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A FRACTIONAL STEP SCHEME FOR THE OPTIMAL CONTROL OF A NONLINEAR POPULATION DYNAMICS WITH DIFFUSION

BY

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Abstract. In this paper it is proved the convergence of an iterative scheme of Lie- Trotter type for the optimal harvesting problem governed by a nonlinear population dynamics with diffusion.

1. Introduction. Assume that a population is free to move in Ω, an open and bounded subset of R^N, with $\partial\Omega$ of C^1-class. The dynamics of population is described by a function $y(a,x,t)$ which represents the density of the individuals of age a, at time t and at the position x.

Here we shall study the following model

$$(1.1)\qquad \begin{cases} y_t + y_a + \mu(a)y - \Delta_x y = -vy \ , \ \text{if } y(a,x,t) < \psi(a,x) \\ y_t + y_a + \mu(a)y - \Delta_x y \le -vy \ , \ \text{if } y(a,x,t) = \psi(a,x) \\ \dfrac{\partial y}{\partial \nu} + \alpha y = 0 \ , \ \text{in } (0,A)\times\partial\Omega\times(0,T) \ (\alpha \ge 0) \\ y(0,x,t) = \displaystyle\int_0^A b(a)y(a,x,t)da \ , \ x\in\Omega, \ t\in(0,T) \\ y(a,x,0) = y_0(a,x) \ , \ a\in(0,A), \ x\in\Omega \end{cases}$$

The significances of these terms are: μ(a) is the rate of mortality, characteristic of the species; $b(a)$ is the rate of fertility; y_0 is the initial density. Birth is described by the "renewal equation" $(1.1)_4$.

The boundary condition $(1.1)_3$ shows that we have a migration from Ω to outside which is proportional to the density.

If the density $y(a,x,t)$ is less than a critical density $\psi(a,x)$, then the population dynamics is described by the equation $(1.1)_1$ and if $y(a,x,t)$ is equal to the critical density $\psi(a,x)$, then appears an additional mortality due to the over-population and to the limited food sourses. In this case the dynamics is described by the inequality $(1.1)_2$. This fact makes that $y(a,x,t)$ always remains less or equal than the critical density $\psi(a,x)$.

If the term $-vy$ is replaced by f, we obtain the model discussed in M.G. GARRONI and M. LANGLAIS (1982), where the authors have studied the existence, uniqueness and positivity of the solution.

In our model $v(a,x,t)$ represents the harvesting effort (which is an additional rate of mortality) and $v(a,x,t)y(a,x,t)$ is the harvested density.

Assume that v is a solution of (1.2)

$$(1.2)\qquad \begin{cases} v_t - \lambda v = u(a,x,t)\ ,\ (a,x,t) \in (0,A)\times\Omega\times(0,T) \\ v(a,x,0) = 0\ ,\ (a,x) \in (0,A)\times\Omega, \end{cases}$$

where $u \in U = \{h \in L^2((0,A)\times\Omega\times(0,T));\ 0 \le h(a,x,t) \le L_1\ a.e.\}$.

The significance of (1.2) is obvious if we take for example $\lambda = 0$ (the initial value of the harvesting effort is 0 and its derivative v_t is bounded).

This paper is concerned with the convergence of an iterative scheme of Lie–Trotter type for the following optimal control problem (which is an "optimal harvesting" problem)

(P) Maximize

$$\int_0^T \int_0^A \int_\Omega v(a,x,t)y(a,x,t)dxdadt,$$

where y is the solution of (1.1) and v is the solution of (1.2). For other "optimal harvesting" problems we refer to M. BROKATE (1985), M.E. GURTIN and L.F. MURPHY $(1981)_{A,B}$.

The main result of this paper amounts to saying that in certain situation problem (P) can be approximated for $\varepsilon \to 0$ by the following sequence of optimal control problems

($\mathbf{P}_\varepsilon$) Maximize

$$\int_0^T \int_0^A \int_\Omega v(a,x,t)y(a,x,t)dxdadt,$$

where y is the solution of

$$(1.3)\qquad \begin{cases} y_t + y_a + \mu(a)y - \Delta_x y = -vy \ , \ \text{in } (0,A)\times\Omega\times(i\varepsilon,(i+1)\varepsilon) \\ \dfrac{\partial y}{\partial \nu} + \alpha y = 0 \ , \ \text{in } (0,A)\times\partial\Omega\times(i\varepsilon,(i+1)\varepsilon) \\ y(0,x,t) = \displaystyle\int_0^A b(a)y(a,x,t)da \ \text{in } \Omega\times(i\varepsilon,(i+1)\varepsilon) \\ \qquad\qquad y^+(a,x,i\varepsilon) = Py^-(a,x,i\varepsilon) \ \text{in } (0,A)\times\Omega \\ y(a,x,0) = y_0(a,x) \ \text{in } (0,A)\times\Omega \end{cases}$$

$i = 0,1,...,N-1$, $\varepsilon = \dfrac{T}{N}$ (here $(Ph)(a,x)$=min $(h(a,x),\psi(a,x))$ and v is a solution of (1.2).

On $(i\varepsilon,(i+1)\varepsilon)$, problem (1.3) is linear. The iterative scheme (1.3) provides a theoretic algorithm for obtaining the solutions to problem (P).

In S. ANIŢA (1990) we have studied the convergence of such an iterative scheme in the case of a convex cost function and in the case when $-vy$ was repalced by v. In that case the main result can be proved by using the Lie–Trotter formula for nonlinear semigroups. Anyway the idea of considering the approximating problems (P_ε) has been given by Lie–Trotter formula (see also S. ANIŢA (1988) and V. BARBU (1988)).

The main additional difficulties in our present paper are due to the nonconvexity of the cost functional and to the presence of the bilinear term $-vy$ in (1.1).

2. Preliminaries. We shall use the following hypotheses:

1. $\mu \in C([0,A]), \mu(a) \geq 0$ in [0,A], $b \in L^\infty(0,A)$, $b(a) \geq 0$ a.e. in (0,A).

2.

$$\begin{cases} y_0 \in L^2(0,A;H^2(\Omega)) \\ \dfrac{\partial y_0}{\partial \nu} + \alpha y_0 = 0 \ \text{a.e. in } (0,A)\times\partial\Omega \\ y_0(0,x) = \displaystyle\int_0^A b(a)y_0(a,x)da \ \text{a.e. in } \Omega \\ y_0(a,x) \geq 0 \ \text{a.e. in } (0,A)\times\Omega \end{cases}$$

The critical density satisfies

$$\begin{cases} \psi \in L^2(0,A;H^2(\Omega)) \\ \dfrac{\partial \psi}{\partial a} \in L^\infty((0,A)\times\Omega) \\ \dfrac{\partial \psi}{\partial \nu} + \alpha\psi = 0 \text{ a.e. in } (0,A)\times\partial\Omega \\ \psi(0,x) = \displaystyle\int_0^A b(a)\psi(a,x)da \text{ a.e. in } \Omega \\ \psi(a,x) \geq y_0(a,x) \text{ a.e. in } (0,A)\times\Omega \end{cases}$$

In S. ANIŢA (1990) we have proved the following existence result:

Theorem 2.1. *There exists y, a unique solution of* (1.1), *i.e.* $y \in L^2((0,A)\times(0,T);H^2(\Omega)), y_t, y_a \in L^2((0,A)\times\Omega\times(0,T)), y(a,x,t) \leq \psi(a,x)$ *a.e. in* $(0,A)\times\Omega\times(0,T)$ *and y satisfies* (1.1) *a.e.*

It also follows that (1.3) has a unique solution (the solution is non-negative) (see also M.G. GARRONI and M. LANGLAIS (1982)).

Using Arzelà's theorem and Aubin's compactness theorem (see the Appendix) it follows that (P) and (P_ε) have optimal solutions.

Denote by $\Phi(v) = \displaystyle\int_0^T \int_0^A \int_\Omega v(a,x,t)y(a,x,t)dxdadt$, where y is the solution of (1.1) corresponding to v.

3. The main result.

Theorem 3.1. *If* $(v_\varepsilon^*, y_\varepsilon^*)$ *is an optimal pair for* (P_ε), *then*

$$\lim_{\varepsilon \searrow 0} \Phi(v_\varepsilon^*) = Max\ \Phi \tag{3.1}$$

and

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_0^A \int_\Omega v_\varepsilon^* y_\varepsilon^* dxdadt = Max\ \Phi \tag{3.2}$$

Moreover, every limit point of $\{v_\varepsilon^*\}$ *is an optimal control for problem* (P).

Proof. v_ε^* is corresponding to u_ε^* subject to (1.2)

$$(v_\varepsilon^*(a,x,t) = e^{\lambda t}\int_0^t e^{-\lambda s}u_\varepsilon^*(a,x,s)ds, t \in [0,T],\ a.e.\ (a,x) \in (0,A)\times\Omega)$$

Since $\{u_\varepsilon^*\}$ is bounded in $L^2(0,T;L^2((0,A)\times\Omega))$, it follows that on a subsequence (also denoted by $\{u_\varepsilon^*\}$ $u_\varepsilon^* \rightharpoonup^w u^*$ in $L^2((0,A)\times\Omega\times(0,T))$ and $v_\varepsilon^*(.,.,t) = e^{\lambda t}\int_0^t e^{-\lambda s}u_\varepsilon^*(.,.,s)ds \to e^{\lambda t}\int_0^t e^{-\lambda s}u^*(.,.,s)ds = v^*(.,.,t)$, $t\in[0,T]$ in $L^2((0,A)\times\Omega)$.

This implies that $v_\varepsilon^* \to v^*$ strongly in $L^2((0,A)\times\Omega\times(0,T))$.

The boundedness of $\{v_\varepsilon^*\}$ in $L^\infty((0,A)\times\Omega\times(0,T))$ implies that on a subsequence, also denoted by v_ε^* we have that

$$v_\varepsilon^* \rightharpoonup^{w^*} v^* \text{ weakly}^* \text{ in } L^\infty((0,A)\times\Omega\times(0,T))$$

and

$$v_\varepsilon^* \to v^* \text{ in } L^2((0,A)\times\Omega\times(0,T))$$

Since $\{v_\varepsilon^*\}$ is bounded in $L^\infty((0,A)\times\Omega\times(0,T))$, it follows that $\{y_\varepsilon^*\}$ is bounded in $L^2((0,A)\times\Omega\times(0,T))$ and $y_\varepsilon^* \rightharpoonup^w y_1$ weakly in the same space. Using now the strong convergence of $\{v_\varepsilon^*\}$ in $L^2((0,A)\times\Omega\times(0,T))$ and its boundedness in $L^\infty((0,A)\times\Omega\times(0,T))$ we may conclude that

$$v_\varepsilon^* y_\varepsilon^* \rightharpoonup^w v^* y_1^* \text{ weakly in } L^2((0,A)\times\Omega\times(0,T)) \tag{3.3}$$

We postpone for the time being the verification of the lemma below

Lemma. *By* (3.3) *follows that*

$$y_\varepsilon^*(t) \to y^*(t) \text{ *strongly in* } L^2((0,A)\times\Omega) \tag{3.4}$$

for all $t\in[0,T]$, *where* y^* *is the solution of* (3.5)

$$\begin{cases} y_t + y_a + \mu y - \Delta_x y = -v^* y_1 \text{ a.e. in} \\ \qquad \{(a,x,t)\in(0,A)\times\Omega\times(0,T); y(a,x,t) < \psi(a,x)\} \\ y_t + y_a + \mu y - \Delta_x y \le -v^* y_1 \text{ a.e. in} \\ \qquad \{(a,x,t)\in(0,A)\times\Omega\times(0,T); y(a,x,t) = \psi(a,x)\} \\ \dfrac{\partial y}{\partial \nu} + \alpha y = 0 \text{ a.e. in } (0,A)\times\partial\Omega\times(0,T) \\ y(0,x,t) = \displaystyle\int_0^A b(a)y(a,x,t)da \text{ a.e. in } \Omega\times(0,T) \\ y(a,x,0) = y_0(a,x) \text{ a.e. in } (0,A)\times\Omega \end{cases} \tag{3.5}$$

Since $y_\varepsilon^*(t) \to y^*(t)$ strongly in $L^2((0,A)\times\Omega)$ for all $t\in[0,T]$, we may conclude that $y_\varepsilon^* \rightharpoonup^w y^*$ weakly in $L^2((0,A)\times\Omega\times(0,T))$. Hence

$y^*(a,x,t) = y_1(a,x,t)$ a.e. in $(0,A)\times\Omega\times(0,T)$ and y^* is the solution of (1.1) corresponding to v^*.

Denote by Φ_ε the cost functional corresponding to (P_ε). Since

$$y_\varepsilon^*(t) \to y^*(t) \text{ strongly in } L^2((0,A)\times\Omega), \forall t\in[0,T]$$

and

$$v_\varepsilon^* \to v^* \text{ strongly in } L^2((0,A)\times\Omega\times(0,T))$$

it follows that

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_0^A\int_\Omega v_\varepsilon^* y_\varepsilon^* dxdadt = \Phi(v^*)$$

Let $(\tilde v,\tilde y)$ be an optimal pair for (P). We get $\Phi_\varepsilon(\tilde v)\le\Phi_\varepsilon(v_\varepsilon^*)$. Since $y_\varepsilon^{\tilde v}(t)\to\tilde y(t)$ strongly in $L^2((0,A)\times\Omega)$, $\forall t\in[0,T]$ ($y_\varepsilon^{\tilde v}$ is the solution of (1.3) corresponding to $\tilde v$).

We infer that

$$\Phi(\tilde v) = \lim_{\varepsilon\searrow 0}\Phi_\varepsilon(\tilde v) \le \lim_{\varepsilon\searrow 0}\Phi_\varepsilon(v_\varepsilon^*) = \Phi(v^*)$$

We may conclude that

$$\lim_{\varepsilon\searrow 0}\Phi_\varepsilon(v_\varepsilon^*) = \Phi(v^*) = \text{Max }\Phi$$

Denote by $y^{v_\varepsilon^*}$ the solution of (1.1) corresponding to v_ε^*.

Consider $Y = L^2((0,A)\times\Omega)$, $V = L^2(0,A;H^1(\Omega))$ and denote by $\|\cdot\|_Y$ and $\|\cdot\|_V$ their usual norms. We get

$$\|y_t^{v_\varepsilon^*}\|_{L^\infty(0,T;Y)} \le c\ ,\forall\varepsilon>0 \tag{3.6}$$

$$\|y^{v_\varepsilon^*}\|_{L^\infty(0,T;V)} \le c\ ,\forall\varepsilon>0 \tag{3.7}$$

$$\|y_a^{v_\varepsilon^*}\|_Y \le c(t)\ ,\forall\varepsilon>0 \text{ and } t\in[0,T] \tag{3.8}$$

Using Arzelà's theorem it follows that $\{y^{v_\varepsilon^*}(t)\}$ is compact in Y,$\forall t\in[0,T]$. This fact implies via Arzelà's theorem that $\{y^{v_\varepsilon^*}\}$ is compact in C([0,T];Y). Hence

$$y^{v_\varepsilon^*}(t)\to y^*(t)\ ,t\in[0,T]\ \text{ (uniformly on a subsequence)}$$

and we infer that

$$\lim_{\varepsilon \searrow 0} \Phi(v_\varepsilon^*) = \operatorname{Max} \Phi$$

Proof of the Lemma. We set $v_\varepsilon^*{=}v_\varepsilon$, $y^{v_\varepsilon^*} = y_\varepsilon$. By (1.3) we obtain

$$\begin{aligned} &\left\|y_\varepsilon^-((i+1)\varepsilon)\right\|_Y^2 + \int_{Q_\varepsilon^i} |\nabla_x y_\varepsilon|^2 dxdadt + \\ &+\alpha \int_{\Sigma_\varepsilon^i} y_\varepsilon^2 dxdadt \le \varepsilon L + \|y_\varepsilon^+(i\varepsilon)\|_Y^2 + c \int_{Q_\varepsilon^i} y_\varepsilon^2 dxdadt \ , \end{aligned}$$

where $Q_\varepsilon^i = (0,A)\times\Omega\times(i\varepsilon,(i+1)\varepsilon), \Sigma_\varepsilon^i = (0,A)\times\partial\Omega\times(i\varepsilon,(i+1)\varepsilon)$. We also obtain

$$\begin{aligned} &\int_{i\varepsilon}^{(i+1)\varepsilon} \left\|\frac{\partial y_\varepsilon}{\partial t}\right\|_Y^2 dt + \|\nabla_x y_\varepsilon^-((i+1)\varepsilon)\|_Y^2 + \\ &+\alpha \int_0^A \int_{\partial\Omega} |y_\varepsilon^-((i+1)\varepsilon)|^2 dxda \le \\ &\le \|\nabla_x y_\varepsilon^+(i\varepsilon)\|_Y^2 + \alpha \int_0^A \int_{\partial\Omega} |y_\varepsilon^+(i\varepsilon)|^2 dxda + \varepsilon L \end{aligned} \tag{3.9}$$

Combining these last two relations we obtain

$$\sum_{i=0}^{N-1} \int_{i\varepsilon}^{(i+1)\varepsilon} \left\|\frac{\partial y_\varepsilon}{\partial t}\right\|_Y^2 dt + \|y_\varepsilon(t)\|_V^2 \le c \tag{3.10}$$

We get

$$\sum_{i=0}^{N-1} \|y_\varepsilon^+(i\varepsilon) - y_\varepsilon^-(i\varepsilon)\|_Y = \sum_{i=0}^{N-1} \|Py_\varepsilon^-(i\varepsilon) - y_\varepsilon^-(i\varepsilon)\|_Y$$

and $y_\varepsilon^- = z_\varepsilon(i\varepsilon) + \theta_\varepsilon(i\varepsilon)$, where

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial t} + \dfrac{\partial z_\varepsilon}{\partial a} + \mu z_\varepsilon - \Delta_x z_\varepsilon = -v_\varepsilon y_\varepsilon \text{ a.e. in} \\ \qquad\qquad (0,A)\times\Omega\times((i-1)\varepsilon, i\varepsilon) \\ \dfrac{\partial z_\varepsilon}{\partial \nu} + \alpha z_\varepsilon = 0 \text{ a.e. in } (0,A)\times\partial\Omega\times((i-1)\varepsilon, i\varepsilon) \\ z_\varepsilon(0,x,t) = \displaystyle\int_0^A b(a) z_\varepsilon(a,x,t) da \text{ a.e. in } \Omega\times((i-1)\varepsilon, i\varepsilon) \\ z_\varepsilon(a,x,(i-1)\varepsilon) = 0 \text{ a.e. in } (0,A)\times\Omega \end{cases} \tag{3.11}$$

$$(3.12)\qquad \begin{cases} \dfrac{\partial\theta_\varepsilon}{\partial t}+\dfrac{\partial\theta_\varepsilon}{\partial a}+\mu\theta_\varepsilon-\Delta_x\theta_\varepsilon=0 \text{ a.e. in} \\ \qquad\qquad (0,A)\times\Omega\times((i-1)\varepsilon,i\varepsilon) \\ \dfrac{\partial\theta_\varepsilon}{\partial\nu}+\alpha\theta_\varepsilon=0 \text{ a.e. in } (0,A)\times\partial\Omega\times((i-1)\varepsilon,i\varepsilon) \\ \theta_\varepsilon(0,x,t)=\displaystyle\int_0^A b(a)\theta_\varepsilon(a,x,t)da \text{ a.e. in } \Omega\times((i-1)\varepsilon,i\epsilon) \\ \theta_\varepsilon(a,x,(i-1)\varepsilon)=Py_\varepsilon^-(a,x,(i-1)\varepsilon) \text{ a.e. in } (0,A)\times\Omega \end{cases}$$

Consider now the operator

$$(3.13)\qquad \mathcal{A}h=-\frac{\partial h}{\partial a}-\mu(a)h+\Delta_x h\ ,$$

$$D(\mathcal{A})=\{h\in Y;h_a\in Y,h\in L^2(0,A;H^2(\Omega)),h(0,x)=\int_0^A b(a)h(a,x)da$$

$$a.e.\ in\ \Omega,\frac{\partial h}{\partial\nu}+\alpha h=0 \text{ a.e. in } (0,A)\times\partial\Omega\}$$

$\mathcal{A}$ is the generator of a C_0-semigroup $S(t)$, $t\geq 0$. If we denote by $K=\{h\in Y;$ $h\geq 0\ a.e.\}$, we get $S(t)K\subset K$, $\forall t\geq 0$.

$$Py_\varepsilon^-(i\varepsilon)-\theta_\varepsilon(i\varepsilon)=P[S(\varepsilon)y_\varepsilon^-((i-1)\varepsilon)-$$
$$-\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau]-\theta_\varepsilon(i\varepsilon)=$$
$$=P[\theta_\varepsilon(i\varepsilon)-\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau]-\theta_\varepsilon(i\varepsilon)$$

If $\theta_\varepsilon(i\varepsilon)(a,x)\leq\psi(a,x)$, then

$$P[\theta_\varepsilon(i\varepsilon)-\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau](a,x)=y_\varepsilon^-(i\varepsilon)$$

and

$$(3.14)\qquad \begin{aligned}&\|Py_\varepsilon^-(i\varepsilon)-\theta_\varepsilon(i\varepsilon)\|_Y=\|y_\varepsilon^-(i\varepsilon)-\theta_\varepsilon(i\varepsilon)\|_Y= \\ &=\left\|\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau\right\|_Y\leq\varepsilon\tilde{L}\end{aligned}$$

If $\theta_\varepsilon(i\varepsilon)(a,x)>\psi(a,x)$, then

$$0\leq w=\{P[\theta_\varepsilon(i\varepsilon)-\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau]-\theta_\varepsilon(i\varepsilon)\}(a,x)=$$
$$=[-\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau](a,x)\ ,$$

for $y_\varepsilon^-(i\varepsilon)(a,x) \le \psi(a,x)$ and is less than or equal to

$$-\left[\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau\right](a,x)$$

for $y_\varepsilon^-(i\varepsilon)(a,x) > \psi(a,x)$.

We infer that

$$\|Py_\varepsilon^-(i\varepsilon)-\theta_\varepsilon(i\varepsilon)\|_Y \le \|S(\varepsilon)\psi-\psi\|_Y + \tag{3.15}$$
$$+\left\|\int_0^\varepsilon S(\varepsilon-\tau)v_\varepsilon((i-1)\varepsilon+\tau)y_\varepsilon((i-1)\varepsilon+\tau)d\tau\right\|_Y \le \varepsilon\tilde{L}\ ,$$

because $\psi \in D(\mathcal{A})$.

We obtain that

$$\|Py_\varepsilon^-(i\varepsilon)-y_\varepsilon^-(i\varepsilon)\| \le \|Py_\varepsilon^-(i\varepsilon)-\theta_\varepsilon(i\varepsilon)\|_Y + \|z_\varepsilon(i\varepsilon)\|_Y \le \varepsilon\bar{L} \tag{3.16}$$

Hence

$$\sum_{i=0}^{N-1}\|y_\varepsilon^+(i\varepsilon)-y_\varepsilon^-(i\varepsilon)\|_Y \le c\ ,\forall\varepsilon>0 \tag{3.17}$$

and the total variation of y_ε on $[0,T]$ is uniformly bounded. As before we can prove that $\{y_\varepsilon(t)\}$ is compact for $\forall\ t\in[0,T]$ in Y. We may therefore apply the Helly-Foiaş theorem to conclude that on a subsequence,again denoted by $\{y_\varepsilon\}$ we have

$$y_\varepsilon(t)\to y^*(t) \tag{3.18}$$

strongly in Y, $\forall t\in[0,T]$ and weakly star in $L^\infty((0,A)\times(0,T);H^1(\Omega))$. Let q be an element of $L^2(0,A;H^1(\Omega))$, $q(a,x)\le\psi(a,x)$ a.e. in $(0,A)\times\Omega$. $\forall s\in[i\varepsilon,(i+1)\varepsilon)$, $t\in[k\varepsilon,(k+1)\varepsilon)$, $i\le k$:

$$\frac{1}{2}(\|y_\varepsilon(t)-q\|_Y^2-\|y_\varepsilon(s)-q\|_Y^2)+\int_s^t\int_0^A\int_\Omega(y_\varepsilon)_a(y_\varepsilon-q)dxdad\tau+$$
$$+\frac{1}{2}\sum_{j=i+2}^{k}(\|y_\varepsilon^-(j\varepsilon)-q\|_Y^2-\|y_\varepsilon^+((j-1)\varepsilon)-q\|_Y^2)+$$
$$+\frac{1}{2}(\|y_\varepsilon^-((i+1)\varepsilon)-q\|_Y^2-\|y_\varepsilon^+(k\varepsilon)-q\|_Y^2)+$$
$$+\int_s^t\int_0^A\int_\Omega\mu y_\varepsilon(y_\varepsilon-q)dxdad\tau=-\int_s^t\langle\nabla_x y_\varepsilon(\tau),\nabla_x(y_\varepsilon(\tau)-q)\rangle d\tau-$$
$$-\alpha\int_s^t\int_0^A\int_{\partial\Omega}y_\varepsilon(\tau)(y_\varepsilon(\tau)-q)dxdad\tau-$$
$$-\int_s^t\int_0^A\int_\Omega v_\varepsilon(\tau)y_\varepsilon(\tau)(y_\varepsilon(\tau)-q)dxdad\tau$$

Letting ε tends to 0, it follows that

$$\begin{aligned}
&\frac{1}{2}(\|y^*(t)-q\|_Y^2-\|y^*(s)-q\|_Y^2)+\\
&+\int_s^t\int_0^A\int_\Omega (y_a^*+\mu y^*)(y^*-q)dxdad\tau+\\
&+\int_s^t \langle \nabla_x y^*(\tau), \nabla_x(y^*(\tau)-q)\rangle d\tau+\\
&+\alpha\int_s^t\int_0^A\int_{\partial\Omega} y^*(\tau)(y^*(\tau)-q)dxdad\tau+\\
&+\int_s^t\int_0^A\int_\Omega v^*(\tau)y^*(\tau)(y^*(\tau)-q)dxdad\tau\le 0
\end{aligned} \tag{3.19}$$

($(y_\varepsilon)_a \rightharpoonup^w y_a^*$ weakly in $L^2((0,A)\times\Omega\times(0,T))$ on a subsequence,because $\{(y_\varepsilon)_a\}$ is bounded in $L^2((0,A)\times\Omega\times(0,T))$). It is easy to show that $y^*(a,x,t)\le\psi(a,x)$ a.e. in $(0,A)\times\Omega\times(0,T)$. Now if we take $q=y^*(s)$ in (3.19) and use Gronwall's lemma we find after some computation that

$$\|y^*(t)-y^*(s)\|_Y\le c|t-s|\ ,\ \forall s,t\in[0,T] \tag{3.20}$$

Hence y^*:[0,T]$\to Y$ is absolutely continuous and $y^*\in W^{1,2}([0,T];Y)$.

Dividing (3.19) by $t-s$ and letting s tend to t we obtain that y^* is a variational solution of (1.1) (see also S. ANIŢA (1990)).We have proved in S. ANIŢA (1990) that (1.1) admits an unique variational solution which is also a solution in the sense of Theorem 2.1., thereby completing the proof of the lemma and of Theorem 3.1.

Using the methods of V. BARBU (1984) we obtain for (P_ε) the following maximum principle:

Theorem 3.2. *If $(v_\varepsilon^*, y_\varepsilon^*)$ (u_ε^* corresponds to v_ε^*) is an optimal pair for (P_ε) then there exists p a solution for* (3.21) :

$$\begin{cases}
p_t+p_a-\mu p+\Delta_x p=v_\varepsilon^*(1+p)-b(a)p(0,x,t)\\
\qquad\qquad a.e.\ in\ (0,A)\times\Omega\times(i\varepsilon,(i+1)\varepsilon)\\
\dfrac{\partial p}{\partial\nu}+\alpha p=0\ \ a.e.\ in\ (0,A)\times\partial\Omega\times(i\varepsilon,(i+1)\varepsilon)\\
p(A,x,t)=0\ \ a.e.\ in\ \Omega\times(i\varepsilon,(i+1)\varepsilon)\\
p(a,x,T)=0\ \ a.e.\ in\ (0,A)\times\Omega\\
\qquad p^-(a,x,(i+1)\varepsilon)=p^+(a,x,(i+1)\varepsilon)\ \ a.e.\ in\\
\qquad\qquad [(y_\varepsilon^*)^+(a,x,(i+1)\varepsilon)<\psi(a,x)]\\
p^-(a,x,(i+1)\varepsilon)=0\ \ a.e.\ in\ [(y_\varepsilon^*)^+(a,x,(i+1)\varepsilon)=\psi(a,x)]
\end{cases} \tag{3.21}$$

and

$$u_\varepsilon^*(a,x,t)=\begin{cases}0\ , & for\ \displaystyle\int_t^T e^{\lambda\tau}(1+p)y_\varepsilon^*d\tau<0\\ L_1, & for\ \displaystyle\int_t^T e^{\lambda\tau}(1+p)y_\varepsilon^*d\tau>0\end{cases}\tag{3.22}$$

(*see also* S. ANIŢA (1990)).

This last result allow us to solve numerically the approximating problem (P_ε).

4. Appendix. The compactness result of J.P. Aubin:

Theorem. *Let B_0,B and B_1 be three Banach spaces such that*

$$B_0\subset B\subset B_1\tag{A.1}$$

algebraically and topologically. Suppose that the embedding $B_0\subset B$ is compact.

Let M be a bounded subset of $L^{p_0}(0,T;B_0)$ such that

$$\frac{dv}{dt}\in L^{p_1}(0,T;B_1)\ ;\ \left\|\frac{dv}{dt}\right\|_{L^{p_1}(0,T;B_1)}\le c\ ,\ \forall v\in M\tag{A.2}$$

where $c>0$ is a constant independent of v and $1<p_i<+\infty$, $i=0,1$ are given.

In these conditions M is relatively compact in $L^{p_0}(0,T;B)$.

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