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A SET–VALUED INTEGRAL

BY

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Introduction. The purpose of this paper is to define an integral for multifunctions with values in the family of all nonempty compact convex subsets of a real Banach algebra with respect to a multimeasure. The multimeasure is here in the sense of [2] (i.e. it is a multifunction φ defined on a measurable space $(S, \mathcal{A})$ and having values in the family of all nonempty compact convex subsets of a real Banach algebra such that $\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2)$, whenever $E_1 \cap E_2 = \emptyset$).

First, the integral of simple multifunctions is defined and some properties of the integral are presented. Then we define φ–integrable multifunctions F by means of a sequence of simple multifunctions which is said to be a defining sequence for F. In this case we define the integral of F as the limit of integrals of a defining sequence for F and prove some properties of the integral. Finally, we get a convergence theorem of Vitaly type for this kind of integral.

Note also that our integral includes both the Brooks integral [2] and the integral described in [9]. Exactly, if the multifunctions to be integrated are single valued then our integral becomes the Brooks integral and if the multimeasure is single valued then our integral becomes the Martellotti–Sambucini integral [9].

Terminology and Notations. Let S be a nonempty set, let $\mathcal{A}$ be an algebra of subsets of S, let $(X, \|\cdot\|)$ be a real commutative Banach algebra. $\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ is the family of all nonempty compact convex subsets of X and D is the usual Hausdorff metric defined on $\mathcal{P}_{kc}$. We define $|\!|A|\!| = D(A, 0),\ A \in \mathcal{P}_{kc}$, where 0 is used in place of $\{0\} \in \mathcal{P}_{kc}$. If $A, B \in \mathcal{P}_{kc}, \lambda \in \mathbb{R}$,

$$(*) \qquad A + B = \{x + y \mid x \in A, y \in B\},$$

$A \cdot B = \{xy \mid x \in A, y \in B\},\ \lambda A = \{\lambda x \mid x \in A\}$.

Let $\mathcal{K} \subset \mathcal{P}_{kc}$ be a semigroup with identity 0 under the operation $(*)$, satisfying the conditions:

(K_1). $A \cdot B$ is convex for every $A, B \in \mathcal{K}$.
(K_2). $A \cdot (B + C) = A \cdot B + A \cdot C,\ \forall A, B, C \in \mathcal{K}$.

It is easy to see that the following classes of sets:

1. $\mathcal{K} = \{\{x\} \mid x \in X\}$ for X like above and
2. $\mathcal{K} = \{A \in \mathcal{P}_{kc} \mid A \subset I\!R_+\}$ for $X = I\!R$

are such semigroups.

Assume $\varphi : \mathcal{A} \longrightarrow \mathcal{P}_{kc}$. φ is said to be a multimeasure (cf. [2]) if $\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2)$, whenever $E_1 \cap E_2 = \emptyset$. The total variation of φ is the non–negative (possibly infinite) set function $v(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$v(\varphi, A) = \sup \sum_{i=1}^{n} \|\varphi(E_i)\|,$$

where the supremum is taken over all finite partitions $\{E_i\}_{i=1}^n$ of $A \in \mathcal{A}$, with $E_i \in \mathcal{A}$ for all $i = 1, ..., n$.

If φ is a multimeasure then $v(\varphi, \cdot)$ is finitely additive.

Let $F, G : S \to \mathcal{P}_{kc}$ be multifunctions. We denote by $\widetilde{D}(F, G) : S \to I\!R$ the function defined on S as follows:

$$(\widetilde{D}(F, G))(s) = D(F(s), G(s)),\ \forall s \in S.$$

Throughout this paper, $\mathcal{A}$ will be an algebra of subsets of S, $\varphi : \mathcal{A} \longrightarrow \mathcal{K}$ a multimeasure and ν the total variation of φ. We shall assume $\nu(S) < \infty$.

A multifunction $F : S \longrightarrow \mathcal{K}$ is said to be a *simple multifunction* if $F = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$, where $C_i \in \mathcal{K}$, $A_i \in \mathcal{A}$, $i = 1, ..., n$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and 1_{A_i} is the characteristic function of A_i.

The Integral. Definition 1. If $F : S \to \mathcal{K}$ is a simple multifunction $F = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$ then "*the integral of* F *over* E" is defined to be

$$\int_E F\, d\varphi = \sum_{i=1}^{n} C_i \cdot \varphi(A_i \cap E),\ E \in \mathcal{A}.$$

Since $C_i \in \mathcal{K}$ for all $i = 1, ..., n$ the integral of F is independent of the representation chosen for F.

The following lemma will be needed.

Lemma 2. *For every* $A, B, C \in \mathcal{P}_{kc}$,

$$D(A \cdot B, A \cdot C) \leq \|A\| \cdot D(B, C).$$

The next theorem establishes some properties of simple multifunctions.

Theorem 3. *Let* $F, G : S \longrightarrow \mathcal{K}$ *be simple multifunctions. Then:*

a) $D\left(\int_E F\, d\varphi, \int_E G\, d\varphi\right) \leq \int_E \widetilde{D}(F, G) d\nu, \quad \forall E \in \mathcal{A}.$

b) $\int_{E_1 \cup E_2} F\, d\varphi = \int_{E_1} F\, d\varphi + \int_{E_2} F\, d\varphi$, *whenever* $E_1 \cap E_2 = \emptyset$, $E_1, E_2 \in \mathcal{A}$ (*if we put* $\Gamma(E) = \int_E F\, d\varphi$, $\forall E \in \mathcal{A}$, *then* Γ *is a multimeasure).*

Proof. a) Let $F, G : S \to \mathcal{K}$ be simple multifunctions $F = \sum_{i=1}^{n} B_i \cdot 1_{A_i}$, $G = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$. From Lemma 2 we have

$$D\left(\int_S F\, d\varphi, \int_S G\, d\varphi\right) = D\left(\sum_{i=1}^{n} B_i \cdot \varphi(A_i), \sum_{i=1}^{n} C_i \cdot \varphi(A_i)\right) \leq$$
$$\leq \sum_{i=1}^{n} D(B_i \cdot \varphi(A_i), C_i \cdot \varphi(A_i)) \leq \sum_{i=1}^{n} \|\varphi(A_i)\| \cdot D(B_i, C_i) \leq$$
$$\leq \sum_{i=1}^{n} \nu(A_i) \cdot D(B_i, C_i) = \int_S \widetilde{D}(F, G) d\nu.$$

b) Let F be a simple multifunction $F = \sum_{i=1}^{n} B_i \cdot 1_{A_i}$ and $E_1, E_2 \in \mathcal{A}$, $E_1 \cap E_2 = \emptyset$. Then we have

$$\int_{E_1 \cup E_2} F\, d\varphi = \sum_{i=1}^{n} B_i \cdot \varphi((E_1 \cup E_2) \cap A_i) =$$
$$= \sum_{i=1}^{n} B_i \cdot \varphi(E_1 \cap A_i) + \sum_{i=1}^{n} B_i \cdot \varphi(E_2 \cap A_i) = \int_{E_1} F\, d\varphi + \int_{E_2} F\, d\varphi.$$

Theorem 4. *Let $F : S \longrightarrow \mathcal{K}$ be a simple multifunction and $\Gamma(E) = \int_E F\, d\varphi$, $\forall E \in \mathcal{A}$. Then the total variation of Γ on E, $v(\Gamma, E) = \int_E |F| d\nu$, $\forall E \in \mathcal{A}$, where $|F| = \widetilde{D}(F, \theta)$ and θ is the multifunction defined by $\theta(s) = 0$, $\forall s \in S$.*

Proof. Let $F : S \to \mathcal{K}$ be a simple multifunction $F = \sum_{i=1}^{n} B_i \cdot 1_{A_i}$, $A_i \in \mathcal{A}$, $B_i \in \mathcal{K}$, $i = 1, ..., n$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, $S = \bigcup_{i=1}^{n} A_i$. Then $\int_S F\, d\varphi = \sum_{i=1}^{n} B_i \cdot \varphi(A_i)$.

Let E_j, $j = 1, ..., m$, be disjoint sets in $\mathcal{A}$ with $S = \bigcup_{j=1}^{m} E_j$. Then we have

$$\sum_{j=1}^{m} |\Gamma(E_j)| = \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} B_i \cdot \varphi(E_j \cap A_i) \right\| \leq \sum_{j=1}^{m} \sum_{i=1}^{n} |B_i| \cdot |\varphi(E_j \cap A_i)| =$$
$$= \sum_{i=1}^{n} |B_i| \cdot \sum_{j=1}^{m} |\varphi(E_j \cap A_i)| \leq \sum_{i=1}^{n} |B_i| \cdot \nu(A_i) = \int_S |F| d\nu$$

which shows that

$$v(\Gamma, S) \leq \int_S |F| d\nu. \tag{1}$$

Now, let E_k^i, $k = 1, ..., m_i$, be disjoint sets in $\mathcal{A}$ with $A_i = \bigcup_{k=1}^{m_i} E_k^i$ such that $\sum_{k=1}^{m_i} |\varphi(E_k^i)| > \nu(A_i) - \frac{\varepsilon}{M}, i = 1, ..., n$, where ε is a given positive number and $M = \sum_{i=1}^{n} |B_i|$. Then

$$v(\Gamma, S) \geq \sum_{i=1}^{n} \sum_{k=1}^{m_i} |\Gamma(E_k^i)| = \sum_{i=1}^{n} \sum_{k=1}^{m_i} |B_i| \cdot |\varphi(E_k^i)| =$$
$$= \sum_{i=1}^{n} |B_i| \cdot \sum_{k=1}^{m_i} |\varphi(E_k^i)| > \sum_{i=1}^{n} |B_i| \cdot \nu(A_i) - \varepsilon =$$
$$= \int_S |F| d\nu - \varepsilon,$$

which proves that

$$v(\Gamma, S) \geq \int_S |F| \nu. \tag{2}$$

From (1) and (2) results $v(\Gamma, S) = \int_S |F| d\nu$.

Corollary 5. *If $F : S \longrightarrow \mathcal{K}$ is a simple multifunction then* $D\left(\int_E F\, d\varphi, 0\right) \leq \int_E \widetilde{D}(F, \theta) d\nu$, $\forall E \in \mathcal{A}$.
(*The inequality is also written as* $\left\| \int_E F\, d\varphi \right\| \leq \int_E |F| d\nu,\ \forall E \in \mathcal{A}$.)

Definiton 6. A multifunction $F : S \longrightarrow \mathcal{K}$ is said to be *φ–totally measurable* if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \longrightarrow \mathcal{K}$ satisfying the conditions:

1) $\widetilde{D}(F_n, F)$ is ν–measurable (in the sense of [7]), $\forall n \in I\!N$.

2) $\widetilde{D}(F_n, F)$ converges to 0 in ν–measure (in the sense of [7]) ($\widetilde{D}(F_n, F) \xrightarrow{\nu} 0$).

Remark 7. If F is φ–totally measurable then F is also totally measurable (with respect to ν) in the sense of [9].

The next result follows from Definition 6.

Proposition 8. Let $F, G : S \longrightarrow \mathcal{K}$ be multifunctions such that F is φ–totally measurable and G is simple. Then $\widetilde{D}(F, G)$ is ν–measurable.

Definition 9. Let $F : S \longrightarrow \mathcal{K}$ be a φ–totally measurable multifunction. F is said to be *φ–integrable* on S if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \longrightarrow \mathcal{K}$ satisfying the conditions:

1) $\widetilde{D}(F_n, F)$ is ν–measurable, $\forall n \in I\!N$.

2) $\widetilde{D}(F_n, F) \xrightarrow{\nu} 0$.

3) $\lim\limits_{n,m \to \infty} \int_S \widetilde{D}(F_n, F_m) d\nu = 0$.

Such a sequence $(F_n)_n$ of simple multifunctions will be said to be a *defining sequence* for F.

For each $E \in \mathcal{A}, D\left(\int_E F_n\, d\varphi, \int_E F_m\, d\varphi\right) \leq \int_E \widetilde{D}(F_n, F_m)d\nu \leq$ $\leq \int_S \widetilde{D}(F_n, F_m)d\nu$.

Since $\lim_{n,m\to\infty} \int_S \widetilde{D}(F_n, F_m)d\nu = 0$, the sequence $\left\{\int_E F_n\, d\varphi\right\}_n$ is Cauchy in $\mathcal{P}_{kc}$.

The completeness of $\mathcal{P}_{kc}$ (cf. [5], Theorem II-14, p.47) implies that $\lim_{n\to\infty} \int_E F_n\, d\varphi$ exists in $\mathcal{P}_{kc}$.

The above inequality shows that the limit is uniform with respect to $E \in \mathcal{A}$.

Definiton 10. Let $F : S \longrightarrow \mathcal{K}$ be a φ–integrable multifunction with defining sequence $(F_n)_n$. Define $\int_E F\, d\varphi = \lim_{n\to\infty} \int_E F_n\, d\varphi$ to be the *integral of F over E, $E \in \mathcal{A}$.*

Theorem 11. *If $F : S \longrightarrow \mathcal{K}$ is a φ–integrable multifunction and $\Gamma(E) = \int_E F\, d\varphi$, $\forall E \in \mathcal{A}$, then Γ is a multimeasure.*

Proof. It results straightforward from Definitions 9 and 10. In fact, let $(F_n)_n$ be a defining sequence for F and $E_1, E_2 \in \mathcal{A}$, $E_1 \cap E_2 = \emptyset$. Then we have

$$\int_{E_1\cup E_2} F\, d\varphi = \lim_{n\to\infty} \int_{E_1\cup E_2} F_n\, d\varphi =$$
$$= \lim_{n\to\infty} \int_{E_1} F_n\, d\varphi + \lim_{n\to\infty} \int_{E_2} F_n\, d\varphi = \int_{E_1} F\, d\varphi + \int_{E_2} F\, d\varphi.$$

Proposition 12. Let $F : S \to \mathcal{K}$ be a φ–integrable multifunction with defining sequence $(F_n)_n$. Then:

a) The function $\widetilde{D}(F, \theta)$ is ν–integrable and
$\int_E \widetilde{D}(F, \theta)d\nu = \lim_{n\to\infty} \int_E \widetilde{D}(F_n, \theta)d\nu,\ \forall E \in \mathcal{A}$.

b) $\lim_{n\to\infty} \int_S \widetilde{D}(F_n, F)d\nu = 0$.

Proof. It follows once again from Definitions 9 and 10.

Remark 13. The Brooks, Martellotti–Sambucini and Dunford integrals are obtained like particular cases from the integral defined in this paper. So:

a) If $X = I\!R$, $\mathcal{K} = \{\{x\} \mid x \in I\!R\}$ and $F : S \to \mathcal{K}$, $F = \{f\}$, then we get the Brooks integral [2] for the function f with respect to the multimeasure φ.

b) If φ is single valued, $\varphi = \{\mu\}$, , then we get the Martellotti–Sambucini integral [9] for F with respect to the measure μ.

c) If $X = I\!R$, $\mathcal{K} = \{\{x\} \mid x \in I\!R\}$, $F : S \to \mathcal{K}$, $F = \{f\}$ and $\varphi = \{\mu\}$ (so the multifunction and the multimeasure are both single valued) then the integral here defined becomes the Dunford integral [7].

The next theorem extends the result of Theorem 3-a) to φ–integrable multifunctions.

Theorem 14. *Let $F, G : S \longrightarrow \mathcal{K}$ be φ–integrable multifunctions. Then*

$$D\left(\int_E F\, d\varphi, \int_E G\, d\varphi\right) \leq \int_E \widetilde{D}(F,G)d\nu,\ \forall E \in \mathcal{A}.$$

Proof. Let $(F_n)_n, (G_n)_n$ be defining sequences for F, G respective. Then, for all sufficiently large n we have

$$D\left(\int_E F\, d\varphi, \int_E G\, d\varphi\right) \leq 2\varepsilon + \int_E \widetilde{D}(F_n,G_n)d\nu. \tag{3}$$

Since $|\widetilde{D}(F,G) - \widetilde{D}(F_n,G_n)| \leq \widetilde{D}(F,F_n) + \widetilde{D}(G,G_n)$, $\widetilde{D}(F,G)$ is ν–limit of a sequence of simple functions so it is ν–measurable.

Also the sequence $\{\widetilde{D}(F_n,G_n)\}_n$ is Cauchy in $L^1(S,\mathcal{A},\nu)$ since

$$\int_S |\widetilde{D}(F_n,G_n) - \widetilde{D}(F_m,G_m)|d\nu \leq \int_S \widetilde{D}(F_n,F_m)d\nu + \int_S \widetilde{D}(G_n,G_m)d\nu.$$

Therefore $\widetilde{D}(F,G)$ is ν–integrable and

$$\lim_{n\to\infty} \int_S \widetilde{D}(F_n,G_n)d\nu = \int_S \widetilde{D}(F,G)d\nu.$$

We make $n \to \infty$ in (3) and the statement is proved.

Theorem 15. *If $F : S \longrightarrow \mathcal{K}$ is a φ–integrable multifunction, then $D\left(\int_E F\, d\varphi, 0\right) \leq \int_E \widetilde{D}(F,\theta)d\nu$, $\forall E \in \mathcal{A}$. Moreover, if we put $\Gamma(E) =$*

$\int_E F\,d\varphi$, $\forall E \in \mathcal{A}$, *then the total variation of* Γ *on* E, $v(\Gamma, E) = \int_E \widetilde{D}(F,\theta)d\nu = \int_E |F| d\nu$, $\forall E \in \mathcal{A}$.

Proof. Let $(F_n)_n$ be a defining sequence for F. Then

$$\lim_{n\to\infty} D\left(\int_E F_n\,d\varphi, 0\right) = D\left(\int_E F\,d\varphi, 0\right).$$

From b)–Proposition 12 we have

$$D\left(\int_E F\,d\varphi, 0\right) = \lim_{n\to\infty} D\left(\int_E F_n\,d\varphi, 0\right) \leq$$
$$\leq \lim_{n\to\infty} \int_E \widetilde{D}(F_n,\theta)d\nu \leq \int_E \widetilde{D}(F,\theta)d\nu.$$

We now prove that $v(\Gamma, E) = \int_E \widetilde{D}(F,\theta)d\nu = \int_E |F| d\nu, \forall E \in \mathcal{A}$.

We consider $\Gamma_n(E) = \int_E F_n\,d\varphi$, $\forall E \in \mathcal{A}$, $n \in I\!N$ and $\varepsilon > 0$.

Then there is n_0 such that $\int_E \widetilde{D}(F_n, F)d\nu < \varepsilon$, $\forall n \geq n_0$. We can choose $A_1, ..., A_m, B_1, ..., B_q$ such that

$$v(\Gamma, E) - \sum_{i=1}^{m} |\Gamma(A_i)| < \varepsilon,\ v(\Gamma_n, E) - \sum_{j=1}^{q} |\Gamma_n(B_j)| < \varepsilon.$$

Let (E_i) be the partition of E generated by the sets A_i, B_j. Then $|v(\Gamma, E) - v(\Gamma_n, E)| < 3\varepsilon$ and so $v(\Gamma, E) = \int_E \widetilde{D}(F,\theta)d\nu$.

Theorem 16. *If* $F : S \longrightarrow \mathcal{K}$ *is a* φ*–integrable multifunction and* $\Gamma(E) = \int_E F\,d\varphi$, $\forall E \in \mathcal{A}$, *then* $\Gamma << \nu$ (*i.e.* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $v(\Gamma, E) < \varepsilon$ *for all* $E \in \mathcal{A}$ *with* $\nu(E) < \delta$).

Proof. From Theorem 15 we have

$$v(\Gamma, E) = \int_E \widetilde{D}(F,\theta)d\nu,\ \forall E \in \mathcal{A}. \tag{4}$$

Let M be the total variation of Γ. Since the function $\widetilde{D}(F,\theta)$ is ν–integrable (cf. a)–Proposition 12),

$$\lim_{\nu(E)\to 0} v(M,E)=0. \tag{5}$$

Then the conclusion of the theorem follows immediately from (4) and (5).

Theorem 17. *Let $(F_n)_n,(G_n)_n$ be two defining sequences for a φ–integrable multifunction $F:S\to\mathcal{K}$. Then*

$$\int_E F\,d\varphi=\lim_{n\to\infty}\int_E F_n\,d\varphi=\lim_{n\to\infty}\int_E G_n\,d\varphi,\ \forall E\in\mathcal{A}.$$

Proof. It easily results that $\widetilde{D}(F_n,G_n)\xrightarrow{\nu}0$. Let $\varepsilon>0$ and $A_n=\{s\in S\mid D(F_n(s),G_n(s))>\varepsilon\}$.

We will prove that $\lim_{n\to\infty} D\left(\int_E F_n\,d\varphi,\int_E G_n\,d\varphi\right)=0,\ \forall E\in\mathcal{A}$.

For all sufficiently large $n>n_0$,

$$D\left(\int_E F_n\,d\varphi,\int_E G_n\,d\varphi\right)\le\int_E\widetilde{D}(F_n,G_n)d\nu=$$
$$=\int_{E\cap A_n}\widetilde{D}(F_n,G_n)d\nu+\int_{E-A_n}\widetilde{D}(F_n,G_n)d\nu\le$$
$$\le\int_{E\cap A_n}\widetilde{D}(F_n,G_n)d\nu+\varepsilon\nu(E).$$

Using the Vitali–Hahn–Saks Theorem, by a standard proof, it follows that $\lim_{n\to\infty}\int_{E\cap A_n}\widetilde{D}(F_n,G_n)d\nu=0$ (see Theorem 2.8 in [9]).

Definition 18. A finite or countable family of pairwise disjoint sets $(E_i)_i\subset\mathcal{A}$ will be called a ν–exhaustion of S if for each $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that $\nu\left(S-\bigcup_{i=1}^{n_0}E_i\right)<\varepsilon$.

Proposition 19. *Let $(E_i)_{i\in\mathbb{N}}\subset\mathcal{A}$ be a ν-exhaustion of S such that $S=\bigcup_{i\in\mathbb{N}}E_i$. Let $(C_n)_n$ be a sequence in $\mathcal{K}$ such that there exists $r>0$ such that $|C_n|\le r$, $\forall n\in\mathbb{N}$. We define*

$$F=\sum_{i=1}^{\infty}C_i 1_{E_i}. \tag{$**$}$$

Such definition is formal, in the sense that for every $s \in S$ there exists a unique $i = i(s) \in I\!N$ such that $s \in E_{i(s)}$ and so $F(s) = C_{i(s)}$ (hence F is $\mathcal{K}$–valued). Then F defined as in $(**)$ is φ-integrable.

Proof. It easily results that $F_n = \sum_{i=1}^{n} C_i \cdot 1_{E_i}$ is a defining sequence for F.

Theorem 20. (The convergence theorem of Vitaly type) *Let $F : S \longrightarrow \mathcal{K}$ be a multifunction, $(F_n)_n$ be a sequence of φ–integrable multifunctions $F_n : S \longrightarrow \mathcal{K}$ and $\Gamma_n(E) = \int_E F_n \, d\varphi$, $\forall E \in \mathcal{A}$, $n \in I\!N$ with the conditions:*

(i) $\widetilde{D}(F_n, F) \xrightarrow{\nu} 0$.

(ii) $\Gamma_n \ll \nu$, *uniform with respect to* $n \in I\!N$.

Then F is φ–integrable and $\int_E F \, d\varphi = \lim_{n\to\infty} \int_E F_n \, d\varphi, \ \forall E \in \mathcal{A}$.

Proof. It easily follows that

$$\lim_{n,m\to\infty} \nu(\{s \in S; D(F_n(s), F_m(s)) > \varepsilon\}) = 0, \ \forall \varepsilon > 0.$$

From (ii), the sequence $\left(\int_E F_n \, d\varphi\right)_n$ is Cauchy in $\mathcal{P}_{kc}$, $\forall E \in \mathcal{A}$.

To prove that F is φ–integrable one has to find a defining sequence $(G_n)_n$ for F. Since F_n is φ–integrable for all n, there is a defining sequence $(G_k^n)_{k\in I\!N}$ for F_n. Then we have for every $n \in I\!N$:

1. $\widetilde{D}(G_k^n, F_n)$ is ν–measurable, $\forall k \in I\!N$.

2. $\widetilde{D}(G_k^n, F_n) \xrightarrow{\nu} 0$, for $k \longrightarrow \infty$.

3. $\lim_{k,p\to\infty} \int_S \widetilde{D}(G_k^n, G_p^n) d\nu = 0$.

Let $H_{n,k} = \left\{s \in S; D(G_k^n(s), F_n(s)) > \frac{1}{2^n}\right\}$.

Then for all $n \in I\!N$ there is $\bar{k}(n) \in I\!N$ such that $\nu(H_{n,k}) < \frac{1}{2^n}$ and $\int_S \widetilde{D}(G_k^n, F_n) d\nu \leq \frac{1}{2^n}$, $\forall k \geq \bar{k}$.

Let $B_n = H_{n,\bar{k}(n)}$. Then $\lim_{n\to\infty} \nu(B_n) = 0$ and if we put $G_n = G_{\bar{k}(n)}^n$ then $D(G_n(s), F(s)) \leq \frac{1}{2^n}, \forall s \in cB_n$. Then $\widetilde{D}(G_n, F) \xrightarrow{\nu} 0$. In

fact, since $|\widetilde{D}(G_m, G_n) - \widetilde{D}(G_m, F)| \leq \widetilde{D}(G^n_{\overline{k}(n)}, F_n) + \widetilde{D}(F_n, F)$, $\widetilde{D}(G_n, F)$ is ν–totally measurable and so $\widetilde{D}(G_n, F) \xrightarrow{\nu} 0$. We now prove that $\lim_{n,k\to\infty} \int_S \widetilde{D}(G_n, G_k)d\nu = 0$.

Since for all sufficiently large n, k, $\int_S \widetilde{D}(G_n, F_n)d\nu \leq \frac{1}{2^n}$, $\int_S \widetilde{D}(G_k, F_k)d\nu \leq \frac{1}{2^k}$ and from (ii), $\int_S \widetilde{D}(F_n, F_k)d\nu \leq \varepsilon$, from a standard decomposition of S, we have $\int_S \widetilde{D}(G_n, G_k)d\nu \leq \varepsilon + \frac{1}{2^n} + \frac{1}{2^k}$.

Thus F is φ–integrable.

Moreover, $\lim_{n\to\infty} D\left(\int_E F_n\, d\varphi, \int_E G_n\, d\varphi\right) = 0$, $\forall E \in \mathcal{A}$. In fact,

$$D\left(\int_E G_n\, d\varphi, \int_E F_n\, d\varphi\right) \leq \int_S \widetilde{D}(G_n, F_n)d\nu \leq \frac{1}{2^n},$$

therefore $\int_E F\, d\varphi = \lim_{n\to\infty} \int_E G_n\, d\varphi = \lim_{n\to\infty} \int_E F_n\, d\varphi$. This completes the proof.

The following result is an immediate consequence of Theorem 20.

Corollary 21. (Lebesgue dominated convergence theorem) *Let $(F_n)_n$ be a sequence of φ–integrable multifunctions $F_n : S \to \mathcal{K}$. Let F be a multifunction $F : S \longrightarrow \mathcal{K}$ such that $\widetilde{D}(F_n, F) \xrightarrow{\nu} 0$. Suppose that there exists $g \in L^1(S, \mathcal{A}, \nu)$ such that $|F_n| \leq |g|$, $\forall n \in I\!N$. Then F is φ–integrable and $\int_E F\, d\varphi = \lim_{n\to\infty} \int_E F_n\, d\varphi, \forall E \in \mathcal{A}$.*

Proposition 22. *Let $(F_n)_n$ be a sequence of φ-integrable multifunctions that converges to F, uniformly with respect to $s \in S$. Then F is φ-integrable and*

$$\int_E F\, d\varphi = \lim_{n\to\infty} \int_E F_n\, d\varphi, \quad \forall E \in \mathcal{A}.$$

Proof. The proof is analogous to that of Theorem 20. In fact, if $(G^n_k)_{k\in I\!N}$ is a defining sequence of simple multifunctions for F_n, for every $\varepsilon > 0$, $n \in I\!N$ there exists $k(n, \varepsilon) \in I\!N$ such that, for $k > k(n, \varepsilon)$

$$\{s \in S \mid D(G^n_k(s), F_n(s)) > \varepsilon\} = \emptyset$$

and, for n suitable large

$$\{s \in S \mid D(F(s), F_n(s)) > \varepsilon\} = \emptyset.$$

Then $G_n = G^n_{k\left(n, \frac{1}{2^n}\right)}$ is a defining sequence for F.

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