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SOME PROPERTIES OF THE PSEUDO ALMOST PERIODIC FUNCTIONS ON LOCALLY COMPACT GROUPS

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Abstract. A generalization from $\mathbb{R}$ to a locally compact Hausdorff Abelian group of the pseudo almost periodic functions is given. The mean value and the corresponding Fourier series of a pseudo almost periodic function are defined. The paper presents several properties of the Fourier series corresponding to a pseudo almost periodic function such as Parseval's equality, and a mean convergence property. Finally, in the special case of a σ–compact locally compact Hausdorff Abelian group, a density property of the C^*–algebra of pseudo almost periodic functions with respect to the supremum norm is established.

In his PhD thesis ([6]), C. Z h a n g defined the pseudo almost periodic functions on $\mathbb{R}$ and studied the mean value and the corresponding Fourier series for such functions. In an earlier paper ([2]), we have generalized these concepts to a Hausdorff locally compact σ-compact Abelian group G, we defined the concept of the mean value, the Fourier series associated to a pseudo almost periodic function, and we established some of their properties. In this paper we generalize the notion of a pseudo almost periodic function to a locally compact Hausdorff Abelian group G and we establish, in the special case of a σ-compact locally compact Hausdorff Abelian group G, a density property of the C^*-algebra of pseudo almost periodic functions with respect to the supremum norm.

In Sections 1 and 2 we use a locally compact Hausdorff Abelian group G and in Section 3, a σ-compact locally compact Hausdorff Abelian group G.

In Section 1 we recall the definitions of an almost periodic function and of a weak almost periodic function on G and an auxiliary result necessary for defining a space functions, denoted $PAP_0(G)$, studied in the second section.

The basic space considered in the second section is the space $PAP(G)$ of pseudo almost periodic functions defined on G. The main definitions and results about this functions reffer to the mean value and the Fourier series of a pseudo almost periodic function; some of these properties are the Parseval's equality and a mean convergence property.

In Section 3 a density property of the space $PAP(G)$ is proved. For this purpose we first introduce the notions of an ergodic zero set and the space of bounded continuous functions on G with an ergodic zero support. Denoting this space by $\mathcal{C}^0_{erg}(G)$ we show that $\mathcal{C}^0_{erg}(G)$ is uniformly dense in $PAP_0(G)$.

1. Terminology and notations. Let G be a locally compact Hausdorff Abelian group and λ the Haar measure on G. We denote by $\mathcal{C}(G)$ the C^*-algebra of continuous, bounded complex-valued functions on G and by $\mathcal{B}(G)$ the C^*-algebra of bounded complex–valued functions on G, both endowed with the supremum norm. Throughout this paper, $\|\cdot\|$ denotes the supremum norm on $\mathcal{B}(G)$. For $f\in\mathcal{B}(G)$ and $a \in G$, the translate of f by a is the function $R_af(x) = f(xa),\ x\in G$. A subset $\mathcal{F}$ of $\mathcal{B}(G)$ is said to be translation invariant if $R_a\mathcal{F} \subset \mathcal{F}$, for all $a\in G$.

Definition 1.1. ([1],[5]) A bounded complex–valued function g on G is called *an almost periodic function* if the family of functions $\{R_ag \mid a\in G\}$ is relatively compact in $\mathcal{B}(G)$ with respect to the supremum norm.

Definition 1.2. ([4]) A function $g \in \mathcal{B}(G)$ is said to be *a weak almost periodic function* if the family of functions $\{R_ag \mid a\in G\}$ is relatively compact in the weak topology of the space $\mathcal{B}(G)$.

Let us denote by $AP(G)$ the set of all continuous almost periodic functions on G and by $WAP(G)$ the set of all continuous weak almost periodic functions on G.

The following theorem is an existence result of a net of sets which plays an important role in this paper.

Theorem 1.1. ([3]) *There exists an increasing net $(H_\alpha)_{\alpha\in I}$ of open, relatively compact subsets of G, with $G = \bigcup_{\alpha\in I} H_\alpha$, such that for each $x\in G$,*

we have:

$$\lim_{\alpha} \frac{\lambda(xH_\alpha \Delta H_\alpha)}{\lambda(H_\alpha)} = 0.$$

In the following, we consider a net $(H_\alpha)_{\alpha \in I}$ whose existence is assured by Theorem 1.1.

Denote by $\hat{G}$, the group of the characters of the group G and by $\langle \hat{G} \rangle$, the subspace generated by the set $\hat{G}$ in the Banach space $\mathcal{C}(G)$.

2. The Fourier series of a pseudo almost periodic function on G. Let us denote

$$PAP_0(G) = \{\varphi \in \mathcal{C}(G) | \lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} | \varphi(x) | \, d\lambda(x) = 0\}.$$

It is easy to see that the set $PAP_0(G)$ is a translation invariant C^*-subalgebra of $\mathcal{C}(G)$.

Definition 2.1. A function $f \in \mathcal{C}(G)$ is called *a pseudo almost periodic function* if $f = g + \varphi$, where $g \in AP(G)$ and $\varphi \in PAP_0(G)$.

Let us denote by $PAP(G)$ the set of all pseudo almost periodic functions.

In the following lemma we prove that the hypothesis on G to be σ-compact (the case treated in [2]) is not necessary.

Lemma 2.1. *If $f \in PAP(G)$ and if g is its almost periodic component, we have*

$$g(G) \subset \overline{f(G)},$$

where $\overline{f(G)}$ is the closure of the subset $f(G)$ of C.

Therefore $\|f\| \geq \|g\| \geq \inf_{x \in G} | g(x) | \geq \inf_{x \in G} | f(x) |$.

Proof. Let us assume that $g(G) \not\subset \overline{f(G)}$; then there would be an $x_0 \in G$ and $\varepsilon > 0$ such that

$$\inf_{s \in G} | g(x_0) - f(s) | > \varepsilon.$$

Since g is an almost periodic function, according to Theorem 7.1. of [1], there exists a set $A = \{x_1, \ldots, x_p\} \subset G$ such that, for a fixed $x \in G$, we find $x_k \in A$ with the property that for all $a \in G$ we have

$$| g(xa) - g(x_k a) | < \frac{\varepsilon}{2}.$$

Letting $a = x_k^{-1}x_0$, we obtain

$$| g(xx_k^{-1}x_0) - g(x_0) | < \frac{\varepsilon}{2}.$$

Moreover, we have

$$| R_{x_0}\varphi(xx_k^{-1}) | = | R_{x_0}f(xx_k^{-1}) - R_{x_0}g(xx_k^{-1}) | \geq$$

$$\geq | R_{x_0}f(xx_k^{-1}) - g(x_0) | - | g(x_0) - g(x_0xx_k^{-1}) | > \frac{\varepsilon}{2}.$$

We obtain that, for all $x \in G$,

$$\sum_{i=1}^{p} | R_{x_0}\varphi(xx_i^{-1}) | > \frac{\varepsilon}{2};$$

so, for all $\alpha \in I$

$$\frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \sum_{i=1}^{p} | R_{x_0}\varphi(xx_i^{-1}) | \, d\lambda(x) > \frac{\varepsilon}{2}$$

and therefore

$$\lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \sum_{i=1}^{p} | R_{x_0}\varphi(xx_i^{-1}) | \, d\lambda(x) \geq \frac{\varepsilon}{2}. \tag{1}$$

On the other hand, $\varphi \in PAP_0(G)$; hence $R_{x_0x_i^{-1}}\varphi \in PAP_0(G),\ i = \overline{1,p}$. It follows that, for all $i \in \{1, \ldots, p\}$,

$$\lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} | R_{x_0}\varphi(xx_i^{-1}) | \, d\lambda(x) = 0$$

and therefore

$$\lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} \sum_{i=1}^{p} | R_{x_0}\varphi(xx_i^{-1}) | \, d\lambda(x) = 0. \tag{2}$$

It is obvious that (2) contradicts (1), which proves the Lemma 2.1.

In the sequel, we state Theorems 2.1, 2.2 and Propositions 2.1, 2.2. Their proofs are similar to the proofs of the same results from the σ-compact case [2].

Theorem 2.1. a) *The set $PAP(G)$ is a translation invariant C^*-algebra under the norm, including the constant functions;*

b) *The space of the pseudo almost periodic functions is the direct sum of the space of the almost periodic functions and $PAP_0(G)$ ($PAP(G) = AP(G) \oplus PAP_0(G)$).*

Proposition 2.1. *If $f \in PAP(G)$, then the limit*

$$\lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} f(x)d\lambda(x)$$

exists and is finite.

Definition 2.2. For $f \in PAP(G)$, we shall call the *mean value of the function f*, and we shall denote it by $M(f)$, the limit

$$M(f) = \lim_{\alpha} \frac{1}{\lambda(H_\alpha)} \int_{H_\alpha} f(x)d\lambda(x).$$

Theorem 2.2. *The mean value $M : PAP(G) \to C$ has the following properties:*

a) $M(\alpha f) = \alpha M(f)$, *for* $\alpha \in C$;
b) $M(f + g) = M(f) + M(g)$;
c) *If* $f \equiv 1$ *then* $M(f) = 1$;
d) *If* $f \geq 0$ *then* $M(f) \geq 0$;
e) $\mid M(f) \mid \leq M(\mid f \mid)$;
f) $M(\overline{f}) = \overline{M(f)}$;
g) $M(R_a f) = M(f)$, *for* $a \in G$;
h) *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of functions in* $PAP(G)$ *and* $f \in PAP(G)$ *with* $\|f_n \to f\| \to 0$ *as* $n \to \infty$ *then,* $M(f_n) \to M(f)$ *as* $n \to \infty$.

The following proposition immediately follows from the similar result established for $AP(G)$ [4].

Proposition 2.2. *If $f \in PAP(G)$ there exists an at most countable set of characters of the group G, denoted by $\{\gamma_n \in \hat{G} \mid n \in \mathbb{N}\}$, such that $M(f \cdot \overline{\gamma}_n) \neq 0$, $n \in \mathbb{N}$.*

Definition 2.3. Let f be a function in $PAP(G)$ and let $\{\gamma_n \in \hat{G} \mid n \in I\!N\}$ be the set of characters of the group G such that $M(f \cdot \overline{\gamma}_n) \neq 0$, $n \in I\!N$. We define *the Fourier series of the pseudo almost periodic function* f by

$$f(x) \sim \sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

where

$$a_n = M(f \cdot \overline{\gamma}_n), \quad n \in I\!N.$$

The complex numbers a_n, $n \in I\!N$ are called *the Fourier coefficients of the function* f.

Lemma 2.2. *Let f be a function in $PAP(G)$ and let*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

be the Fourier series of the function f. Then, for all $p \in I\!N$, we have

$$0 \leq M(\mid f - \sum_{k=1}^{p} a_k \gamma_k \mid^2) = M(\mid f \mid^2) - \sum_{k=1}^{p} \mid a_k \mid^2 .$$

Proof. According to the Theorem 2.2 we obtain, for all $p \in I\!N$,

$$0 \leq M(\mid f - \sum_{k=1}^{p} a_k \gamma_k \mid^2) = M[(f - \sum_{k=1}^{p} a_k \gamma_k) \cdot (\overline{f} - \sum_{k=1}^{p} \overline{a}_k \overline{\gamma}_k)] =$$

$$= M(\mid f \mid^2) - \sum_{k=1}^{p} \overline{a}_k M(f \cdot \overline{\gamma}_k) - \sum_{k=1}^{p} a_k M(\overline{f} \cdot \gamma_k) + \sum_{h=1}^{p} \sum_{k=1}^{p} a_h \cdot \overline{a}_k M(\overline{\gamma}_h \cdot \gamma_k).$$

It is easy to see that $M(\gamma_k \cdot \overline{\gamma}_h) = \delta_{k,h}$, for all $k, h \in I\!N$.

It follows that

$$M(\mid f - \sum_{k=1}^{p} a_k \gamma_k \mid^2) = M(\mid f \mid^2) - \sum_{k=1}^{p} \mid a_k \mid^2 .$$

Theorem 2.3. (Parseval's equality). *Let f be a function in $PAP(G)$ and let*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

be the Fourier series of the function f. Then we have

$$M(\mid f \mid^2) = \sum_{n=1}^{\infty} \mid a_n \mid^2 .$$

Proof. Let us suppose that $f = g + \varphi$, with $g \in AP(G)$ and $\varphi \in PAP_0(G)$. It immediately follows that

$$M(\mid f \mid^2) = M(\mid g \mid^2).$$

On the other hand, the functions f and g have the same Fourier series and, applying Theorem 15.2 of [4], we obtain

$$M(\mid g \mid^2) = \sum_{n=1}^{\infty} \mid a_n \mid^2 .$$

Thus, the property in the statement holds.

Now, combining Lemma 2.2 and Theorem 2.3, we find the next corollary.

Corollary 2.1. (Mean convergence) *Let f be a function in $PAP(G)$ and let*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

be the Fourier series of the function f. Then

$$\lim_{N \to \infty} M(\mid f - \sum_{k=1}^{N} a_k \gamma_k \mid^2) = 0.$$

3. A density property of the C^*-algebra $PAP(G)$. According to [5, Th. 18.14, p.255] we are able to use a sequence $(H_n)_{n \in I\!N}$ instead of the net $(H_\alpha)_{\alpha \in I}$ from Sections 1 and 2.

Theorem 3.1. ([5]) *There exists an increasing sequence $(H_n)_{n \in I\!N}$ of open relatively compact subsets of G with $G = \cup_{n=1}^{\infty} H_n$ such that, for each $x \in G$,*

$$\lim_{n \to \infty} \frac{\lambda(xH_n \Delta H_n)}{\lambda(H_n)} = 0.$$

Definition 3.1. A closed subset C of G is said to be an *ergodic zero set* in G if

$$\lim_{n\to\infty} \frac{\lambda(H_n \cap C)}{\lambda(H_n)} = 0.$$

Lemma 3.1. *Let φ be a function in $\mathcal{C}(G)$. Then $\varphi \in PAP_0(G)$ if and only if, for every $\varepsilon > 0$, the set $C_\varepsilon = \{x \in G \mid |\varphi(x)| \geq \varepsilon\}$ is an ergodic zero subset of G.*

Proof. For the "only if" part, we consider a function $\varphi \in PAP_0(G)$, a positive number ε, and a natural number n. We have

$$\begin{aligned}\frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x) = \frac{1}{\lambda(H_n)} \int_{H_n \setminus C_\varepsilon} \mid \varphi(x) \mid d\lambda(x) + \\ + \frac{1}{\lambda(H_n)} \int_{H_n \cap C_\varepsilon} \mid \varphi(x) \mid d\lambda(x) \geq \\ \geq \frac{1}{\lambda(H_n)} \int_{H_n \cap C_\varepsilon} \mid \varphi(x) \mid d\lambda(x) \geq \varepsilon \frac{\lambda(H_n \cap C_\varepsilon)}{\lambda(H_n)}.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{\lambda(H_n \cap C_\varepsilon)}{\lambda(H_n)} \leq \frac{1}{\varepsilon} \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x);$$

hence

$$\lim_{n\to\infty} \frac{\lambda(H_n \cap C_\varepsilon)}{\lambda(H_n)} = 0.$$

We now prove the "if" part. Let φ be a function in $\mathcal{C}(G)$, ε a positive number and n a natural number. Hence, there exists a number M in $(0, \infty)$ such that $\mid \varphi(x) \mid \leq M$ for all $x \in G$.

We have

$$\begin{aligned}\frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x) = \frac{1}{\lambda(H_n)} \int_{H_n \setminus C_\varepsilon} \mid \varphi(x) \mid d\lambda(x) + \\ + \frac{1}{\lambda(H_n)} \int_{H_n \cap C_\varepsilon} \mid \varphi(x) \mid d\lambda(x) \leq \\ \leq \varepsilon \frac{\lambda(H_n \setminus C_\varepsilon)}{\lambda(H_n)} + M \frac{\lambda(H_n \cap C_\varepsilon)}{\lambda(H_n)} \leq \varepsilon + M \frac{\lambda(H_n \cap C_\varepsilon)}{\lambda(H_n)}.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x) \leq \varepsilon \quad \text{for all } \varepsilon > 0$$

and, therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x) = 0.$$

Proposition 3.1. a) *A closed subset of an ergodic zero set in G is an ergodic zero set;*

b) *Any intersection of ergodic zero sets is an ergodic zero set;*

c) *Every finite union of ergodic zero sets is an ergodic zero set;*

d) *The translate of an ergodic zero set is an ergodic zero set.*

Proof. The first three assertions immediatelly follow from the definition of an ergodic zero set. Let us prove d).

Let C be an ergodic zero set in G and let x be in G. According to the properties of the Haar measure we obtain

$$\frac{\lambda(xC \cap H_n)}{\lambda(H_n)} = \frac{\lambda(x(C \cap x^{-1}H_n))}{\lambda(H_n)} = \frac{\lambda(C \cap x^{-1}H_n)}{\lambda(H_n)} =$$

$$= \frac{\lambda((C \cap x^{-1}H_n) \cap H_n)}{\lambda(H_n)} + \frac{\lambda((C \cap x^{-1}H_n) \backslash H_n)}{\lambda(H_n)} \leq$$

$$\leq \frac{\lambda(C \cap H_n)}{\lambda(H_n)} + \frac{\lambda(x^{-1}H_n \backslash H_n)}{\lambda(H_n)} \leq \frac{\lambda(C \cap H_n)}{\lambda(H_n)} + \frac{\lambda(x^{-1}H_n \Delta H_n)}{\lambda(H_n)}, \ \forall n \in \mathbb{N}.$$

It follows that

$$\lim_{n\to\infty} \frac{\lambda(xC \cap H_n)}{\lambda(H_n)} = 0.$$

Remark 3.1. It follows from Lemma 3.1 that $PAP_0(G)$ is the set of all bounded continuous complex-valued functions f on G, such that for each $\varepsilon > 0$ there exists an ergodic zero set C in G (depending on f and ε) with $\mid f(x) \mid < \varepsilon$ for all $x \in G \backslash C$.

Let $\mathcal{C}^0_{erg}(G)$ be the set of all bounded continuous complex–valued functions f on G, such that there exists an ergodic zero set C in G (depending on f) with $f(x) = 0$ for all $x \in G \backslash C$.

In other words, $\mathcal{C}^0_{erg}(G)$ is the set of all bounded continuous functions on G with ergodic zero support.

Theorem 3.2. *The subalgebra $\mathcal{C}^0_{erg}(G)$ of the algebra $PAP_0(G)$ is uniformly dense in $PAP_0(G)$.*

Proof. It is easy to see that $\mathcal{C}^0_{erg}(G)$ is a subalgebra of $PAP_0(G)$.

Let f be a function in $PAP_0(G)$, and ε be a positive number.

Thus, the set $C_\varepsilon = \{x \in G \mid |f(x)| \geq \frac{\varepsilon}{2}\}$ is an ergodic zero set. It follows that

$$\lim_{n\to\infty} \frac{\lambda(C_\varepsilon \cap H_n)}{\lambda(H_n)} = 0$$

and therefore

$$\lim_{n\to\infty} \frac{\lambda((G\backslash C_\varepsilon) \cap H_n)}{\lambda(H_n)} = 1. \tag{6}$$

Taking into account that $G\backslash C_\varepsilon$ is an open set, hence Haar measurable, we can find a closed subset $F_\varepsilon \subset G\backslash C_\varepsilon$ such that

$$\lambda((G\backslash C_\varepsilon)\backslash F_\varepsilon) < 1.$$

Therefore

$$\lim_{n\to\infty} \frac{\lambda(((G\backslash C_\varepsilon)\backslash F_\varepsilon) \cap H_n)}{\lambda(H_n)} = 0. \tag{7}$$

We clearly have

$$\frac{\lambda((G\backslash C_\varepsilon) \cap H_n)}{\lambda(H_n)} = \frac{\lambda(((G\backslash C_\varepsilon)\backslash F_\varepsilon) \cap H_n)}{\lambda(H_n)} + \frac{\lambda(F_\varepsilon \cap H_n)}{\lambda(H_n)}$$

Combining the above equality with (6) and (7), we obtain

$$\lim_{n\to\infty} \frac{\lambda(F_\varepsilon \cap H_n)}{\lambda(H_n)} = 1.$$

Taking into account that F_ε and C_ε are disjoint closed sets in G (G is a T_4 space), we find an open set D_ε such that

$$F_\varepsilon \subset D_\varepsilon \subset \overline{D}_\varepsilon \subset G\backslash C_\varepsilon$$

(where $\overline{D}_\varepsilon$ is the closure in G of the set D_ε).

It follows from these inclusions and from the equalities

$$\lim_{n\to\infty} \frac{\lambda(F_\varepsilon \cap H_n)}{\lambda(H_n)} = \lim_{n\to\infty} \frac{\lambda((G\backslash C_\varepsilon) \cap H_n)}{\lambda(H_n)} = 1,$$

that

$$\lim_{n\to\infty} \frac{\lambda(D_\varepsilon \cap H_n)}{\lambda(H_n)} = \lim_{n\to\infty} \frac{\lambda(\overline{D}_\varepsilon \cap H_n)}{\lambda(H_n)} = 1.$$

Furthermore,

$$\lim_{n\to\infty} \frac{\lambda((G\backslash D_\varepsilon) \cap H_n)}{\lambda(H_n)} = 0,$$

hence $G\backslash D_\varepsilon$ is an ergodic zero set.

Now applying Uryson's theorem, we find a continuous function $g : G \to [0, 1]$, such that $g(x) = 1$ for all $x \in C_\varepsilon$ and $g(x) = 0$ for all $x \in \overline{D}_\varepsilon$.

Next we consider the continuous function $F : G \to C$ defined by $F(x) = f(x) \cdot g(x)$ for all $x \in G$. It is obvious that $F(x) = f(x)$ for all $x \in C_\varepsilon$ and $F(x) = 0$ for all $x \in \overline{D}_\varepsilon$.

We obtain that $F(x) = 0$ for all $x \in D_\varepsilon = G\backslash(G\backslash D_\varepsilon)$, and, since $G\backslash D_\varepsilon$ is an ergodic zero set, it follows that $F \in \mathcal{C}^0_{erg}(G)$.

On the other hand, for all $x \in G$, we have $| F(x) | \leq | f(x) |$ and then it follows that

$$| f(x) - F(x) | < \varepsilon$$

for all $x \in G$.

Corollary 3.1. *The set $PAP(G)$ is the smallest C^*-subalgebra of the C^*-algebra $\mathcal{C}(G)$ containing the characters of the group G and the functions of the algebra $\mathcal{C}^0_{erg}(G)$.*

Proof. Let A be a C^*-subalgebra of the C^*-algebra $\mathcal{C}(G)$, containing the characters of the group G and the functions of the algebra $\mathcal{C}^0_{erg}(G)$. It is known that

$$\overline{\langle \hat{G} \rangle} = AP(G),$$

where $\overline{\langle \hat{G} \rangle}$ is the closure of the set $\langle \hat{G} \rangle$ with respect to the supremum norm ([1]); combining this fact with the hypothesis that A is a Banach algebra, we obtain the inclusion $AP(G) \subset A$. Now, applying Theorem 3.2, it follows that

$$\overline{\mathcal{C}^0_{erg}(G)} = PAP_0(G),$$

where $\overline{\mathcal{C}^0_{erg}(G)}$ is the closure of the set $\mathcal{C}^0_{erg}(G)$ with respect to the supremum norm and, in view of the same property of the algebra A, we conclude that $PAP_0(G) \subset A$. So, $PAP(G) \subset A$.

Remark 3.2. We notice that the space $PAP(G)$ is the direct sum of the spaces $\overline{\langle\hat{G}\rangle}, \overline{\mathcal{C}^0_{erg}(G)}$ and we denote that by:

$$PAP(G) = \overline{\langle\hat{G}\rangle} \oplus \overline{\mathcal{C}^0_{erg}(G)}$$

with respect to the supremum norm.

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