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ASYMPTOTIC BEHAVIOUR AND GROWTH PROPERTIES OF DIFFERENTIAL SYSTEMS

BY

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1. Introduction. The nonlinear variation of parameters formula has been useful in the study of nonlinear perturbation problems in ordinary differential equations. Various stability and asymptotic behaviour problems have been solved in [2], [3], [4], [8] by employing this method. After the defining of the concept of L^p–stability by Strauss [9], Morchalo [8] extended it to so– called L^p–stability in variation and has studied it using the above menthod. Our approach is based on the Alekseev formula [1], [6, p.78], [7, p.82] and the theory of integral inequalities to obtain some results on the asymptotic behaviour and growth properties of solutions of nonlinear perturbed systems.

2. Notation and general setting. The systems which we study are

$$x' = f(t, x) \tag{2.1}$$

and its perturbed

$$y' = f(t, y) + g(t, y). \tag{2.2}$$

In (2.1) and (2.2), x, y, f and g are n–vectors. We assume that f and g are continuous form $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$ where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Furthemore, we assume that the Jacobian matrix f'_x is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$. The solution of the initial value problem which consists of the system (2.1) and the point (t_0, x_0) will be denoted by $x(t; t_0, x_0)$. Similarly, $y(t; t_0, x_0)$ will denote a solution of (2.2) which passes

through the point (t_0, y_0). The symbol $|\cdot|$ will denote some convenient norm on $\mathbb{R}^n$ as well as a corresponding consistent matrix norm.

If $x(t; t_0, x_0)$ is a solution of system (2.1), the variational system of (2.1) associated with $x(t; t_0, x_0)$ is

$$z' = f'_x(t, x(t; t_0, x_0)) \cdot z. \tag{2.3}$$

We denote by $\phi(t, t_0, x_0)$ the fundamental solution matrix of (2.3) that is $n \times n$ identity matrix at $t = t_0$.
The nonlinear variation of parameters formula can be written as

$$y(t; t_0, x_0) = x(t; t_0, x_0) + \int_{t_0}^{t} \phi(t, s, y(s; t_0, x_0)) \cdot g(s, y(s; t_0, x_0))ds. \tag{2.4}$$

For details, see [1], [3], or [6, p.78].
We note that under the above hypotheses on $f(t, x)$, for two solutions of (2.1), $x(t; t_0, x_1)$ and $x(t; t_0, x_2)$ we have [6, Theorem 2.6.4]

$$x(t; t_0, x_2) - x(t; t_0, x_1) = [\int_0^1 \phi(t; t_0, x_1 + s(x_2 - x_1))ds](x_2 - x_1). \tag{2.5}$$

A noteworthy particular case of (2.5) is obtained for $f(t, 0) \equiv 0$, namely

$$x(t; t_0, x_0) = [\int_0^1 \phi(t, t_0, sx_0)ds] \cdot x_0, \quad \text{for} \quad t \geq t_0. \tag{2.6}$$

We now give the definitions of different kinds of stability and of the concept of slowly–grawing, in terms of the behaviour of solutions of (2.1) and of the variational system (2.3).

Definition 2.1. *The solution $x = 0$ of (2.1) is said to be:*

a) Uniformly stable (US) if for every $\varepsilon > 0$ and any $t_0 \geq 0$ there exists $\delta(\varepsilon) > 0$ such that $|x_0| < \delta$ and $t \geq t_0$ imply $|x(t, t_0, x_0)| < \varepsilon$;

b) Uniformly $L^p_{\omega,\sigma}$–stable ($UL^p_{\omega,\sigma} - S$) if it is (US) and for any solution $x(t; t_0, x_0)$, $|x_0| < H \leq \infty$ we have $\sup_{t \geq t_0}[\omega(t)^{-1} \int_{t_0}^{t} \sigma(s)|x(s; t_0, x_0)|^p ds] < \infty$, $p \geq 1$, where $\omega : [t_0, \infty) \to \mathbb{R}_+$ is a monotone nondecreasing function and $\sigma : [t_0, \infty) \to \mathbb{R}_+$;

c) Uniformly stable in variation (USV) if there exists constant $M \geq 1$ *such that* $|\phi(t, t_0, x_0)| \leq M$ *for all* $t \geq t_0 \geq 0$ *whenever* $|x_0| < H \leq \infty$*;*

d) Uniformly $L^p_{\omega,\sigma}$*–stable in variation* $(UL^p_{\omega,\sigma} - SV)$ *if it is (USV) and for any solution* $x(t; t_0, x_0)$, $|x_0| < H \leq \infty$, *we have*

$$\sup_{t \geq t_0}[\omega(t)^{-1} \int_{t_0}^{t} \sigma(s)|x(s; t_0, x_0)|^p ds] < \infty;$$

e) Exponentially asymptotically stable in variation (EASV) if there exist constants $L \geq 1$ *and* $\lambda > 0$ *such that* $|\phi(t, t_0, x_0)| \leq Le^{-\lambda(t-t_0)}$ *for all* $t \geq t_0 \geq 0$, *whenever* $|x_0| < H \leq \infty$*;*

f) Generalized uniformly slowly–growing in variation (GUSGV) if there exists a constant $L \geq 1$ *and a continuous function* $\alpha : I\!R_+ \to I\!R$ *such that* $|\phi(t, t_0, x_0)| \leq L \exp(\int_{t_0}^{t} \alpha(\tau) d\tau)$ *for all* $t \geq t_0 \geq 0$, *whenever* $|x_0| < H \leq \infty$*;*

Remark 2.1. We note that the space $L^p_{\omega,\sigma}([t_0, \infty))$ of all functions $x : [t_0, \infty) \to I\!R^n$ satisfying the condition $||x||_{p,\omega,\sigma} \equiv \sup_{t \geq t_0}[\omega(t)^{-1} \int_{t_0}^{t} \sigma(s) |x(s)|^p ds] < \infty$, for given as above functions ω and σ, is a normed linear space with the norm $||x||_{p,\omega,\sigma}$. Observe that for bounded functions ω and σ te space $L^p_{\omega,\sigma}([t_0, \infty)) \equiv L^p([t_0, \infty))$, where $L^p([t_0, \infty))$ corresponds to the functions $\omega(t) \equiv \sigma(t) \equiv 1$. When $\sigma \equiv 1$ we shall denote $L^p_{\omega,\sigma}([t_0, \infty)) \equiv L^p_\omega([t_0, \infty))$.
Adequately, we shall have the concepts of $(UL^p_\omega - S)$, $(UL^p - S)$, $(UL^p_\omega - SV)$ and $(UL^p - SV)$.

Remark 2.2. Note that (GUSGV) becomes (EASV) when $\alpha(t) = -\lambda$, where λ is a positive constant.

3. Relatioships between $(UL^p_{\omega,\sigma} - SV)$ **and** $(UL^p_{\omega,\sigma} - S)$**.** This section is devoted to the relationship between the $(UL^p_{\omega,\sigma} - SV)$of the zero solution of system (2.1) and the $(UL^p_{\omega,\sigma} - S)$ of the zero solution of system (2.2) for $p \geq 1$. At the same time will be given conditions assuring the belonging to the space $L^p_\omega([t_0, \infty))$ of all solutions of system (2.2).
Now we can formulate

Theorem 3.1. *Assume that the solution* $x = 0$ *of (2.1) is* $(UL^1_{\omega,\sigma} - SV)$ *and that*

$$|g(t, y)| \leq \sigma(t)|y| \text{ for } t \geq t_0 \geq 0 \text{ and } |y| < H \leq \infty, \text{ where} \tag{3.1}$$

$$\sigma(t) \in L^1([0, \infty)).$$

Then the zero solution of (2.2) is $(UL^1_{\omega,\sigma} - S)$.

Proof. We shall prove first that the solution $y = 0$ of (2.2) is (US). Indeed, for any $0 < \varepsilon < H$ let $\delta(\varepsilon) < \varepsilon/Me^{KM}$ and $|x_0| < \delta(\varepsilon)$, where $K = \int_0^\infty \sigma(t)dt$.
Suppose by contrary, that there exists $t_1 \geq t_0$ such that $|y(t_1; t_0, x_0)| = \varepsilon$ and $|y(t; t_0, x_0)| < \varepsilon$ on $[t_0, t_1)$.
By variation of constants formula and by using (2.6) and (3.1) we have

$$|y(t;t_0,x_0)| \leq |x(t;t_0,x_0)| + M\int_{t_0}^{t} \sigma(s)|y(s;t_0,x_0)| \leq \tag{3.2}$$

$$\leq M|x_0| + M\int_{t_0}^{t} \sigma(s)|y(s;t_0,x_0)|ds,$$

on $(t_0, t_1]$.
From here, by applying the Gronwall inequality, one obtains

$$|y(t;t_0,x_0)| \leq M|x_0|\exp[M\int_{t_0}^{t}\sigma(s)ds] \leq M|x_0|\exp(MK) \leq$$
$$\leq M\delta(\varepsilon)\exp(MK), \text{ on } [t_0,t_1].$$

Therefore, $|y(t_1; t_0, x_0)| < \varepsilon$ which is a contradiction. Thus the zero solution of (2.2) is (US).

To show that $\sup_{t\geq t_0}[\omega(t)^{-1}\int_{t_0}^{t}\sigma(s)|y(s;t_0,x_0)|ds] < \infty$ whenever $|x_0| < H \leq \infty$, we shall use (3.2) from where

$$\int_{t_0}^{\tau}\sigma(t)|y(t;t_0,x_0)|dt \leq \int_{t_0}^{\tau}\sigma(t)|x(t;t_0,x_0)|dt+$$
$$+M\int_{t_0}^{\tau}\sigma(t)\int_{t_0}^{t}\sigma(s)|y(s;t_0,x_0)|dsdt =$$
$$= \omega(\tau)\omega(\tau)^{-1}\int_{t_0}^{\tau}\sigma(t)|x(t;t_0,x_0)|dt+$$
$$+M\int_{t_0}^{\tau}[\sigma(t)\omega(t)\omega(t)^{-1}\int_{t_0}^{t}\sigma(s)|y(s;t_0,x_0)|ds]dt,$$

for an arbitrary $\tau \geq t_0$.
Then, from the monotony of $\omega(t)$ and the definition of the $(UL_{\omega,\sigma} - SV)$ of the solution $x = 0$ it follows

$$\int_{t_0}^{\tau} \sigma(t)|yt; t_0, x_0)|dt \leq \overline{K}\omega(\tau) + \omega(\tau)M \int_{t_0}^{\tau} \sigma(t)\omega(t)^{-1} \int_{t_0}^{t} \sigma(s)|y(s; t_0, x_0)|dsdt,$$

where $\overline{K} = \sup_{\tau \geq t_0} [\omega(\tau)^{-1} \int_{t_0}^{\tau} \sigma(t)|x; t_0, x_0)|dt]$.
From here, by Gronwall's inequality we get

$$\omega(\tau)^{-1} \int_{t_0}^{\tau} \sigma(t)|y(t; t_0, x_0)|dt \leq K \exp(M \int_{t_0}^{\tau} \sigma(t)dt \leq K \exp(KL),$$

where $L = \int_0^{\infty} \sigma(t)dt < \infty$.
Consequently, $\sup_{\tau \geq t_0} [\omega(\tau)^{-1} \int_{t_0}^{\tau} \sigma(t)|y(t; t_0, x_0)|dt] < \infty$, i.e. the zero solution of (2.2) is $(UL^1_{\omega,\sigma} - S)$.

Corollary 3.1. *Assume that the solution $x = 0$ of (2.1) is $(UL^1_{\omega} - SV)$ with $\omega(t) \geq 1$, and that*

(3.3) $|g(t, y)| \leq a(t)|y|^m$, $0 < m < 1$, *for all $t \geq t_0 \geq 0$ and $|y| < H \leq \infty$.*

Furthemore, we assume that $v(t) \equiv [\int_0^t a(s)^r dr]^{1/r} \in L^1([0, \infty))$, $r = 1/(1 - m)$. Then all solutions of (2.2) belong to $L^1_{\omega}([t_0, \infty))$.

Proof. By similar arguments used in the proof of Theorem 3.1 we have

$$|y(t; t_0, x_0)| \leq |x(t; t_0, x_0)| + M[\int_{t_0}^{t} a(s)^r ds]^{1/r} (\int_{t_0}^{t} |y(s; t_0, x_0)|ds)^m.$$

From here and the monotony of $\omega(t)$ for arbitrary $\tau \geq t_0 \geq 0$ one obtains

$$\int_{t_0}^{\tau} |y(t; t_0, x_0)|dt \leq \omega(\tau)\omega(\tau)^{-1} \int_{t_0}^{\tau} |x(t; t_0, x_0)|dt + M\omega(\tau) \int_{t_0}^{t} v(t)\cdot$$

$$\cdot[\omega(t)^{-1}\int_{t_0}^{t}|y(s;t_0,x_0)|ds]^m dt \leq$$

$$\leq K\omega(\tau)+M\omega(\tau)\int_{t_0}^{\tau}v(t)[\omega(t)^{-1}\int_{t_0}^{t}|y(s;t_0,x_0)|ds]^m dt,$$

where $K=\sup_{\tau\geq t_0}[\omega(\tau)^{-1}\int_{t_0}^{\tau}|x(t;t_0,x_0)|ds]$.

Using Gronwall's inequality [7, Th.13.4, Ex.1.3.1] we get

$$[\omega(\tau)^{-1}\int_{t_0}^{\tau}|y(t;t_0,x_0)|dt]\leq[K^{1-m}+(1-m)M\int_{t_0}^{\tau}v(t)dt],\ \tau\geq t_0\geq 0,$$

from where it follows the formulated result.

Theorem 3.2. *Assume that the solution $x=0$ of (2.1) is $(UL^p_{\omega,\sigma}-SV)$ and that the perturbation $g(t,y)$ satisfies*

(3.4) $|g(t,y)|\leq a(t)|y|$ *for* $t\geq t_0\geq 0$ *and* $|y|<H\leq\infty$, *where* $a(t)\in L^1([0,\infty))$

and, in addition, together with $\sigma(t)$ satisfying

(3.5) $v(t)\equiv\sigma(t)\cdot[\int_{t_0}^{t}a(s)^q\cdot\sigma(s)^{-q/p}ds]^{p/q}\in L^1([t_0,\infty)),\ p^{-1}+q^{-1}=1.$

Then the zero solution of (2.2) is $(UL^p_{\omega,\sigma}-S)$.

Proof. According to the definition of the $(UL^p_{\omega,\sigma}-SV)$ and (3.4), the solution $y=0$ is (US) by the same argument as in the proof of Theorem 3.1.

To show that $\sup_{\tau\geq t_0}[\omega(\tau)^{-1}\int_{t_0}^{\tau}\sigma(s)|y(s;t_0,x_0)|^p ds]<\infty$ whenever $|x_0|<H\leq\infty$, we shall use the variation of constants formula and (3.4) to obtain

$$|y(t;t_0,x_0)|^p\leq 2^{p-1}|x(t;t_0,x_0)|^p+2^{p-1}M^p(\int_{t_0}^{t}a(s)|y(s;t_0,x_0)|ds)^p\leq$$

$$\leq 2^{p-1}|x(t;t_0,x_0)|^p + 2^{p-1}M^p(\int_{t_0}^{t} a(s)\sigma(s)^{-1/p}\sigma(s)^{1/p}|y(s;t_0,x_0)|ds)^p \leq$$

$$2^{p-1}|x(t;t_0,x_0)|^p + 2^{p-1}M^p \int_{t_0}^{t} a(s)^q\sigma(s)^{-q/p}ds)^{p/q} \int_{t_0}^{t} \sigma(s)|y(s;t_0,x_0)|)^p ds.$$

From here we get

$$\sigma(t)|y(t;t_0,x_0)|)^p \leq 2^{p-1}\sigma(t)|x(t;t_0,x_0)|^p + 2^{p-1}\cdot M^p v(t)\cdot$$

$$\int_{t_0}^{t} \sigma(s)\cdot|y(s;t_0,x_0)|)^p ds.$$

For arbitrary $\tau \geq t_0 \geq 0$, one obtains

$$\int_0^{\tau} \sigma(t)|y(t;t_0,x_0)|)^p dt \leq 2^{p-1}\omega(\tau)\omega(\tau)^{-1} \int_{t_0}^{\tau} \sigma(t)|x(t;t_0,x_0)|)^p dt +$$

$$+2^{p-1}M^p\omega(\tau) \int_{t_0}^{\tau} v(t)\omega(t)^{-1} \int_0^{t} \sigma(s)|y(s;t_0,x_0)|^p ds dt \leq$$

$$\leq K\omega(\tau) + 2^{p-1}M^p\omega(\tau) \int_{t_0}^{\tau} v(t)\omega(t)^{-1} \int_{t_0}^{t} \sigma(s)|y(s;t_0,x_0)|^p ds dt,$$

where $K = \sup_{\tau\geq t_0}[\omega(\tau)^{-1}\int_{t_0}^{\tau}\sigma(t)|x(t;t_0,x_0)|^p dt]2^{p-1} < \infty$.

By Gronwall's inequality it follows $\omega(\tau)^{-1}\int_{t_0}^{\tau}\sigma(t)|y(t;t_0,x_0)|^p dt \leq$

$K\cdot\exp[2^{p-1}M^p\int_{t_0}^{\tau}\sigma(t)v(t)dt] \leq K\exp[2^{p-1}M^pL]$, for any $\tau \geq t_0 \geq 0$, where $L = \int_0^{\infty} v(t)dt < \infty$. Therefore the zero solution of (2.2) is $(UL^p_{\omega,\sigma} - S)$.

Corollary 3.2. *Assume that the solution $x = 0$ of (2.1) is $(UL^p_{\omega,\sigma} - SV)$ with $\omega(t) \geq 1$, and that (3.3) holds. Furthemore we assume that $v(t) \equiv \sigma(t)[\int_{t_0}^{t} a(s)^r \cdot \sigma(s)^{-mr/p} ds]^{1/r} \in L^1([0,\infty))$, $r = p/(p-m)$. Then all solutions of (2.2) belong to $L^p([0,\infty))$.*

Proof. We only note that, this time, we have

$$|y(t;t_0,x_0)|^p \leq 2^{p-1}|x(t;t_0,x_0)|^p + 2^{p-1}M^p[\int_{t_0}^{t} a(s)\cdot\sigma(s)^{-m/p}\cdot\sigma(s)^{m/p}\cdot$$

$$\cdot|y(s;t_0,x_0)|^m ds]^p \leq 2^{p-1}|x(t;t_0,x_0)|^p + 2^{p-1}M^p[\int_{t_0}^{t} a(s)^r\sigma(s)^{-mr/p}ds]^{1/r}$$

$$[\int_{t_0}^{t}\sigma(s)|y(s;t_0,x_0)|^p]ds]^m.$$

Further, a reasoning similar to one used to prove Theorem 3.2 will be handled.

4. Relationship between (GUSGV) and $(UL^p - S)$. In this section we study the relationship between the (GUSGV) of the zero solution of system (2.1) and the $(UL^p - S)$ of the zero solution of system (2.2). At the same time well shall establish when all solutions of (2.2) belong to $L^p([t_0,\infty))$.
We recall that the solution $x = 0$ of (2.1) is (GUSGV) if there exists a constant $L \geq 1$ and a continuous function $\alpha : [t_0,\infty) \to I\!R$ such that
(4.1)

$$|\phi(t;t_0,x_0)| \leq L\exp(\int_{t_0}^{t}\alpha(\tau)d\tau) \text{ for all } t \geq t_0 \geq 0, \text{ whenever } |x_0| < H \leq \infty.$$

With respect to the function $\alpha(t)$ we make the assumption that there exists a positive constant$K$ such that if $t \geq t_0 \geq 0$, then

$$\int_{t_0}^{t}\exp(\int_{s}^{t}\alpha(\tau)d\tau)ds \leq K. \tag{4.2}$$

We remark that (4.2) and Lemma 1 [5, p.68] yields to

$$\lim_{t\to\infty} \exp\Big(\int_{t_0}^{t} \alpha(\tau)d\tau\Big) = 0. \tag{4.3}$$

Now we are able to formulate

Theorem 4.1. *Assume that the solution $x = 0$ of (2.1) is (GUSGV) and that (4.2) and (3.4) hold with $\sup_{t\geq 0} a(t) \leq \gamma < (KL)^{-1}$. Furthemore we assume that*

$$\exp\Big(p\int_{t_0}^{t} \alpha(\tau)d\tau\Big) \in L^1([t_0,\infty)), \tag{4.4}$$

$$\int_{t_0}^{t} a(s)^q \exp\Big(-q\int_{t_0}^{t} \alpha(\tau)d\tau\Big)ds \leq K_1 \text{ for all } t \geq t_0 \geq 0,\ p^{-1}+q^{-1}=1. \tag{4.5}$$

Then the zero solution of (2.2) is $(UL^p - S)$.
Moreover, for every $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that for a solution $y(t; t_0, x_0)$ of (2.2) for which $|x_0| < \delta$ the relation $\int_{t_0}^{\infty} |y(s; t_0, x_0)|^p ds < \varepsilon$ holds.

Proof. We first prove the (US) of the zero solution of (2.2). From (4.3) there exists a positive constant N such that $\exp\int_0^t \alpha(\tau)d\tau \leq N$ for all $t \geq 0$.
For any $0 < \varepsilon < H$ and $t_0 \geq 0$ let $\delta = \delta(\varepsilon) < \min\{\frac{(1-\gamma KL)\varepsilon}{LN}, \varepsilon\}$. Consider the solution $y(t; t_0, x_0)$ of (2.2) such that $|x_0| < \delta$. Suppose, by contrary, that there exists $t_1 > t_0$ such that $|y(t_1; t_0, x_0)| = \varepsilon$ and $y(t; t_0, x_0) < \varepsilon$ on $[t_0, t_1)$.
For all $t \in [t_0, t_1)$ we have

$$|y(t; t_0, x_0)| \leq L\exp\Big(\int_{t_0}^{t} \alpha(\tau)d\tau\Big)\cdot|x_0| + L\int_{t_0}^{t}\Big[\exp\Big(\int_{s}^{t} \alpha(\tau)d\tau\Big)\Big]\cdot a(s)\cdot$$

$$\cdot|y(s;t_0,x_0)|ds < LN\delta + LK\gamma\varepsilon < (1-\gamma KL)\varepsilon + \gamma KL\varepsilon = \varepsilon.$$

Therefore $|y(t_1;t_0,x_0)| < \varepsilon$, which is a contraction. Thus the zero solution of (2.2) is (US). Then, by using a reasoning closed to that from the proof of Theorem 3.2 we obtain

$$|y(t;t_0,x_0)|^p \le 2^{p-1}L^p|x_0|\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau) + 2^{p-1}L^p\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau).$$

$$\cdot[\int_{t_0}^{t}\exp(-q\int_{t_0}^{s}\alpha(\tau)d\tau)\cdot a(s)^q ds]^{p/q}\cdot\int_{t_0}^{t}|y(s;t_0,x_0)|^p ds.$$

Taking into account (4.4) and (4.5) it follows that, for an arbitrary $u \ge t_0$, we have

$$\int_{t_0}^{u}|y(t;t_0,x_0)|^p dt \ge K_2|x_0|^p + K_3\int_{t_0}^{u}[\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)\cdot\int_{t_0}^{t}|y(s;t_0,x_0)|^p ds]dt,$$

where $K_2 = 2^{p-1}L^p\int_{t_0}^{\infty}\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)dt$, $K_3 = 2^{p-1}L^pK_1^{p/q}$.
In view of the Gronwall inequality we obtain the desired result.

Remark 4.1. By using (2.6) it is a simple matter to show that (4.2) and (4.4) imply the $(UL^p - SV)$ of the zero solution of (2.2).

Remark 4.2. Theorem 4.1 takes a particulary simple form when $\alpha(t) = -\lambda$, where λ is a positive constant. We note that in this case the (GUSGV) of the zero solution of (2.1) reduces to the its (EASV).

Corollary 4.3. *Assume that the solution $x = 0$ of (2.1) is (EASV), $\sup_{t\ge 0} a(t) \le \gamma < (KL)^{-1}$ and $\int_{t_0}^{t} a(s)^q e^{\lambda qs}ds \le K_1 < \infty$ for all $t \ge t_0 \ge 0$. Then the conclusion of Theorem 4.1 occur.*

Corollary 4.4. *Assume that the solution $x = 0$ of (2.1) is (GUSGV) and that (3.3) and (4.4) hold. Furthemore we assume that*

$$\int_{t_0}^{t} a(s)^r\exp(-r\int_{t_0}^{t}\alpha(\tau)d\tau)ds \le K_1 \text{ for all } t \ge t_0 \ge 0, \tag{4.6}$$

where $r = p/(p-m)$.

Then all solutions of (2.2) belong to $L^p([t_0,\infty))$.

Proof. By taking into account (2.4) and using Hölder's inequality we have

$$|y(t;t_0,x_0)|^p \le 2^{p-1}L^p|x_0|^p \exp(p\int_{t_0}^{t}\alpha(\tau)d\tau + 2^{p-1}L^p \exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)\cdot$$

$$\cdot[\int_{t_0}^{t} a(s)^r \exp(-r\int_{t_0}^{s}\alpha(\tau)d\tau)ds]^{p/r}\cdot[\int_{t_0}^{t}|y(s,t_0,x_0)|^p ds]^m.$$

From here, for an arbitrary $u \ge t_0$, one obtains

$$\int_{t_0}^{u}|y(t;t_0,x_0)|^p dt \le K_2|x_0|^p + K_3\int_{t_0}^{u}\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)\cdot[\int_{t_0}^{t}|y(s;t_0,x_0)|^p ds]^m dt,$$

where

$$K_2 = 2^{p-1}L^p\int_{t_0}^{\infty}\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)dt,\ K_3 = 2^{p-1}L^pK_1^{p/r} = 2^{p-1}L^pK_1^{p-m}.$$

An application of the Bihari inequality [7, Theorem 1.3.4] yields to

$$\int_{t_0}^{u}|y(t;t_0,x_0)|^p dt \le [(K_2|x_0|^p)^{1-m} + (1-m)K_3\int_{t_0}^{u}\exp(p\int_{t_0}^{t}\alpha(\tau)d\tau)dt]^{1/(1-m)},$$

from where one obtains the desired result.
We now preent an example which satisfies all conditions from Theorem 4.1.

Example 4.1. Let us consider the scalar differential equation

$$x' = -(2t+\frac{1}{t})x - t^2x^3,\ x_(t_0) = x_0,\ t_0 \ge 2^{-1}, \tag{4.7}$$

and its corresponding perturbation equation

$$y' = -(2t+\frac{1}{t})y - t^2y^3 + (e^{-2t^2-y^2}\cdot y\sin y)/2(1+t)(1+y^2). \tag{4.8}$$

The solution of (4.7) is given by

$$x(t;t_0,x_0)=e^{-\int\limits_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau}\cdot x_0[1+x_0^2t_0(1-e^{-(t^2-t_0^2)}]^{-1/2}$$

and

$$\phi(t,t_0,x_0)=e^{-\int\limits_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau}\cdot[1+x_0^2t_0(1-e^{-(t^2-t_0^2)})]^{-3/2}\leq e^{-\int\limits_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau}$$

for all $t\geq t_0\geq 0$ and $x_0\in I\!R$.
We note that all conditions of the Theorem 4.1 occur with $L=K=1,\ p=q=2,\ a(t)=\frac{e^{-2t^2}}{2}(1+t)^{-1},\ \sup\limits_{t\geq 0}a(t)<1$. Consequently, the zero solution of (4.8) is uniformly L^2 stable.
In special case $p=1$ the correspoding result takes the following form

Theorem 4.2. *Assume that the solution $x=0$ of (2.1) is (GUSGV) and that (4.2) and (3.4) hold with $\sup\limits_{t\geq 0}a(t)\leq\gamma<(KL)^{-1}$. Furthermore we assume that*

$$\exp(\int\limits_{t_0}^{t}\alpha(\tau)d\tau)\in L^1([t_0,\infty)), \tag{4.9}$$

$$a(t)\cdot\exp(-\int\limits_{t_0}^{t}\alpha(\tau)d\tau)\leq K_1<\infty. \tag{4.10}$$

Then the zero solution of (2.2) is (UL^1-S). Moreover, for every $\varepsilon>0$ there exists $\delta(\varepsilon,t_0)>0$ such that for a solution $y(t;t_0,x_0)$ of (2.2) for which $|x_0|<\delta$ the relation $\int\limits_{t_0}^{\infty}|y(t;t_0,x_0)|dt<\varepsilon$ holds.

Proof. The uniform stability of the zero solution of (2.2) follows by the same arguments made in the proof of Theorem 4.1. Then we have

$$|y(t;t_0,x_0)|\leq L\exp(\int\limits_{t_0}^{t}\alpha(\tau)d\tau)|x_0|+L\exp(\int\limits_{t_0}^{t}\alpha(\tau)d\tau)\cdot$$

$$\cdot \int_{t_0}^{t} \exp(-\int_{t_0}^{s} \alpha(\tau)d\tau) \cdot a(s) \cdot |y(s;t_0,x_0)|ds.$$

From here, for an arbitrary $u \geq t_0 \geq 0$, one obtains

$$\int_{t_0}^{u} |y(t;t_0,x_0)|dt \leq K_2|x_0| + K_3 \int_{t_0}^{u} \exp(\int_{t_0}^{t} \alpha(\tau)d\tau) \int_{t_0}^{t} |y(s;t_0,x_0)|dsdt,$$

where $K_2 = L\int_{t_0}^{\infty} \exp(\int_{t_0}^{t} \alpha(\tau)d\tau)dt$ and $K_3 = LK_1$.

By Gronwall's inequality one obtains the desired result.

Corollary 4.5. *Assume that the solution $x = 0$ of (2.1) is (GUSGV) and that (3.3) and (4.9) hold. Furthemore we assume that*

$$a^r(t)\exp(-r\int_{t_0}^{t} \alpha(\tau)d\tau) \leq K_1 < \infty \text{ for all } t \geq t_0 \geq 0, \tag{4.11}$$

$$\text{where } r = 1/(1-m).$$

Then all solutions of (2.2) belong to $L^1([t_0,\infty))$.

The proof follows by using an reasoning related to those from proofs of Theorem 4.2 and Corollary 4.4.

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