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# EXISTENCE RESULTS FOR CERTAIN HIGHER ORDER INTEGRODIFFERENTIAL EQUATIONS

BY

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**1. Introduction.** In the present paper we consider the nonlinear integrodifferential equations of the forms:

$$(r(t)x^{(n-1)}(t))' = F(t, x(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times \times \int_0^{s_{n-1}} f(s_n, x(s_n)) ds_n ds_{n-1} \cdots ds_1), \tag{1.1}$$

with the given initial conditions

$$x(0) = x_0, \quad x^{(i-1)}(0) = 0, \quad i = 2, 3, \ldots, n, \tag{1.2}$$

and

$$(r(t)x'(t))^{(n-1)} = F(t, x(t), \int_0^t \frac{1}{r(s_1)} \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n-1}} \times \times f(s_n, x(s_n)) ds_n ds_{n-1} \cdots ds_1), \tag{1.3}$$

with the given initial conditions

$$x(0) = x_0, \quad (r(0)x'(0))^{(i-2)} = 0, \quad i = 2, 3, \ldots, n, \tag{1.4}$$

and

(1.5)
$$(r(t)x^{(n)}(t))^{(n)} = F(t, x(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} \cdots \times$$
$$\times \int_0^{\sigma_{n-1}} f(\sigma_n, x(\sigma_n)) d\sigma_n d\sigma_{n-1} \cdots d\sigma_1 \times$$
$$\times ds_n ds_{n-1} \cdots ds_1),$$

with the given initial conditions

(1.6)
$$x(0) = x_0, \quad x^{(i-1)}(0) = 0, \quad i = 2, 3, \ldots, n,$$
$$(r(0)x^{(n)}(0))^{(i-1)} = 0, \quad i = 1, 2, \ldots, n,$$

where $r(t)$ is a real–valued positive and sufficiently smooth fonction defined for $t \in [0, T]$, $f: [0, T] \times R \longrightarrow R$, $F: [0, T] \times R \times R \longrightarrow R$ are continuous functions and $x_0$ is a given real constant and $R$ denotes the set of real numbers.

In the past few years, many authors have studied the oscillatory and asymptotic behavior of the solutions of various special versions of the above equations with or without delay arguments, see [4, 6, 13] and the references cited therein. However, it seems to us that, global existence results for the general equations of the above forms are not widely discussed in the literature. The main purpose of this paper is to study the global existence of solutions of the above equations by using a simple and classical application of the topological transversality theorem of G r a n a s [3, p. 61], known as Leray – Schauder alternative. In fact, our result in this paper are motivated by the interesting results obtained on the various special versions of such equations in [1, 2, 5, 7–14] with or without delay arguments and the given initial or boundary conditions.

**2. Statement of results.** Our existence theorems are based on the application of the following theorem which is a version of the topological transversality theorem given by G r a n a s in [3, p. 61].

**Theorem G.** *Let $B$ be a Banach space of a normed linear space $E$ and assume $0 \in B$. Let $L: B \longrightarrow B$ be a completely continuous operator and let*

$$U(L) = \{x \in B: x = \lambda Lx \quad for\ some \quad 0 < \lambda < 1\}.$$

*Then either $U(L)$ is bounded or $L$ has a fixed point.*

Our main results are given in the following theorems.

**Theorem 1.** *Assume that:*
(*H*) *There exist continuous functions* $p, q\colon [0,T] \longrightarrow R_+ = [0,\infty)$ *such that*

$$|F(t,x,z)| \le p(t)(|x| + |z|),$$
$$|f(t,x)| \le q(t)g(|x|),$$

*for every* $t \in [0,T]$ *and* $x, z \in R$, *where* $g\colon R_+ \longrightarrow (0,\infty)$ *is a continuously differentiable function such that* $g'(u) \ge 0$ *for* $u \ge 0$.

*Then the problem (1.1)–(1.2) has a solution* $x$ *defined on* $[0,T]$ *provided* $T$ *satisfies*

$$\int_0^T M_1(s_1)ds_1 < \int_{|x_0|}^{\infty} \frac{ds_1}{s_1 + g(s_1)}, \tag{2.1}$$

*where*

$$M_1(t) = \int_0^t \int_0^{s_2} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} [p(s_n) + q(s_n)] \times \times ds_n ds_{n-1} \cdots ds_2, \tag{2.2}$$

*for* $t \in [0,T]$.

**Theorem 2.** *Assume that the hypothesis* (*H*) *in Theorem 1 holds. Then the problem (1.3)–(1.4) has a solution* $x$ *defined on* $[0,T]$ *provided* $T$ *satisfies*

$$\int_0^T M_2(s_1)ds_1 < \int_{|x_0|}^{\infty} \frac{ds_1}{s_1 + g(s_1)}, \tag{2.3}$$

*where*

$$M_2(t) = \frac{1}{r(t)} \int_0^t \int_0^{s_2} \cdots \int_0^{s_{n-1}} [p(s_n) + q(s_n)] ds_n ds_{n-1} \cdots ds_2, \tag{2.4}$$

*for* $t \in [0,T]$.

**Theorem 3.** *Assume that the hypothesis* (*H*) *in Theorem 1 holds. Then the problem (1.5)–(1.6) has a solution* $x$ *defined on* $[0,T]$ *provided* $T$ *satisfies*

$$\int_0^T M_3(s_1)ds_1 < \int_{|x_0|}^{\infty} \frac{ds_1}{s_1 + g(s_1)}, \tag{2.5}$$

*where*

$$M_3(t) = \int_0^t \int_0^{s_2} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{\sigma_1} [p(\sigma_n) + q(\sigma_n)] \times \times d\sigma_n d\sigma_{n-1} \cdots d\sigma_1 ds_n ds_{n-1} \cdots ds_2, \tag{2.6}$$

*for* $t \in [0, T]$.

**3. Proofs of theorems 1–3.** Since the proofs of Theorems 1–3 resemble one another, we give the details of Theorem 1 only, the proofs of Theorem 2 and 3 can be completed by following the proof of Theorem 1 with suitable modifications.

To prove the existence of a solution of (1.1)–(1.2) we apply Theorem G. First we establish the priori bounds for the initial value problem $(1.1)_\lambda$–(1.2), $\lambda \in (0, 1)$, where

$$(r(t)x^{(n-1)}(t))' = \lambda F(t, x(t), \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times \times \int_0^{s_{n-1}} f(s_n, x(s_n)) ds_n ds_{n-1} \cdots ds_1). \tag{$1.1)_\lambda$}$$

Let $x(t)$ be a solution of $(1.1)_\lambda$ – (1.2), then it satisfies the equivalent integral equation

$$x(t) = x_0 + \lambda \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} F(t_n, x(t_n), \int_0^{t_n} \times \times \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, x(s_n)) \times \times ds_n ds_{n-1} \cdots ds_1) dt_n dt_{n-1} \cdots dt_1. \tag{3.1}$$

From (3.1) and using the hypothesis $(H)$ we have

$$|x(t)| \leq |x_0| + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[|x(t_n)| + + \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_n} q(s_n) g(|x(s_n)|) \times \times ds_n ds_{n-1} \cdots ds_1] dt_n dt_{n-1} \cdots dt_1. \tag{3.2}$$

Define a function $z(t)$ by the right side of (3.2), then it is easy to observe that

$$(r(t)z^{(n-1)}(t))' = p(t)[|x(t)| + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times \times \int_0^{s_{n-1}} q(s_n)g(|x(s_n)|)ds_n ds_{n-1} \cdots ds_1]. \tag{3.3}$$

We denote that, since $g$ is differentiable on $(0, +\infty)$ and $g'(u) \geq 0$ on $\boldsymbol{R}_+$, the function $g$ is monofonically increasing on $(0, +\infty)$. Using the fact that $|x(t)| \leq z(t)$ in (3.3) we have

$$(r(t)z^{(n-1)}(t))' \leq p(t)[z(t) + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \times \times \int_0^{s_{n-1}} q(s_n)g(z(s_n))ds_n ds_{n-1} \cdots ds_1]. \tag{3.4}$$

Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q(s_n)g(z(s_n)) \times \times ds_n ds_{n-1} \cdots ds_1. \tag{3.5}$$

From (3.5) and using the facts that $(r(t)z^{(n-1)}(t))' \leq p(t)v(t)$ and $z(t) \leq \leq v(t)$, we observe that

$$\begin{aligned} (r(t)v^{(n-1)}(t))' &= (r(t)z^{(n-1)}(t))' + q(t)g(z(t)) \\ &\leq p(t)v(t) + q(t)g(v(t)) \\ &\leq [p(t) + q(t)](v(t) + g(v(t))). \end{aligned} \tag{3.6}$$

From (3.6) and using the facts that $v(t) \geq 0$, $(v(t) + g(v(t))) > 0$, $(v(t) + g(v(t)))' \geq 0$, $(r(t)v^{(n-1)}(t)) \geq 0$ for $t \in [0, T]$, we observe that

$$\frac{(r(t)v^{(n-1)}(t))'}{v(t) + g(v(t))} \leq [p(t) + q(t)] + \frac{(v(t) + g(v(t)))'(r(t)v^{(n-1)}(t))}{(v(t) + g(v(t)))^2}$$

i.e.

$$\left(\frac{r(t)v^{(n-1)}(t)}{v(t) + g(v(t))}\right)' \leq [p(t) + q(t)]. \tag{3.7}$$

By setting $t = s_n$ in (3.7) and integrating with respect to $s_n$ from 0 to $t,\ t \in [0,T]$ and using the fact that $r(0)v^{(n-1)}(0) = 0$, we obtain

$$\frac{r(t)v^{(n-1)}(t)}{v(t)+g(v(t))} \le \int_0^t [p(s_n)+q(s_n)]ds_n,$$

which implies

$$\frac{v^{(n-1)}(t)}{v(t)+g(v(t))} \le \frac{1}{r(t)}\int_0^t [p(s_n)+q(s_n)]ds_n. \tag{3.8}$$

Again, as above, from (3.8) we observe that

$$\left(\frac{v^{(n-2)}(t)}{v(t)+g(v(t))}\right)' \le \frac{1}{r(t)}\int_0^t [p(s_n)+q(s_n)]ds_n. \tag{3.9}$$

By setting $t = s_{n-1}$ in (3.9) and integrating with respect to $s_{n-1}$ from 0 to $t,\ t \in [0,T]$ and using the fact that $v^{(n-2)}(0) = 0$, we get

$$\frac{v^{(n-2)}(t)}{v(t)+g(v(t))} \le \int_0^t \frac{1}{r(s_{n-1})}\int_0^{s_{n-1}} [p(s_n)+q(s_n)]ds_n ds_{n-1}.$$

Continuing in this way, we obtain

$$\frac{v'(t)}{v(t)+g(v(t))} \le M_1(t), \tag{3.10}$$

where $M_1(t)$ is defined by (2.2). By setting $t = s_1$ in (3.10) and integrating with respect to $s_1$ from 0 to $t,\ t \in [0,T]$ and using (2.1) we have

$$\begin{aligned}\int_{|x_0|}^{v(t)} \frac{ds_1}{s_1+g(s_1)} &\le \int_0^t M_1(s_1)ds_1 \\ &\le \int_0^T M_1(s_1)ds_1 < \int_{|x_0|}^{\infty} \frac{ds_1}{s_1+g(s_1)}.\end{aligned} \tag{3.11}$$

From (3.11) we conclude that there is a constant $Q$ independent of $\lambda \in (0,1)$ such that $v(t) \le Q$ and hence $z(t) \le Q$ for $t \in [0,T]$. Thus we have $|x(t)| \le Q$, for $t \in [0,T]$, and consequently $|x_0| = \sup\{|x(t)|: t \in [0,T]\} \le Q$.

We define $B = C([0,T],R)$ to be the Banach space of all continuous functions from $[0,T]$ into $R$ endowed with the sup–norm $|x|_0 =$

$= \sup\{|x(t)|: t \in [0,T]\}$. In the second step, we rewrite the initial value problem (1.1)–(1.2) as follows. If $y \in B$ and $x(t) = y(t) + x_0,\ t \in [0,T]$, it is easy to see that $y$ satisfies

$$\begin{aligned} y(0) &= y_0 = 0, \\ y(t) &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} F(t_n, y(t_n) + x_0, \int_0^{t_n} \times \\ &\quad \times \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n) + x_0) \times \\ &\qquad \times ds_n ds_{n-1} \cdots ds_1) dt_n dt_{n-1} \cdots dt_1, \end{aligned}$$

if and only if $x$ satisfies (1.1)–(1.2).

Define $L: B_0 \longrightarrow B_0,\ B_0 = \{y \in B: y_0 = 0\}$ by

$$\begin{aligned} Ly(t) = &\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} F(t_n, y(t_n) + x_0, \int_0^{t_n} \times \\ &\times \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, y(s_n) + x_0) \times \\ &\qquad \times ds_n ds_{n-1} \cdots ds_1) dt_n dt_{n-1} \cdots dt_1, \end{aligned} \tag{3.12}$$

for $t \in [0,T]$. Then $L$ is clearly continuous. Now we shall prove that $L$ is completely continuous.

Let $\{w_k\}$ be a bounded sequence in $B_0$, i.e. $|w_k|_0 \le b$, for all $k$, where $b$ is a positive constant. From (3.12) and using the hypothesis $(H)$ and letting $M_1^* = \sup\{M_1(t): t \in [0,T]\}$, we have

$$\begin{aligned} |Lw_k(t)| \le &\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[|w_k(t_n)| + |x_0| + \\ &+ \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q(s_n) \times \\ &\times g(|w_k(s_n)| + |x_0|) ds_n ds_{n-1} \cdots ds_1] \times \\ &\times dt_n dt_{n-1} \cdots dt_1 \end{aligned} \tag{3.13}$$

$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[b + |x_0| +$$
$$+ \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q(s_n) g(b + |x_0|) \times$$
$$\times ds_n ds_{n-1} \cdots ds_1] dt_n dt_{n-1} \cdots dt_1$$
$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[b + |x_0| +$$
$$+ g(b + |x_0|) \int_0^{t_n} M_1(s_1) ds_1] dt_n dt_{n-1} \cdots dt_1$$
$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[b + |x_0| +$$
$$+ g(b + |x_0|) M_1^* T] dt_n dt_{n-1} \cdots dt_1$$
$$\leq [b + |x_0| + g(b + |x_0|) M_1^* T] \int_0^t M_1(t_1) dt_1$$
$$\leq [b + |x_0| + g(b + |x_0|) M_1^* T] M_1^* T.$$

Hence from (3.13) we have

$$|Lw_k|_0 \leq [b + |x_0| + g(b + |x_0|) M_1^* T] M_1^* T.$$

This means that $\{Lw_k\}$ is uniformly bounded.

Now we shall show that the sequence $\{Lw_k\}$ is equicontinuous. Let $0 \leq \sigma_1 \leq \sigma_2 \leq T$. Then

$$(3.14) \quad |Lw_k(\sigma_2) - Lw_k(\sigma_1)|$$
$$\leq \int_{\sigma_1}^{\sigma_2} \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} |F(t_n, w_k(t_n) + x_0,$$
$$\int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} f(s_n, w_k(s_n) + x_0) \times$$
$$\times ds_n ds_{n-1} \cdots ds_1)| dt_n dt_{n-1} \cdots dt_1$$

$$\leq \int_{\sigma_1}^{\sigma_2} \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[|w_k(t_n)| + |x_0| +$$

$$+ \int_0^{t_n} \int_0^{s_1} \cdots \int_0^{s_{n-2}} \frac{1}{r(s_{n-1})} \int_0^{s_{n-1}} q(s_n) g(|w_k(s_n)| + |x_0|) \times$$

$$\times ds_n ds_{n-1} \cdots ds_1] dt_n dt_{n-1} \cdots dt_1$$

$$\leq \int_{\sigma_1}^{\sigma_2} \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[b + |x_0| +$$

$$+ g(b + |x_0|) \int_0^{t_n} M_1(s_1) ds_1] dt_n dt_{n-1} \cdots dt_1$$

$$\leq \int_{\sigma_1}^{\sigma_2} \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)[b + |x_0| +$$

$$+ g(b + |x_0|) M_1^* T] dt_n dt_{n-1} \cdots dt_1$$

$$\leq \int_{\sigma_1}^{\sigma_2} [b + |x_0| + g(b + |x_0|) M_1^* T] M_1(t_1) dt_1$$

$$\leq \int_{\sigma_1}^{\sigma_2} [b + |x_0| + g(b + |x_0|) M_1^* T] M_1^* dt_1.$$

From (3.14) we conclude that $\{Lw_k\}$ is equicontinuous and hence by Arzela – Ascoli theorem the operator $L$ is completely continuous.

Moreover, the set $U(L) = \{y \in B_0 : y = \lambda Ly,\ \lambda \in (0,1)\}$ is bounded, since for every $y$ in $U(L)$ the function $x = y + x_0$ is a solution of $(1.1)_\lambda$–(1.2) for which we have proved $|x|_0 \leq Q$ and hence $|y|_0 \leq Q + |x_0|$. Now an application of Theorem G, the operator $L$ has a fixed point in $B_0$. This means that the problem (1.1)–(1.2) has a solution. This completes the proof of Theorem 1.

In concluding this paper, we note that the results obtained in Theorems 1–3 can be extended very easily to the equations of the forms (1.1)–(1.2), (1.3)–(1.4) and (1.5)–(1.6) when the functions $f$ and $F$ depends on the delay arguments, under appropriate initial conditions. For similar results for first order differential delay equations, see [8, 10, 11].

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