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A NOTE ON SOME EXTENSIONS OF HARDY'S INEQUALITY

BY

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Abstract. In the present note we establish some new extensions of the well known Hardy's integral inequality which claim their origin to a certain variant of Hardy's inequality given by Izumi and Izumi.

1. Introduction. The classical integral inequality due to Hardy (see,[4],[5,p.242]) states that for $f(x) \geq 0$ and $p > 1$

$$\int_0^\infty \left\{ \frac{1}{x} F(x) \right\}^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \tag{1}$$

where $F(x) = \int_0^x f(t)dt$. In the course of attempts of proving an inequality sharper than the inequality established by A s k e y and W a i n g e r [1], in 1968, I z u m i and I z u m i [6, Theorem 2] proved the following interesting variant of the inequality (1).

Let $p > 1$ and $s < -1$ and let f be a nonnegative and integrable on $(0, \pi)$. If $x^s f^p(x)$ is integrable, then

$$\int_0^\pi x^s G^p(x) dx \leq \left(\frac{p}{-s-1} \right)^p \int_0^\pi x^s \left| f\left(\frac{x}{2}\right) - f(x) \right|^p dx, \tag{2}$$

where $G(x) = \int_{\frac{x}{2}}^x \frac{1}{t} f(t)dt$.

A number of papers which deals with the various generalizations and extensions of inequalities (1) and (2) have appeared in the literature, see [1-10] and the references therein. The main purpose of this note is to establish some new extensions of the inequality (2) by using a fairly elementary analysis.

2. Main results. In what follows, we assume that all the integrals exist on the respective domains of their definitions.

Our main result is established in the following theorem.

Theorem 1. *Let $p > 1, m > 1$ be constants. Let f be a nonnegative and integrable function on $(0, a), 0 < a < \infty$. If $F(x)$ is defined by*

$$F(x) = \int_{\frac{x}{2}}^{x} \frac{1}{t} \left\{ \int_{\frac{t}{2}}^{t} \frac{f(s)}{s} ds \right\} dt, \quad x \in (0, a), \tag{3}$$

then

$$\int_0^a x^{-m} F^p(x) dx \leq \left(\frac{p}{m-1} \right)^{2p} \int_0^a x^{-m} |f(x) - f\left(\frac{x}{4}\right)|^p dx. \tag{4}$$

Proof. Integrating the left side in (4) by parts we have

$$\int_0^a x^{-m} F^p(x) dx = -\frac{a^{-m+1}}{(m-1)} F^p(a) + \tag{5}$$

$$+ \left(\frac{p}{m-1} \right) \int_0^a x^{-m+1} F^{p-1}(x) \left[\frac{1}{x} \int_{\frac{x}{2}}^{x} \frac{f(s)}{s} ds - \frac{1}{2} \frac{1}{\frac{x}{2}} \int_{\frac{x}{4}}^{\frac{x}{2}} \frac{f(s)}{s} ds \right] dx.$$

From (5) and using the Hölder's inequality with indices $\frac{p}{p-1}, p$ we observe that

$$\int_0^a x^{-m}F^p(x)dx \le \left(\frac{p}{m-1}\right)\int_0^a x^{-m}F^{p-1}(x)\Big|\int_{\frac{x}{2}}^{x}\frac{f(s)}{s}ds-$$

$$-\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{f(s)}{s}ds\Big|dx \le$$

$$\le \left(\frac{p}{m-1}\right)\int_0^a x^{-m}F^{p-1}(x)\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}dx =$$

$$= \left(\frac{p}{m-1}\right)\int_0^a \left[\left\{x^{-m}\right\}^{\frac{p-1}{p}}F^{p-1}(x)\right]\cdot$$

$$\cdot\left[\left\{x^{-m}\right\}^{-\left(\frac{p-1}{p}\right)+1}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}\right]dx \le$$

$$\le \left(\frac{p}{m-1}\right)\left[\int_0^a x^{-m}F^p(x)dx\right]^{\frac{p-1}{p}}\cdot$$

$$\cdot\left[\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right]^{\frac{1}{p}}.$$

Dividing both sides of (6) by the first integral factor on the right side in (6) and taking the pth power of both sides we get

$$\int_0^a x^{-m}F^p(x)dx \le \left(\frac{p}{m-1}\right)^p\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx. \tag{7}$$

Now, integrating by parts the integral on the right side in (7) we have

$$\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx = -\frac{a^{-m+1}}{m-1}\left\{\int_{\frac{a}{4}}^{a}\frac{|f(s)|}{s}\right\}ds+ \tag{8}$$

$$+\left(\frac{p}{m-1}\right)\int_0^a x^{-m+1}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^{p-1}\cdot$$

$$\cdot\left[\frac{1}{x}|f(x)|-\frac{1}{4}\frac{1}{\frac{x}{4}}|f\left(\frac{x}{4}\right)|\right]dx.$$

From (8) and using the Hölder's inequality with indices $\frac{p}{p-1}, p$ we observe that

$$\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx \leq \tag{9}$$

$$\leq\left(\frac{p}{m-1}\right)\int_0^a\left[\left\{x^{-m}\right\}^{\frac{p-1}{p}}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^{p-1}\right]\cdot$$

$$\cdot\left[\left\{x^{-m}\right\}^{-\left(\frac{p-1}{p}\right)+1}|f(x)-f\left(\frac{x}{4}\right)|\right]dx\leq$$

$$\leq\left(\frac{p}{m-1}\right)\left[\int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right\}^p dx\right]^{\frac{p-1}{p}}\cdot$$

$$\cdot\left[\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right]^{\frac{1}{p}}.$$

Dividing both sides of (9) by the first integral factor on the right side in (9) and taking the pth power of both sides we get

$$(10)\quad \int_0^a x^{-m}\left\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\right\}^p dx \le \left(\frac{p}{m-1}\right)^p \int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx.$$

Using (10) in (7) we get the desired inequality in (4). This completes the proof of the theorem.

We next establish the following slightly different versions of Theorem 1 which we believe are of independent interest.

Theorem 2. *Let p, m and f be as defined in Theorem 1. If $F(x)$ is defined by*

$$(11)\qquad F(x) = \int_0^x \frac{1}{t}\left\{\int_{\frac{t}{2}}^t \frac{f(s)}{s}ds\right\}dt, \quad x \in (0,a),$$

then

$$(12)\qquad \int_0^a x^{-m}F^p(x)dx \le \left(\frac{p}{m-1}\right)^{2p} \int_0^a x^{-m}|f(x)-f\left(\frac{x}{2}\right)|^p dx.$$

Theorem 3. *Let p, m and f be as defined in Theorem 1. If $F(x)$ is defined by*

$$(13)\qquad F(x) = \int_{\frac{x}{2}}^x \frac{1}{t}\left\{\int_0^t \frac{f(s)}{s}ds\right\}dt, \quad x \in (0,a),$$

then

$$(14)\qquad \int_0^a x^{-m}F^p(x)dx \le \left(\frac{p}{m-1}\right)^{2p} \int_0^a x^{-m}|f(x)-f\left(\frac{x}{2}\right)|^p dx.$$

The proofs of Theorems 2 and 3 follow by the similar arguments as in the proof of Theorem 1 with suitable modifications and hence we omit the details.

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