ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIV, s.I.a, Matematică, 1998, f1

ON AKTIVIS ALGEBRA OF BOL–MOUFANG TYPE

BY

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Abstract. We study algebras satisfying the Bol–Moufang type identities and obtain algebraic relations which are functions of the Jacobian in the identities. We also obtain invariant forms which are the analogue of the invariant killing forms for Lie algebras for some Bol–Moufang type algebras.

Introduction. Lie algebra, Malcev algebra and Aktivis algebra have been studied by authors like M a l c e v [5], S a g l e [7], H o f m a n n and S t r a m b a c h [3], A k t i v i s [1], and S o l a r i n [10], [11], [12]. The "non–associative"(local) Lie group are called (local) Lie loops or analytical (local) loops.

In this work, we study algebras satisfying identities of Bol–Moufang type. We obtain, for each studied identity, an algebraic relation similar to the Jacobian relation for a Lie algebra. It has been shown that algebras satisfying the Bol–Moufang type identities are anti–commutative algebras except for Bol and Moufang type identities. If in this two last cases the suplementary condition that the square of every element belongs to the right nucleus (left or middle nucleus) is satisfied then we obtain an anti–commutative algebra. Anticommutative algebras satisfy the modified Aktivis identity under the natural operations defined on them [11], [12]. Thus algebras satisfying Bol–Moufang type identities are natural generalization of Lie algebras with respect to the operations defined on them.

Associated with a (local) Lie loop, G is a tangent algebra $L(a) = = T_e(a)$ which carries a bilinear anti–commutative multiplication $(x, y) \longrightarrow [x, y]$ called the commutative bracket and a trilinear operation $(x, y, z) \longrightarrow \langle x, y, z\rangle$ called associator bracket vanishing identically in the case of a (local) Lie group. The two operations are linked by the Aktivis identity

$$\sum[sgn(\sigma).\sigma\langle x,y,z\rangle : \sigma \in S_3] = [[x,y],z] + [[y,z],x] + [[z,x],y] \quad (1)$$

Algebras with a binary and a ternary operations satisfying (1) are called Aktivis algebras.

Definition 1.1. In Lie triple algebra is a vector space together with an anti–commutative binary operation $(x,y) \longrightarrow [x,y]$ and a ternary operation $(x,y,z) \longrightarrow \langle x,y,z\rangle$ which are linked by several algebraic conditions. A Lie algebra A satisfies the condition that

$$xy = -yx \quad \text{and} \quad xy.z + yz.x + zx.y = 0 \quad \text{for all } x,y,z \in A.$$

Definition 1.2. The following identities are called Bol–Moufang type identities and loops and algebras satisfying them are called Bol–Moufang loops and Bol–Moufang algebras:

(i)	$x^2.yz = x^2y.z;$	(vii)	$(xy.z)x = x(y.zx);$
(ii)	$yx^2.z = y.x^2z;$	(viii)	$(xy.z)y = x(y.zy);$
(iii)	$yz.x^2 = y.zx^2;$	(ix)	$(xy.y)z = x(y.yz);$
(iv)	$(yx.x)z = y.x^2z;$	(x)	$(x.xy)z = x(xy.z);$
(v)	$y(x.xz) = yx^62z;$	(xi)	$(xy.z)z = x(yz.z);$
(vi)	$(xy.z)y = x(yz.y);$	(xii)	$(y.zy)x = y(z.yx);$

(called LS, S, RS, RM, LM, Bol, Extra, Moufang, C, LC, RC, and left Bol identity respectively).

Definition 1.3. A multilinear form $f(x_1,x_2,...,x_n)$ on A which satisfies the condition

$$\sum[\sigma f(x_1,x_2,...,x_n) : \sigma \in S_n] = 0$$

will be called an invariant form on A.

Definition 1.4. An algebra $(E,(\cdot,\cdot),\langle\cdot,\cdot,\cdot\rangle)$ over a field F with a bilinear map $(x,y) \longrightarrow [x,y]$ and a trilinear map $(x,y,z) \longrightarrow \langle x,y,z\rangle$ is called a modified Aktivis algebra if and only if

(i) $(x,y) = -(y,x)$

(ii) $\sum[sgn(\sigma).\sigma(x,y,z) : \sigma \in S_3] = k[((x,y),z)+((y,z),x)+((z,x),y)] \quad (2)$

where k is an element in F.

Lemma 1.1. Extra, C, LC, RC, RS, S, LS, RM, LM *algebras are anti–commutative* [11], [12], [13].

Lemma 1.2. *An algebra satisfying the Moufang or Bol identities such that the square of every element in its left (right, middle) nucleus is an anti–commutative algebra* [11].

Lemma 1.3. *An anti–commutative algebra E is a modified Aktivis algebra under its natural operations* [11].

Lemma 1.4. *If A is an algebra satisfying the*

(i) *Extra identity* $(xy.z)x = x(y.zx)$ *then*

$$(xy.z)w - x(y.zw) - x(w.zx) + (wy.z)x = 0 \tag{1a}$$

(ii) *Moufang identity* $(xy.z)y = x(y.zy)$ *then*

$$(xy.z)w - x(y.zw) - x(w.zy) + (xw.z)y = 0 \tag{1b}$$

(iii) C*–identity* $(xy.yz) = x(y.yz)$ *then*

$$(xy.z)w - x(y.zw) - x(z.yw) + (x.zy)w = 0 \tag{1c}$$

(iv) LC*–identity* $(xx.y)z = x(x.yz)$ *then*

$$(xy.z)w - x(y.zw) - y(x.zw) + (yx.z)w = 0 \tag{1d}$$

(v) RC*–identity* $(xy.z)z = x(y.zz)$ *then*

$$(xy.z)w - x(y.zw) - x(y.wz) + (xy.w)z = 0 \tag{1e}$$

(vi) *Bol–identity* $(xy.z)y = x(yz.y)$ *then*

$$(xy.z)w + (xw.z)y - x(wz.y) - x(yz.w) = 0 \tag{1f}$$

(vii) RM*–identity* $(yx.x)z = y.x^2z$ *then*

$$(yx.w)z + (yw.x)z - y(xw.z) - y(wx.z) = 0 \tag{1g}$$

(viii) LM*–identity* $y(x.xz) = yx^2.z$ *then*

$$z(y.xw) + z(x.yw) - (z.yx)w - (z.xy)w = 0 \tag{1h}$$

(ix) LS–*identity* $x^2.yz = x^2y.z$ *then*

$$(xw.y)z - xw.yz + (wx.y)z - wx.yz = 0 \tag{1i}$$

(x) RS–*identity* $yz.x^2 = y.zx^2$ *then*

$$yz.xw + yz.wx - y(z.xw) - y(z.wx) = 0 \tag{1j}$$

(xi) S–*identity* $yx^2.z = y.x^2z$ *then*

$$(y.xw)z + (y.wx)z - y(xw.z) - y(wx.z) = 0 \tag{1k}$$

(vii) *Left Bol identity* $(y.zy)x = y(z.yx)$ *then*

$$(w.zy)w + (y.zw)x - w(z.yx) - y(zwx) = 0 \tag{1l}$$

Proof. The results are easily obtained by linearizing each identity [10], [11], [12], [13].

Main results. In this section the following notations are used:

$$\begin{aligned}
J(x,y,z) &= xy.z + yz.x + zx.y \\
J^R(x,y,z) &= R_xR_yR_z + R_yR_zR_x + R_zR_xR_y \\
J_R(x,y,z) &= R_{xy}R_z + R_{yz}R_x + R_{zx}R_y \\
J^L(x,y,z) &= L_xL_yL_z + L_yL_zL_x + L_zL_xL_y \\
J_L(x,y,z) &= L_xL_{yz} + L_yL_{zx} + L_zL_{xy} \\
G(x,y,z) &= x.yz + y.zx + z.xy
\end{aligned}$$

Theorem 2.1. *If A is an algebra over a field F, satisfying the*

(i) *Extra identity then*

$$J^R(x,y,z) - R_{G(x,y,z)} + L_{J(x,y,z)} - J^L(x,y,z) = 0$$

(ii) *Moufang identity then*

$$J^R(x,y,z) - R_{G(x,y,z)} + J^R(x,z,y) - R_{G(x,z,y)} = 0$$

(iii) C–*identity then*

$$J^R(x,y,z) - R_{G(x,y,z)} + J^R(x,z,y) - R_{G(x,z,y)} = 0$$

(iv) LC–*identity then*

$$J^L(x,y,z) - L_{J(x,y,z)} + J^L(x,z,y) - L_{J(x,z,y)} = 0$$

(v) RC–*identity then*

$$J^R(x,y,z) - R_{G(x,y,z)} + J^R(x,z.y) - R_{G(x,z,y)} = 0$$

(vi) *Bol identity then*

$$J^R(x,y,z) - RJ(x,y,z) + J^R(x,z,y) - R_{J(x,z,y)} = 0$$

(vii) RM–*identity then*

$$J^R(x,y,z) - R_{J(x,y,z)} + J^R(x,z,y) - R_{J(x,z,y)} = 0$$

(viii) LM–*identity then*

$$J^L(x,y,z) - L_{G(x,y,z)} + J^L(x,z,y) - L_{G(x,z,y)} = 0$$

(ix) LS–*identity then*

$$L_{J(x,y,z)} - J_L(x,y,z) + L_{J(x,z,y)} - J_{L(x,z,y)} = 0$$

(x) RS–*identity then*

$$J_R(x,y,z) - R_{G(x,y,z)} + J_R(x,z,y) - R_{G(x,z,y)} = 0$$

(xi) S–*identity then*

$$J_R(x,y,z) - R_{J(x,y,z)} + J_R(x,z,y) - R_{J(x,z,y)} = 0$$

(xii) *Left–Bol identity then*

$$J^{L(x,y,z)} - L_{G(x,y,z)} + J_L(x,z,y) - R_{G(x,z,y)} = 0$$

Proof. The procedure for establishing the result for each identity is the same, we therefore prove for one case only.

(iv) Let A satisfy the Bol identity. Then by $(1f)$ we obtain

$$xR_wR_zR_y + xR_yR_zR_w - xR_{wz.y} - xR_{yz.w} = 0$$

This implies that

$$R_yR_zR_x + R_xR_zR_y - R_{yz.x} - R_{xz.y} = 0 \tag{2a}$$

for all x, y, z in A. Substituting permutations (x, y, z) and (z, x, y) in $(2a)$ we get

$$R_xR_yR_z + R_zR_yR_x - R_{xy.z} - R_{zy.x} = 0 \tag{2b}$$

$$R_zR_xR_y + R_yR_xR_z - R_{zx.y} - R_{yx.z} = 0 \tag{2c}$$

Adding $(2a)$, $(2b)$ and $(2c)$ together gives

$$(R_xR_yR_z + R_yR_zR_x + R_zR_xR_y) + (R_xR_zR_y + R_zR_yR_x + R_yR_xR_z)$$

$$-(R_{xy.z} + R_{yz.x} + R_{zx.y}) - (R_{xz.y} + R_{zy.x} + R_{yx.z}) = 0$$

this implies that

$$J^R(x, y, z) + J^R(x, z, y) - R_{J(x,y,z)} - R_{J(x,z,y)} = 0$$

Hence

$$J^R(x, y, z) - R_{J(x,y,z)} + J^R(x, z, y) - R_{J(x,z,y)} = 0 \tag{3}$$

Remark that the results for some identities are identical vis: Moufang, C– and RC–identities yield identical results; similarly Bol and RM–identity; left Bol and LM–identities.

The rest of this section is devoted to obtain the analogue of the invariant killing forms for Lie algebras for some of the Bol–Moufang type algebras.

Corollary 2.1. *Let A be an algebra over a field F, satisfying the Bol identity. Then*

$$f(x, y, z) \equiv trace\ (J^R(x, y, z) - R_{J(x,y,z)})$$

is an invariant trace form of A.

Proof. Since A satisfies the Bol identity then (3) holds, therefore

$$trace\ (J^R(x, y, z) - R_{J(x,y,z)} + J^R(x, z, y) - R_{J(x,z,y)}) = 0$$

This implies that

$$trace\ (J^R(x, y, z) - R_{J(x,y,z)}) + trace\ (J^R(x, z, y) - R_{J(x,z,y)}) = 0$$

This is $f(x,y,z)+f(x,z,y)=0$ for all x,y,z in A.

Following the same procedure as in Corollary 2.1 $f(x,y,z)$ can be defined for all the twelve Bol–Moufang identities.

Theorem 2.2. *Let A be an algebra over a field F, satisfying respectively*

(i) *Bol identity and consider*

$$f(x,y,z) \equiv trace\ R_{x[R_y,R_z]} = trace\ R_{[y,z]x}$$

(ii) *Left Bol identity and consider*

$$f(x,y,z) \equiv trace\ (L_{y[L_z,L_x]}) = trace\ (L_{y[x,z]})$$

(iii) C*–identity and consider*

$$f(x,y,z) \equiv trace\ (R_{x[y,z]}) = trace\ (R_{x(R_yR_z-R_zR_y)}).$$

Then $f(x,y,z)$ is an invariant trace form on A for the respective identities.

Proof. The procedure for establishing the results for each of the identities is the same, we shall therefore establish for one case only.

(i) Let A satisfy the Bol identity then subtracting $(2a)$ from $(2b)$ and manipulating we get

$$[R_x,[R_y,R_z]] - R_{[z,y]x} - R_{x[R_y,R_z]} = 0 \tag{4}$$

Since trace of a commutator is zero, then (4) gives

$$trace\ (R_{x[R_y,R_z]}) = trace\ R_{[z,y]x}.$$

Let

$$f(x,y.z) \equiv trace\ (R_{x[R_y,R_z]}).$$

Then $f(x,y,z)+f(x,z,y)=0$ for all x,y,z in A.

This reseach is supported by University Research Council of Obafemi Awolowo University, Ile–Ife. Grant 1423LS

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Received: 1.XI.1995
Revised: 30.XII.1997

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