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ON THE GEOMETRY OF TANGENT BUNDLE OF A (PSEUDO-) RIEMANNIAN MANIFOLD

BY

V. OPROIU and N. PAPAGHIUC

0. Introduction. In [5], [6] the authors have studied the properties of a pseudo–Riemannian metric G on the cotangent bundle T^*M of a manifold M by using an arbitrary symmetric nonlinear connection on this bundle. Remark that this pseudo–Riemannian metric G on T^*M is very much similar to the Riemann extension considered in [7], [8], [9].

In [3] the first author has studied some aspects of the differential geometry of the tangent bundle of a Langange manifold when this bundle is endowed with a pseudo–Riemannian metric obtained from the fundamental tensor field by a method similar to the obtaining of the complete lift of a (pseudo–)Riemannian metric on a differentiable manifold. So, the Levi Civita connection associated with the considered pseudo– Riemannian metric has been obtained, next the local coordinate expressions of its curvature tensor field and the corresponding Bianchi identities have been written down.

In the present paper we study the properties of a pseudo– Riemannian metric G of type "complete lift" on the tangent bundle TM of a (pseudo–) Riemannian manifold (M, g). The considered pseudo–Riemannian metric G on TM is defined by using the Levi Civita cannection of the (pseudo–) Riemannian metric g on M and an arbitrary M– tensor field of type (0,2) on the tangent bundle TM. First we show that the geometric properties of (TM, G) to be either flat, or projectively flat, or conformally flat or locally symmetric are expressed only in the terms of the Levi Civita connection of g on M and of the symmetric part of the considered M–tensor field on TM. Next, in the particular case where the considered M–tensor field is independent of the tangential coordinates we get the necessary ans suffi-

cient conditions under which the pseudo–Riemannian manifold (TM, G) is either flat, or projectively flat, or conformally flat, or lacally symmetric. In section 2 we define an almost complex structure J on TM and we get the conditions under which the almost complex manifold with Norden metric (TM, J, G) is a Kaehlerian manifold with Norden metric. In section 3, we consider a naturally defined almost product structure P on TM and study the conditions under which the almost parahermitian manifold (TM, G, P) is a parakaehlerian manifold. Finally, some classes of manifolds whose tangent bundles carry parakaehlerian structures are also presented (Theorems 7, 8, 9 and 10).

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞. We use the well known summation convention, the range for the indices i, j, k, h, l, s, t being always $\{1, 2, \ldots, n\}$. We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

1. The pseudo–Riemannian manifold (TM, G)**.** Let M be an n–dimensional manifold and denote by $\tau: TM \to M$ its tangent bundle. Then TM is a $2n$–dimensional manifold and some special local charts on TM induced from local charts on M may be used. Namely, if $(U, x^i); i = 1, \ldots, n$ is a local chart on M, then the local chart $(\tau^{-1}(U), x^i \circ \tau, y^i); i = 1, \ldots, n$ is defined on TM where $y^1, \ldots, y^n$ are the vector space coordinates of a element from $\tau^{-1}(U)$ with respect to the natural frame $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ defined by the local chart $(U, x^i); i = 1, \ldots, n$. We shall denote, by an abuse of notation, $x^i = x^i \circ \tau$, so x^i are considered simultaneously as local coordinates on M and on TM. The M–tensor fields and the linear M–connections may be considered on TM and the usual tensor fields and linear connections on the base manifold M may be thought of naturally as M–tensor fields and linear M–connections on TM (sec [5], [8]). The tangent bundle TTM of TM has an integrable vector subbundle $VTM = \text{Ker } \tau_*$, called the vertical distribution on TM. A nonlinear connection on TM is defined by a distribution HTM, complementary to VTM in TTM (called the horizontal distribution)

$$TTM = VTM \oplus HTM. \tag{1}$$

In the following we assume that M is a (pseudo–)Riemannian manifold with the (pseudo–)Riemannian metric

$$g = g_{ij}(x) dx^i dx^j$$

and denote by g^{ij} the components of the inverse of the matrix (g_{ij}); $i, j, = 1, \ldots, n, i.e.$ $g_{ik}g^{jk} = \delta^j_i$.

Denote by Γ^k_{ij} the connection coefficients of the Levi Civita connection ∇ of g (i.e. the Christoffel symbols) and let a_{ij}; $i, j = 1, \ldots, n$ be the components of an arbitrary M–tensor field of type (0,2) on TM. We think of g_{ij} as the components of an M–tensor field of type (0,2) on TM and consider the functions

$$N^i_j = \Gamma^i_{jk}y^k + g^{ih}a_{hj}; \quad i, j = 1, \ldots, n. \tag{2}$$

It follows that these functions are the connection coefficients of a non-linear connection which defines a horizontal distribution denoted by HTM on TM. The system of the local vector fields $(\partial_i = \frac{\partial}{\partial y^i})$; $i = 1, \ldots, n$ is a local frame in VTM and the system of local vector fields $(\delta_i = \frac{\delta}{\delta x^i})$; $i = 1, \ldots, n$, where

$$\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} \tag{3}$$

is a local frame in HTM. Then (∂_i, δ_i); $i = 1, \ldots, n$ is a local frame on TM adapted to the direct sum decomposition (1).

The system of local 1–froms $(\delta y^i, dx^i)$; $i = 1, \ldots, n$, where

$$\delta y^i = dy^i + N^i_j dx^j \tag{4}$$

is the dual local frame of the local frame (∂_i, δ_i); $i = 1, \ldots, n$.

Then we have, as usual:

$$\left[\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right] = -\Phi^k_{ji}\frac{\partial}{\partial y^k}; \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k_{ij}\frac{\partial}{\partial y^k}, \tag{5}$$

where

$$\Phi^k_{ji} = \frac{\partial N^k_j}{\partial y^i}; \qquad R^k_{ij} = \frac{\delta N^k_j}{\delta x^i} - \frac{\delta N^k_i}{\delta x^j} \tag{6}$$

and the integrability of the system defined by HTM on TM is equivalent to the vanishing of the components R^k_{ij} on $\tau^{-1}(U)$.

Remark that the components Φ^k_{ji} define a linear M– connection and the components R^k_{ij} define an M–tensor field of type (1,2) on TM.

Consider the following pseudo–Riemannian metric G of complete lift type on TM

$$
\begin{aligned}
G = 2g_{ij}\delta y^i dx^j &= 2g_{ij}dy^i dx^j + 2g_{ij}\Gamma^i_{hk}y^k dx^h dx^j + \\
&+(a_{ij}+a_{ji})dx^i dx^j = 2g_{ij}dy^i dx^j + \\
&+\tfrac{\partial g_{ij}}{\partial x^k}y^k dx^i dx^j + (a_{ij}+a_{ji})dx^i dx^j .
\end{aligned} \tag{7}
$$

Remark that the pseudo–Riemannian metric G depends only on (the pseudo–)Riemannian metric g on M and on the symmetric part of the M–tensor field on TM defined by the components a_{ij}.

Denote by $\widetilde{\nabla}$ the Levi Civita connection of the considered pseudo–Riemannian metric G on TM.

Then the following result is proved by a straightforward computation.

Proposition 1. *The local coordinate expression of $\widetilde{\nabla}$ in the local frame (∂_i, δ_i) adapted to the direct sum decomposition (1) is:*

$$
\widetilde{\nabla}_{\partial_i}\partial_j = 0; \quad \widetilde{\nabla}_{\partial_i}\delta_j = g^{kh}\frac{\partial b_{jh}}{\partial y^i}\frac{\partial}{\partial y^k}; \quad \widetilde{\nabla}_{\delta_i}\partial_j = \left(\Gamma^k_{ij} + g^{kh}\frac{\partial c_{hi}}{\partial y^j}\right)\frac{\partial}{\partial y^k};
$$

$$
\widetilde{\nabla}_{\delta_i}\delta_j = \left(\Gamma^k_{ij} - g^{kh}\frac{\partial c_{ij}}{\partial y^h}\right)\frac{\delta}{\delta x^k} + \frac{1}{2}g^{kh}(R_{ijh} - R_{jhi} - R_{hij})\frac{\partial}{\partial y^k},
$$

where

$$
R_{hij} = g_{hk}R^k_{ij}
$$

and c_{ij} (respectively b_{ij}) denotes the simmetric part (respectively the skew–symmetric part) of a_{ij}, i.e. $c_{ij} = \frac{1}{2}(a_{ij}+a_{ji})$ and $b_{ij} = \frac{1}{2}(a_{ij}-a_{ji})$.

Remark. From Proposition 1 it follows that the esential coefficients of the local coordinate expression of $\widetilde{\nabla}$ in the local adapted frame (∂_i, δ_i); $i = 1, \ldots, n$ are expressed by using only the components of the (pseudo–)Riemannian metric g on M, the coefficients of the Levi Civita connection ∇ of g, the simmetric part and the skew–symmetric part of the M–tensor field defined by the components a_{ij} on TM and the M–tensor field on TM defined by $\frac{1}{2}(R_{ijh} - R_{jhi} - R_{hij})$. On the other hand, taking into account that the pseudo–Riemannian metric G depends only on the components of the

(pseudo–)Riemannian metric g on M and on the simmetric part of the M–tensor field defined by the components a_{ij} on TM we can consider the nonlinear connection $\overline{HTM}$ defined by the connection coefficients $\overline{N}^i_j$:

$$\overline{N}^i_j = \nabla^i_{jk} y^k + g^{ih} c_{hj}; \quad i,j = 1,\dots,n,$$

where c_{ij} denotes the simmetric part of $a_{ij}, i.e.\ \ c_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$. Then the vector fields $(\overline{\delta}_i = \frac{\overline{\delta}}{\delta x^i}; \quad i = 1,\dots,n$, where

$$\overline{\delta}_i = \frac{\overline{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overline{N}^j_i \frac{\partial}{\partial y^j}$$

define a local frame in the horizontal distribution $\overline{HTM}$ and the vector fields $(\partial_i, \overline{\delta}_i) : \ i = 1,\dots,n$ define a local frame in TM adapted to the direct sum decomposition

$$TTM = VTM \oplus \overline{HTM}. \tag{8}$$

Denote by $(\overline{\delta}y^i, dx^i); \quad i = 1\dots,n$ the dual local frame of the local frame $(\partial_i, \overline{\delta}_i); \quad i = 1,\dots,n$. Then we have

$$\overline{\delta}y^i = dy^i + \overline{N}^i_j dx^j$$

and the pseudo–Riemannian metric G defined by (7) on TM becomes

$$G = 2g_{ij}\overline{\delta}y^i dx^j = 2g_{ij} dy^i dx^j + \frac{\partial g_{ij}}{\partial x^k} y^k dx^i dx^j + 2c_{ij} dx^i dx^j.$$

Since we have

$$\delta_i = \overline{\delta}_i - g^{jh} b_{hi} \partial_j,$$

where b_{ij} denotes the skew–symmetric part of a_{ij}, from Proposition 1 we obtain by a straightforward computation.

Proposition 2. *The local coordinate expression of the Levi Civita connection $\overline{\nabla}$ of G in the local frame $(\partial_i, \overline{\delta}_i); \quad i = 1,\dots,n$ adapted to the direct sum decomposition (8) is:*

$$\widetilde{\nabla}_{\partial_i}\partial_j = 0; \quad \widetilde{\nabla}_{\partial_i}\overline{\delta}_j = 0; \quad \widetilde{\nabla}_{\overline{\delta}_i}\partial_j = \left(\Gamma^k_{ij} + g^{kh}\frac{\partial c_{hi}}{\partial y^i}\right)\frac{\partial}{\partial y^k};$$

$$\widetilde{\nabla}_{\overline{\delta}_i}\overline{\delta}j = \left(\Gamma^k_{ij} - g^{kh}\frac{\partial c_{ij}}{\partial y^h}\right)\frac{\overline{\delta}}{\delta x^k} + g^{kh}\overline{R}_{ijh}\frac{\partial}{\partial y^k},$$

where $\overline{R}_{ijk}$ *are defined by*

$$\overline{R}_{hij} = g_{hk}\overline{R}^k_{ij};\quad \overline{R}^k_{ij} = \overline{\delta}_i\overline{N}^k_j - \overline{\delta}_j\overline{N}^k_i.$$

Remark. From Proposition 2 it follows that the esential coefficients of the local coordinate expression of $\widetilde{\nabla}$ in the local adapted frame $(\partial_i, \overline{\delta}_i)$; $i = 1, \ldots, n$ are expresed by using only the (pseudo–)Riemannian metric g on M, the coefficients of the Levi Civita connection ∇ of g, the components of the symmetric M–tensor field defined by c_{ij} and the components $\overline{R}_{kij}$ defined above.

Since the geometric properties of the pseudo–Riemannian manifold (TM, G) to be either flat, or projectively flat, or conformally flat or locally symmetric are independent of the choice of the horizontal distribution HTM or $\overline{HTM}$ on TM, from Proposition 2 we have

Theorem 3. *The geometric properties of the pseudo– Riemannian manifold* (TM, G) *to be either flat, or projectively flat, or conformally flat or locally symmetric which depend only on the metric* G *and its Levi Civita connection* $\widetilde{\nabla}$ *are expressed only in the terms of the nonlinear connection defined by* $\overline{N}^i_j$ *on* TM.

In the following we give the conditions under which the pseudo– Riemannian manifold (TM, G) is respectively flat, projectively flat, conformally flat and locally symmetric, assuming that the M–tensor field defined by the components a_{ij} is independent of the tangential coordinates y^i, i.e. the components a_{ij} define a tensor field of type (0,2) on the base manifold M thought of as an M–tensor field on TM.

We obtain by a straightforward computation the following result.

Theorem 4. *Let* (M, g) *be a (pseudo–)Riemannian manifold and denote by* Γ^k_{ij} *the coefficients of the Levi Civita connection* ∇ *of* g. *Consider on* TM *the nonlinear connection defined by (2), where the components* a_{ij} *define a tensor field of type (0,2) on* M *thought of as an* M*–tensor field on* TM. *Denote by* (TM, G) *the pseudo–Riemannian manifold* TM *endowed with the pseudo–Riemannian metric* G, *where* G *is given by (7). Then we have:*

(i) (TM, G) *is flat if and only if* (M, g) *is flat and the symmetric part* c_{ij} *of* a_{ij} *satisfies the condition*

$$\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) = 0, \tag{9}$$

where $\nabla_i c_{jk}$ are the local components of the covariant derivative of the tensor field on M defined by c_{jk} with respect to ∇.

(ii) *(TM, G) is projectively flat if and only if (TM, G) is flat.*

(iii) *(TM, G) is conformally flat if and only if (M, g) has constant sectional curvature and the simmetric part c_{ij} of a_{ij} satisfies the condition*

$$\begin{aligned}(10)\qquad &\nabla_i(\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_j(\nabla_k c_{il} - \nabla_l c_{ik}) = \\ &= \tfrac{r}{n(n-1)}(c_{jk}g_{il} - c_{jl}g_{ik} - c_{ik}g_{jl} + c_{il}g_{jk}),\end{aligned}$$

where r denotes the scalar curvature of (M, g) and $n > 2$ is the dimension of M.

(iv) *(TM, G) is locally symmetric if and only if (M, g) is locally symmetric and the simmetric part c_{ij} of a_{ij} satisfies the condition*

$$\begin{aligned}(11)\quad &\nabla_h\nabla_i(\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_h\nabla_j(\nabla_k c_{il} - \nabla_l c_{ik}) + \\ &+\nabla_h c_{is}R^s_{jkl} - \nabla_h c_{js}R^s_{ikl} + (\nabla_s c_{hl} - \nabla_l c_{hs})R^s_{kij} - (\nabla_i c_{hs} - \nabla_s c_{hi})R^s_{jlk} + \\ &+ (\nabla_j c_{hs} - \nabla_s c_{hj})R^s_{ilk} - (\nabla_k c_{hs} - \nabla_s c_{hk})R^s_{lij} = 0,\end{aligned}$$

where R^h_{kij} are the local coordinate componenets of the curvature field of the Levi Civita connection on (M, g).

Remarks.

(i) If $c_{ij} = 0$ then the conditions (9), (10), (11) are identically fulfilled.

(ii) If c_{ij} is parallel with respect to ∇, then the conditions (9) and (11) are identically fulfilled too.

(iii) If c_{ij} satisfies the relation $\nabla_i c_{jk} - \nabla_j c_{ik} = \nabla_k \omega_{ij}$, where the components ω_{ij} define a 2–form on M and if (M, g) is flat then the condition (9) is identically verified. In particular, if the 2–form defined by the components ω_{ij} is parallel then the components c_{ij} define a Codazzi tensor field and (9) is identically verified.

2. An almost complex structure on (TM, G). Let (M, g) be a (pseudo–)Riemannian manifold and let (TM, G) be the pseudo– Riemannian manifold with G defined by (7) where the considered nonlinear connection on TM is given by (2). We can define the following almost complex structure J on TM:

$$(12)\qquad J\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i};\quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\delta}{\delta x^i}.$$

Then, according with the terminology from [2], we may verify that G and J define an almost complex structure with Norden metric on TM (or equivalently, (TM, J, G) is a hyperbolic almost Hermitian manifold), i.e.

$$G(JX, JY) = -G(X, Y); \quad X, Y \in \Gamma(TM). \tag{13}$$

On TM we consider the following tensor field F of type (0,3) defined by (see [2])

$$F(X, Y, Z) = G((\widetilde{\nabla}_X J)Y, Z); \quad X, Y, Z \in \Gamma(TM), \tag{14}$$

where $\widetilde{\nabla}$ denotes the Levi Civita connection on (TM, G).

Then the almost complex manifold with Norden metric (TM, J, G) is a Kaehlerian manifold with Norden metric (or an hyperbolic Kaehlerian manifold) if

$$F(X, Y, Z) = 0; \quad X, Y, Z \in \Gamma(TM),$$

or, equivalently, if $\widetilde{\nabla} J = 0$.

By using (12), (14) and Proposition 1 we get by a straightforward computation

$$\begin{aligned}
&F\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}, \tfrac{\partial}{\partial y^k}\right) = F\left(\tfrac{\partial}{\partial y^i}, \tfrac{\delta}{\delta x^j}, \tfrac{\delta}{\delta x^k}\right) = 0;\\
&F\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}, \tfrac{\delta}{\delta x^k}\right) = F\left(\tfrac{\partial}{\partial y^i}, \tfrac{\delta}{\delta x^k}, \tfrac{\partial}{\partial y^j}\right) = \tfrac{\partial b_{kj}}{\partial y^i};\\
&F\left(\tfrac{\delta}{\delta x^i}, \tfrac{\partial}{\partial y^j}, \tfrac{\partial}{\partial y^k}\right) = F\left(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^j}, \tfrac{\delta}{\delta x^k}\right) = \tfrac{\partial c_{ij}}{\partial y^k} + \tfrac{\partial c_{ik}}{\partial y^j};\\
&F\left(\tfrac{\delta}{\delta x^i}, \tfrac{\partial}{\partial y^j}, \tfrac{\delta}{\delta x^k}\right) = F\left(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^k}, \tfrac{\partial}{\partial y^j}\right) = \tfrac{1}{2}(R_{kij} + R_{jki} - R_{ijk}).
\end{aligned} \tag{15}$$

Examining the above relations it follows that the condition $F = 0$ which must be fulfilled for (TM, G) to be a Kaehlerian manifold with Norden metric is reduced to

$$(i)\frac{\partial b_{ij}}{\partial y^k} = 0; \quad (ii)\frac{\partial c_{ij}}{\partial y^k} + \frac{\partial c_{ik}}{\partial y^j} = 0; \quad (iii)R_{kij} + R_{jki} - R_{ijk} = 0. \tag{16}$$

From (16)(i) and (16)(ii) it follows that a_{ij} must be independent of y^k, i.e. the components a_{ij} define a tensor field of type (0,2) on the base manifold M thought of as an M–tensor field on TM. Next, using (2), (3)

and the second relation (6) we obtain by a straightforward computation that the condition (16)(iii) becomes

$$y^h R_{hijk} + \nabla_k c_{ij} - \nabla_j c_{ik} - \nabla_i b_{jk} = 0, \tag{17}$$

where R_{hijk} denotes the local coordinate components of the Riemann–Christoffel tensor of ∇ on M and c_{ij} (respectively b_{ij}) is the symmetric part (respectively the skew–symmetric part) of a_{ij}.

Hence we may state

Theorem 5. *The almost complex manifold with Norden metric (TM, J, G) is a Kaehlerian manifold with Norden metric if and only if the components a_{ij} are independent of y^k, the Levi Civita connection ∇ of g is flat and the symmetric part c_{ij} and the skew–symmetric part b_{ij} of a_{ij} satisfy the condition*

$$\nabla_k c_{ij} - \nabla_j c_{ik} = \nabla_i b_{jk}. \tag{18}$$

Remark. The condition (18) implies that the 2–form defined by the components b_{ij} is closed. Then the condition (18) is equivalent to $\nabla_i a_{jk} = \nabla_j a_{ik}$ and Theorem 5 may be formulated as follows: The almost complex manifold with Norden metric (TM, J, G) is Kaehlerian with Norden metric if and only if the components a_{ij} define a tensor field on the base manifold M, the 2–form defined by the components b_{ij} is closed and the horizontal distribution HTM defined by the nonlinear connection coefficients N^i_j on TM is involutive. Remark also that if c_{ij} define a Codazzi tensor field with respect to ∇ and b_{ij} is parallel with respect to ∇, then the condition (18) is identically satisfied.

3. An almost product structure on (TM, G). In this section we consider a nonlinear connection on TM defined by its connection coefficients of the form (2) and define and almost product structure P on TM determined by the distributions VTM and HTM. Next we obtain the conditions under which the almost parahermitian manifold (TM, G, P) is a parakaehlerian manifold (see [1]).

Consider on TM the almost product structure P natuarlly defined by the direct sum decomposition (1), i.e.

$$P\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^i}; \quad P\left(\frac{\delta}{\delta x^i}\right) = -\frac{\delta}{\delta x^i}. \tag{19}$$

It follows easily that $G(PX,PY)=-G(X,Y);\quad X,Y\in\Gamma(TM)$, therefore (TM,G,P) is an almost parahermitian manifold (see [1]). Define the 2–form Ω associated with the almost parahermitian structure (G,P) on TM by

$$\Omega(X,Y)=G(PX,Y);\quad X,Y\in\Gamma(TM). \tag{20}$$

According to the terminology from [1] we have that (TM,G,P) is a parakaehlerian manifold if $\widetilde{\nabla}\Omega$ vanishes identically on TM. Using (19), (20) and Proposition 1 we obtain by a straightforward computation

$$\begin{aligned}&(\widetilde{\nabla}_{\partial_i}\Omega)(\partial_j,\partial_k)=(\widetilde{\nabla}_{\partial_i}\omega)(\partial_j,\delta_k)=(\widetilde{\nabla}_{\delta_i}\Omega)(\partial_j,\partial_k)=(\widetilde{\nabla}_{\delta_i}\Omega)(\partial_j,\delta_k)=0\\ &(\widetilde{\nabla}_{\partial_i}\Omega)(\delta_j,\delta_k)=2\tfrac{\partial b_{kj}}{\partial y^i};\quad (\widetilde{\nabla}_{\delta_i}\Omega)(\delta_j,\delta_k)=R_{kij}+R_{jki}-R_{ijk}.\end{aligned} \tag{21}$$

Then the condition $\widetilde{\nabla}\Omega=0$ which must be fulfilled for (TM,G,P) to be a parakaehlerian manifold is reduced to

$$(i)\ \ \frac{\partial b_{kj}}{\partial y^i}=0;\quad (ii)\ \ R_{kij}+R_{jki}=R_{ijk}. \tag{22}$$

From (22)(i) it follows that the skew–symmetric part b_{ij} of a_{ij} does not depend on the tangential components y^k, i.e. the components a_{ij} are of the form

$$a_{ij}(x,y)=c_{ij}(x,y)+b_{ij}(x), \tag{23}$$

where $c_{ij}(x,y)$ are the components of a symmetric M–tensor field of type (0,2) on TM and $b_{ij}(x)$ are the components of a skew–symmetric tensor field of type (0,2) on M thouhgt of as a skew–symmetric M–tensor field on TM. Hence we have

Theorem 6. *The almost parahermitian manifold (TM,G,P) where G is defined by (7) and the nonlinear connection is defined by (2) with a_{ij} of the form (23) is a parakaehlerian manifold if and only if the components R_{kij} satisfy the condition (22)(ii).*

In order to obtain some classes of manifolds whose tangent bundles carry parakaehlerian structures, we consider on TM some particular M–tensor fields of the form (23) and we study the conditions under which (TM,G,P) is a parakaehlerian manifold.

Firstly, we consider on TM the M–tensor field defined by the components a_{ij}:

$$a_{ij} = c_{ij} + b_{ij}, \tag{24}$$

where c_{ij} (respectively b_{ij}) are the components of a symmetric (respectively skew–symmetric) tensor field on M thouhgt of as an M–tensor field on TM. Then, using (2), (6) and (24) we get by a straightforward computation

$$R_{kij} = R_{khij}y^h + \nabla_i a_{kj} - \nabla_j a_{ki},$$

where R_{khij} are the local coordinate components of the Riemann–Christoffel tensor field defined by ∇ on (M, g). Next, the condition (22)(ii) becomes

$$R_{khij}y^h + \nabla_i c_{kj} - \nabla_j c_{ki} + \nabla_k b_{ij} = 0, \tag{25}$$

hence we have

Theorem 7. *Let the nonlinear connection N^i_j on TM be defined by (2) where a_{ij} are given by (24). Then (TM, G, P) is a parakaehlerian manifold if and only if the (peudo–)Riemannian manifold (M, g) is flat and the components c_{ij} and b_{ij} satisfy the condition*

$$\nabla_i c_{kj} - \nabla_j c_{ki} = \nabla_k b_{ji}. \tag{26}$$

Remark. The condition (25) is equivalent to

$$R_{kij} + \sum_{(i,j,k)} \nabla_i b_{jk} = 0,$$

where $\sum_{(i,j,k)}$ denotes the sum consisting of three terms obtained by cyclic permutations of i, j, k. Thus, Theorem 7 becomes: Under the hypothesis of Theorem 7 we have that (TM, G, P) is a parakaehlerian manifold if and only if the horizontal distribution HTM on TM is involutive and the 2–form defined by the components b_{ij} is closed.

Now, we consider on TM the M–tensor field defined by the components a_{ij} of the form (23), with

$$a_{ij} = kg_{ih}g_{jl}y^h y^l + t_{ij}, \tag{27}$$

where k is a nonzero constant and the components t_{ij} define and arbitrary tensor field of type (0,2) on M thought of as an M–tensor field on TM. Using (2), (6) and (27) we obtain by a strainghtforward computation

$$R_{ijk} = [R_{iljk} + k(t_{ik}g_{jl} - t_{ij}g_{kl} + t_{jk}g_{il} - t_{kj}g_{il})]y^l + \nabla_j t_{ik} - \nabla_k t_{ij}. \tag{28}$$

By using (28) it follows that the condition (22)(ii) becomes

$$\begin{aligned}[R_{iljk} + k(t_{ik}g_{jl} - t_{ij}g_{kl} + t_{jk}g_{il} - t_{kj}g_{il})]y^l + \\ +\nabla_j h_{ik} - \nabla_k h_{ij} + \nabla_i b_{jk} = 0,\end{aligned} \tag{29}$$

where h_{ij} (respectively b_{ij}) denotes the symmetric part (respectively the skew–symmetric part) of t_{ij}. The condition (29) is equivalent to

$$\begin{aligned}&(i) \quad R^h_{ijk} = k(t_{ik}\delta^h_j - t_{ij}\delta^h_k + t_{jk}\delta^h_i - t_{kj}\delta^h_i);\\ &(ii) \quad \nabla_j h_{ik} - \nabla_k h_{ij} = \nabla_i b_{kj},\end{aligned} \tag{30}$$

where R^h_{ijk} are the local coordinate components of the curvature tensor field of the Levi Civita connection ∇ an M.

By taking into account that ∇ is the Levi Civita connection of g, we have from (30)(i)

$$t_{ij} = \frac{1}{k(n-1)}R_{ij}, \tag{31}$$

where R_{ij} are the local coordinate components of the Ricci tensor defined by the curvature tensor of ∇, i.e. we have $t_{ij} = h_{ij}$ and $b_{ij} = 0$. Thus, the condition (30)(i) implies that the (pseudo–)Riemannian manifold (M, g) has constant sectional curvature, next replacing the expression of t_{ij} from (31) in (30)(i) and using the second Bianchi identity we get that the condition (30)(ii) is identically verified.

Hence we may state

Theorem 8. *Let (M, g) be a (pseudo–)Riemannian manifold with* $\dim M = n > 2$. *Consider on TM the nonlinear connection defined by*

$$N_i^k = \Gamma_{ij}^k y^j + k g_{ih} y^h y^k + \frac{1}{k(n-1)} g^{kh} R_{hi},$$

where Γ_{ij}^k are the connection coefficients of the Levi Civita connection ∇ of g, R_{ij} are the local coordinate components of the Ricci tensor defined by the curvature tensor of ∇ and k is a nonzero constant. The the almost parahermitian manifold (TM, G, P) where G is given by (7) and P is defined by (19) is a parakaehlerian manifold if and only if the (pseudo–)Riemannian manifold (M, g) has constant sectional curvature.

More parakaehlerian structures on tangent bundles can be obtained in the cases of complex and quaternion manifolds.

Let (M, g, F) be a *Kaehler manifold* with the almost complex structure defined by the tensor field F of type (1,1) such that $F^2 = -I$ and denote by ∇ the Levi Civita connection of g such that $\nabla F = 0$ (see [10]). Moreover, we have $g(FX, FY) = g(X, Y)$; $X, Y \in \Gamma(M)$. Consider on TM the M–tensor field defined by the components a_{ij} of the form (23):

$$a_{ij} = k(g_{ih} g_{jk} - g_{is} F_h^s g_{jt} F_k^t) y^h y^k + t_{ij}, \tag{32}$$

where F_i^h are the components of F, k is a nonzero constant and t_{ij} are the components of an arbitrary tensor field of type (0,2) on M thought of as an M–tensor field on TM. In this case, using (2), (6) and (32) we obtain by a straightforward computation

$$\begin{aligned} R_{ijk} = \{ R_{iljk} + k \big[& t_{ik} g_{jl} - t_{ij} g_{kl} + t_{jk} g_{il} - t_{kj} g_{il} + t_{hj} F_k^h F_i^s g_{sl} - \\ & - t_{hk} F_j^h F_i^s g_{sl} + t_{hj} F_i^h F_k^s g_{sl} - t_{hk} F_i^h F_j^s g_{sl} \big] \} y^l + \nabla_j t_{ik} - \nabla_k t_{ij}, \end{aligned}$$

where R_{iljk} are the local coordinate components of the Riemann–Christoffel tensor on (M, g). Using the above expression of R_{ijk} we obtain by a straightforward computation that the condition (22)(ii) which must be fulfilled for (TM, G, P) to be a parakaehlerian manifold is equivalent to the following two conditions

$$\begin{aligned} & (i) \quad R_{kij}^h = k(t_{kj} \delta_i^h - t_{ki} \delta_j^h - t_{ji} \delta_k^h + t_{ij} \delta_k^h + t_{li} F_j^l F_k^h - \\ & \qquad - t_{lj} F_i^l F_k^h + t_{li} F_k^l F_j^h - t_{lj} F_k^l F_i^h); \\ & (ii) \quad \nabla_i h_{jk} - \nabla_j h_{ik} + \nabla_k b_{ij} = 0, \end{aligned} \tag{33}$$

where R^h_{kij} are the local coordinate components of the curvature tensor field of ∇ on M and h_{ij} (respectively b_{ij}) denotes the symmetric part (respectively the skew–symmetric part) of t_{ij}.

From (33)(i) it follows

$$t_{ij} = \frac{1}{k(n^2-4)}[nR_{ij} - 2F^h_i F^k_j R_{hk}]. \tag{34}$$

Replacing this expression of t_{ij} in (33)(i), next using the second Bianchi identity we get that the condition (33)(ii) is identically verified.

Hence we may state

Theorem 9. *Let (M, g, F) be a Kaehler manifold with real dimension $n > 2$ and consider on TM the nonlinear connection defined by*

$$N^k_i = \Gamma^k_{ij}y^j + k(g_{ih}y^k y^h - F^k_l g_{is}F^s_h y^l y^h) + \frac{1}{k(n^2-4)}g^{kh}(nR_{hi} - 2F^l_h F^s_i R_{ls}),$$

where Γ^k_{ij} are the connection coefficients of the Levi Civita connection ∇ of g, R_{ij} are the components of the Ricci tensor defined by the curvature tensor of ∇, F^j_i are the components of F and k is a nonzero constant. Then the almost parahermitian manifold (TM, G, P) where G is defined by (7) and P is defined by (19) is a parakaehlerian manifold if and only if (M, g, F) has constant holomorphic sectional curvature.

Consider now (M, g, S) a *quaternion Kaehler manifold* . Then M is a $4m$–dimensional manifold , S is a subbundle with fibre dimension 3 of the vector bundle of tensors of type (1,1) on M and, locally, S has a canonical base (F_1, F_2, F_3) such that

$$F^2_\alpha = -I; \quad F_\alpha \circ F_\beta = -F_\beta \circ F_\alpha = F_\gamma,$$

where $\alpha = 1, 2, 3$ and (α, β, γ) is any cyclic permutation of (1,2,3). Moreover, we have

$$g(F_\alpha X, F_\alpha Y) = g(X, Y); \quad X, Y \in \Gamma(TM), \quad \alpha = 1, 2, 3.$$

Denote by ∇ the Levi Civita connection of g which preserves S, i.e., locally

$$\nabla F_\alpha = \sum_{\beta=1,2,3} \eta_{\alpha\beta} \oplus F_\beta,$$

where $\eta_{\alpha\beta}$ are locally defined 1– forms adapted with the Levi Civita connection ∇ of g (see [4]) and $\eta_{\alpha\beta} = -\eta_{\beta\alpha}$.

Consider on TM the M–tensor field defined by the components a_{ij} of the form (23):

$$a_{ij} = k\{g_{ih}g_{jk}y^h y^k - \sum_{\alpha=1,2,3} g_{is}(F_\alpha)^s_h g_{jt}(F_\alpha)^t_k y^h y^k\} + t_{ij}, \tag{35}$$

where $(F_\alpha)^k_i$ are the local coordinate components of F_α, k is a nonzero constant and t_{ij} are the components of an arbitrary tensor field of type (0,2) on M thought of as an M–tensor field on TM. Then using (2), (6) and (35) we obtain by a straightforward computation

$$\begin{aligned} R_{ijk} &= \{R_{iljk} + k[t_{ik}g_{jl} - t_{ij}g_{kl} + t_{jk}g_{il} - t_{kj}g_{il} + \\ &+ \sum_{\alpha=1,2,3}(t_{hj}(F_\alpha)^h_k(F_\alpha)^s_i g_{sl} - t_{hk}(F_\alpha)^h_j(F_\alpha)^s_i g_{sl} + \\ &+ t_{hj}(F_\alpha)^h_i(F_\alpha)^s_k g_{sl} - t_{hk}(F_\alpha)^h_i(F_\alpha)^s_j g_{sl})]\}y^l + \nabla_j t_{ik} - \nabla_k t_{ij}. \end{aligned} \tag{36}$$

By using (36) it follows by a straightforward computation that the condition (22)(ii) is equivalent to the following two relations

$$\begin{aligned} &(i) \quad R^h_{kij} = k\{t_{kj}\delta^h_i - t_{ki}\delta^h_j - t_{ji}\delta^h_k + t_{ij}\delta^h_k + \sum_{\alpha=1,2,3}[-(F_\alpha)^l_i(F_\alpha)^h_k t_{lj} + \\ &\qquad + (F_\alpha)^l_j(F_\alpha)^h_k t_{li} + (F_\alpha)^l_k(F_\alpha)^h_j t_{li} - (F_\alpha)^l_k(F_\alpha)^h_i t_{lj}]\}, \\ &(ii) \quad \nabla_i h_{jk} - \nabla_j h_{ik} = \nabla_k b_{ji}, \end{aligned} \tag{37}$$

where h_{ij} (respectively b_{ij}), denotes the symmetric part (respectively the skew–symmetric part) of t_{ij}. From the condition (37)(i) we obtain by a strainghtforward computation

$$t_{ij} = \frac{1}{8km(m+2)}[(2m+3)R_{ij} - \sum_{\alpha=1,2,3}(F_\alpha)^h_i(F_\alpha)^k_j R_{hk}]. \tag{38}$$

Replacing this expression of t_{ij} in (37)(i), next using the second Bianchi identity we get that the condition (37)(ii) is identically verified.

Hence we may state

Theorem 10. *Let (M, g, S) be a quaterninon Kaehler manifold with* dimM > 4. *Consider on TM the nonlinear conection defined by N_i^k, where*

$$N_i^k = \Gamma_{ij}^k y^j + k[g_{ih}y^h y^k - \sum_{\alpha=1,2,3}(F_\alpha)_l^k g_{is}(F_\alpha)_h^s y^l y^h] +$$
$$+\frac{1}{8km(m+2)} g^{kh}[(2m+3)R_{hi} - \sum_{\alpha=1,2,3}(F_\alpha)_h^s(F_\alpha)_i^t R_{st}],$$

where Γ_{ij}^k are the coefficients of the Levi Civita connection ∇ on (M, g, S), k is a nonzero constant and R_{ij} are the components of the Ricci tensor defined by the curvature tensor of ∇. Then the almost parahermitian manifold (TM, G, P) where G is given by (7) and P is defined by (19) is a parakaehlerian manifold if and only if the quaternion Kaehler manifold (M, g, S) has constant Q–sectional curvature (see [4]).

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Faculty of Mathematics
University "Al.I.Cuza", Iaşi
ROMANIA
e–mail: voproiu@uaic.ro
Departament of Mathematics
Technical University, Iaşi
ROMANIA
e-mail: npapag@math.tuiasi.ro