^n = p(r_0^{-1})[(p(t))/(1-p(t))] \text{ (Since } p(t)<1) \\
&< 2p(r_0^{-1})p(t) \text{ (Since } p(t) < 1/2).
\end{aligned}$$

But

$$\begin{aligned}
p(t) &= \rho(t, \theta) = \rho(t + r_0^{-1}r, r_0^{-1}r) = \rho(I, r_0^{-1}r) = \\
&= \rho(r_0^{-1}r_0, r_0^{-1}r) \leq p(r_0^{-1})\rho(r_0, r),
\end{aligned}$$

So,

$$\begin{aligned}
\rho(r^{-1}, r_0^{-1}) &< 2p(r_0^{-1})\rho(r_0, r)p(r_0^{-1}) = \\
&= 2(p(r_0^{-1}))^2\rho(r, r_0) = K\rho(r, r_0), \ K > 0.
\end{aligned}$$

Hence $r \to r^{-1}$ is a continuous map of G into G.

Theorem 9. *Let (X, ρ) be defined as in Theorem 6. Let Z and S denote respectively the set of all topological divisors of zero and that of all singular elements of X. Then $Z \subseteq S$ and bd S is a subset of Z.*

Proof. $Z \subseteq S$ is obvious. To prove the second part, we take $z \in$ bd S. We show that $z \in Z$. Now S being closed, there exists $\{r_n\} \subset G$ such that $\rho(r_n, z) \to 0$ as $n \to \infty$.

Again, as in Theorem 4, we can prove that $p(r_n^{-1}) \to \infty$, as $n \to \infty$. We now write $z = r_n + r_n^0$. Then $p(r_n^0) = \rho(r_n^0, \theta) = \rho(r_n + r_n^0, r_n) = \rho(z, r_n) \to 0$ as $n \to \infty$. Next, let $z_n = [r_n^{-1}/p(r_n^{-1})]$. Then $z_n \not\to 0$ as $n \to \infty$. Also, $zz_n = (zr_n^{-1})/(p(r_n^{-1})) = [((r_n + r_n^0)r_n^{-1})/(p(r_n^{-1}))] = [I/(p(r_n^{-1}))] + r_n^0 z_n$. So,

$$p(zz_n) \leq p[I/(p(r_n^{-1}))] + (p(r_n^0))(p(z_n)).$$

As the multiplication is a continuous operation and as $[1/((p(r_n^{-1})) \to 0$. So, $p[I/(p(r_n^{-1}))] \to p(0I) = p(\theta) = 0$, as $n \to \infty$. Again, $\{z_n\}$ is bounded, $z_n \not\to 0$, but $p(r_n^0) \to 0$, s $n \to \infty$. So $zz_n \to 0$ and hence $z \in Z$.

5. Radical of a semialgebra with s–identity. In this article, it has been shown that the usual concept of radical of an algebra fails for a semialgebra. In fact, it is nolonger a two sided ideal and hence corresponding quotient algebra with respect to this radical cannot be obtained. However, left and right radicals for the semialgebra can be defined and their characterizations can be obtained.

Definition 11. Let X be a semialgebra. A left radical R_ℓ of X is defined as the intersection of all maximal left ideals. A right radical R_r of X is defined as the intersection of all maximal right ideals.

Definition 12. A semialgebra is called semisimple if R_ℓ and R_r are zero ideals of the semialgebra.

Theorem 10. *Let R_ℓ be the left radical of a semialgebra with I as its s–identity. Then R_ℓ consists of precisely those elements r of X such that each $r^0 \in X$ for which $xr + r^0 = I$, for some $x \in X$, is left regular. But r^0 is not right regular.*

Proof. We first prove the following lemmas:

Lemma 1. *If $r \in R_\ell$ and if $r + r^0 = I$, then r^0 is left regular.*

If possible, let r^0 be not left regular. Then $[x(r^0) : x \in X]$ is a proper left ideal of X containing r^0. We now imbed $[x(r^0) : x \in X]$ in a maximal

left ideal M of X. Obviously, $r^0, r \in M$. So $r + r^0 = I$ is impossible, as M is proper. Hence r^0 is left regular.

Lemma 2. *If $r \in R_\ell$ and if $xr + r^0 = I$, for some $x \in X$, then each such $r^0 \in X$, is left regualr but r^0 is not right regular.*

Proof. Since $r \in R_\ell \Longrightarrow xr \in R_\ell$, so by Lemma 1, r^0 is left regular. If possible, let r^0 be right regular also. Then there exists a unique $s \in X$ such that $r^0 s = I$. Now $r^0 + xr = I \Longrightarrow I + (xr)s = s$. As s is unique, so for given r and x, $(xr)s$ is also unique. Hence $I + (xr)s = s$ means that the cancellation law for addition holds in X. But this is not possible in X. So, r^0 is not right regular.

Lemma 3. *Let $r \in X$. Let $r^0 \in X$ be left regular and let r^0 be not right regular. If $xr + r^0 = I$, for some $x \in X$, then $r \in R_\ell$.*

Proof. If possible, let $r \notin R_\ell$. Then r does not belong to some maximal left ideal M of X. So, $\{m + xr : m \in M,\ x \in X\}$ is a left ideal containing both M and r. So, $\{m + rx\}$ is improper. Hence, $xr + m = I$, for some $m \in M$ and some $x \in X$. Therefore, by Lemma 2, m is left regular and m is not right regular. But this implies that m cannot belong to a proper left ideal. Hence, we get a contradiction as $m \in M$. So $r \in R_\ell$.

Proof of the main theorem follows directly from Lemmas 2 and 3.

Theorem 11. *Let R_r be the right radical of a semialgebra X with I as its s–identity. Then R_r consists of precisely those elements $r \in X$ such that each $r^{00} \in X$ for which $rx + r^{00} = I$, for some $x \in X$, is right regular, but r^{00} is not left regular.*

Theorem 12. *Let R_ℓ and R_r be respectively the left and right radical of a semialgebra X with I as its s–identity. Then $R_\ell \cap R_r = \{\theta\}$.*

Proof. Let $r(\neq \theta) \in R_\ell$. Then each $r^0 \in X$ such that $xr + r^0 = I$, for some $x \in X$ is left regular (not right regular). So, there exists a unique $s \in X$ such that $sr^0 = I$. Now, let $r^{00} \in X$ be such an element that $rx + r^{00} = I$, for the same r and x taken above. We show that r^{00} cannot be right regular. If possible, let r^{00} be right regular. Then there exists a unique $s' \in X$ such that $r^{00}s' = I$. Now,

$$\begin{aligned} rx + r^{00} &= I \Longrightarrow (rsx)(rx) + rsx(r^{00}) = rsx \\ &\Longrightarrow r(sxr)x + (rsx)r^{00} = rsx \\ &\Longrightarrow r(sxr)x + rx + (rsx)r^{00} = rsx + rx \\ &\Longrightarrow rsx + (rsx)r^{00} = rsx + rx \\ &[\text{Since } xr + r^0 = I \Longrightarrow sxr + I = s] \\ &\Longrightarrow (rsx)s' + rsx = (rsx)s' + rxs'. \end{aligned}$$

As cancellation law does not hold in X, so $rsx \neq rxs'$. Naturally, any s' can act as the right inverse of r^{00} provided $rxs' \neq rsx$. Thus s' fails to be unique. This is a contradiction. Hence $r \notin R_r$. So $R_\ell \cap R_r = \{\theta\}$.

6. Radical of a metric para–semialgebra with s–identity.

Theorem 13. *The left radical R_ℓ of a complete metric para–semialgebra (X, ρ) with s–identity is a proper closed left ideal of (X, ρ).*

Proof. Let L be any maximal left ideal of (X, ρ). We show that L is closed. If L is not closed, then $L \subset \overline{L}$ (closure of L). As L is proper, so $L \subset S$ (the set of all singular elements). As S is closed, so $\overline{L} \subset S$. Hence $\overline{L}$ is also proper. This contradicts maximality of L, as $L \subset \overline{L}$. So, L is closed. Thus R_ℓ is the intersection of closed ideals. Hence, R_ℓ is closed.

Theorem 14. *The right ideal R_r of a complete metric para–semialgebra with s–identity is a proper closed right ideal.*

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Received: 16.XII.1994
Revised: 20. V.1996

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