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EXTENSIONS OF STABLE VECTOR BUNDLES ON SMOOTH CURVES: LANGE'S CONJECTURE

BY

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Abstract. Let X be a smooth projective curve of genus $g \geq 2$. Fix integers r,s,a and b with $r>0$, $s>0$ and $a/r<b/s$. Here we prove the existence of an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

of stable vector bundles on X with $\operatorname{rank}(E)=r$, $\deg(E)=a$, $\operatorname{rank}(G)=s$ and $\deg(G)=b$ (Lange's conjecture). The same result was proved by B. Russo and M. Teixidor i Bigas.

1. Introduction. Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbf{K}$ with $\operatorname{char}(\mathbf{K}) = 0$. Recently several papers (see e.g. [1], [3], [5], [14]) were devoted to the proof of particular cases of the following conjecture due to H. LANGE ([7], p. 455). Fix integers r, s, a, b with $r > 0$, $s > 0$, and $a/r < b/s$. Is there an exact sequence of stable vector bundles on X

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0 \tag{1}$$

with $\deg(F) = a$, $\operatorname{rank}(F) = r$, $\deg(G) = b$ and $\operatorname{rank}(G) =?$ An affirmative answer to this question for all quadruples (r, s, a, b) will be called Property $(\$, X)$. If we require only the existence on an exact sequence (1) of stable vector bundles with fixed numeri cal data (r, s, a, b) we will call Property $(\$, X, r, s, a, b)$ the affirmative answer to this question. If we require only that E is semistable we obtain the corresponding Properties $(\$\$, X)$ and $(\$\$, X, r, s, a, b)$. Property $(\$, X)$ was proved for a general X in [14]. Property $(\$, X, r, s, a, b)$ was proved for every X if $g \geq (r + s + 1)/2$ and for a few other triples (g, r, s) in [5]. Property $(\$, X, r, s, a, b)$ was proved for all X if $b/s - a/r \leq (g - 1)/\max\{r, s\}$ in [1], Th. 2.1. Property $(\$\$, X)$ was

proved in [3] for X bielliptic and for X with general moduli and in [14], Th. 0.2, for every X. The aim of this paper is the proof of the following result which was independently proved in [12]. The preprint form of this paper was written on September 8^{th}, 1997.

Theorem 0.1. *Let X be a curve of genus $g \geq 2$. If Property $(\$\$, X)$ holds, then Property $(\$, X)$ holds.*

Remark 0.2. Notice that by [14], Th. 0.2, Theorem 0.1 shows that in characteristic 0 Property $(\$, X)$ is true for every smooth curve of genus $g \geq 2$, i.e. it gives an affirmative answer to Lange's conjecture.

At the beginning of section 1 we will prove two easy results (Proposition 1.1 and Corollary 1.2) which will give $(\$, X, r, s, a, b)$ for a huge number of triples (g, r, s) without using $(\$\$, X, r, s, a, b)$ but just [1], Th. 2.1. Only for a few nasty triples (g, r, s) we will use $(\$\$, X, r, s, a, b)$. The proof of 1.1 and 1.2 will be very short and easy. However, our main concern is to show how to use semistability to obtain rather easily stability for this type of problems related to the existence of stable vector bundles with suitable filtrations. Indeed semistability is very well–behaved with respect to specialization and pull–backs and it is often possible to use these good properties of semistability (see for instance respectively the proof of [14], Th. 0.2, and the proofs of [3], Th. 0.1 and Th. 0.2). It was discussed in [7] and [5] how to use Property $(\$, X)$ to study an interesting stratification of the moduli space of stable vector bundles on X.

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1. Proof of 0.1. For all integers x, y with $x>0$ let $M(X; x, y)$ be the moduli scheme of stable vector bundles on X with degree y and rank x. $M(X; x, y)$ is an irreducible smooth variety of dimension $x^2(g-1)+1$. Notice that by the openness of stability Property $(\$, X, r, s, a, b)$ is equivalent to the assertion that the general extension of a general $G \in M(X; s, b)$ by a general $F \in M(X; r, a)$ is stable. Fix $P \in X$. Let $\mathbf{K}_P$ be the skyscraper length 1 sheaf supported by P. Fix vector bundles U and V on X. We will say that V is obtained from U making a negative elementary transformation supported by P and that U is obtained from V making a positive elementary transformation supported by P if there is a surjection $f : U \to \mathbf{K}_P$ with $V \cong Ker(f)$. Notice that in this case we have $\text{rank}\,(U) = \text{rank}\,(V)$, $\deg(U) = \deg(V) + 1$ and that U^* (resp. V^*) is obtained from V^* (resp.

U^*) making a negative (resp. positive) elementary transformation. A general positive (resp. negative) elementary transformation supported by P of a general $A \in M(X; x, y)$ is stable (see e.g. [2], Remark 2.7, or make an easy characteristic 0 proof using the results proved in [10] and [9] quoted in 1.11 below)). Hence for all integers x, y with $x > 0$ making a positive elementary transformation supported by P induces a birational map between $M(X; x, y)$ and $M(X; x, y+1)$.

Proposition 1.1. (Any characteristic) *Assume* $(a+1)/r < (b-1)/s$ *and that Property* $(\$, X, r, s, a+1, b-1)$ *is true. Then Property* $(\$, X, r, s, a, b)$ *is true.*

Proof. Fix a general $U \in M(X; r, a+1)$, a general $V \in M(X; s, b-1)$ and a general extension

$$0 \to U \to H \to V \to 0 \tag{2}$$

By assumption H is stable. Fix $P \in X$. Let H' be the general positive elementary transformation of H supported by P. The bundle H' fits in an exact sequence

$$0 \to U \to H' \to V' \to 0 \tag{3}$$

in which V' may be considered as the general positive elementary transformation of V supported by P. Let H'' be the general bundle obtained from H' by a negative elementary transformation supported by P. The bundle H'' fits in an exact sequence

$$0 \to U' \to H\text{"} \to V' \to 0 \tag{4}$$

in which U' may be considered as a general negative elementary transformation of U. Notice that H may be obtained from H' applying a negative elementary transformation supported by P, i.e. the inverse of the positive elementary transformation used to pass from H to H'. By the openness of stability we may assume H'' stable. Note also that for every subsheaf T of U and every quotient sheaf T' of V' we have $\mu(T) \leq \mu(U) = (a+1)/r < < (b-1)/s \leq \mu(T')$. Hence $h^0(X, \mathrm{Hom}(V', U')) = 0$. Every bundle on a smooth curve of genus ≥ 2 may be deformed to a stable bundle ([11], Prop. 2.6, or [6], Cor. 2.2). Hence, deforming U' and V' to stable bundles and using the theory of the Global Ext-functor ([4] or [8]) we obtain $(\$, X, r, s, a, b)$. Alternatively, to obtain $(\$, X, r, s, a, b)$ we may use that U' and V' are stable by the generality of U and V (see e.g. [2], Remark 2.7, or make an easy

characteristic 0 proof using the results proved in [10] and [9] quoted in 1.11 below).

If $\mathrm{char}(\mathbf{K}) = 0$ Proposition 1.1 allows us to use the very strong result [1], Th. 2.1.

Corollary 1.2. *Fix integers g, r, s, a, b with $g \geq 2$, $r > 0$, $s > 0$ and $a/r < b/s$. Let X be a smooth genus g curve. Assume the existence of an integer $u \geq 0$ with $(a+u)/r < (b-u)/s \leq (a+u)/r + (g-1)/\max\{r,s\}$. Then $(\$, X, r, s, a, b)$ is true.*

Proof. By [1], Th. 2.1, we have $(\$, X, r, s, a+u, b-u)$. Hence we conclude applying u times Proposition 1.1.

Remark 1.3. Notice that Corollary 1.2 covers a huge number of triples (g, r, s), i.e. all triples with $1/r + 1/s \leq (g-1)/\max\{r,s\}$ and in particular most of the cases proved in [5], §3.

Remark 1.4. (Any characteristic) The proof of Proposition 1.1 shows that if $(a+1)/r \leq (b-1)/s$ and Property $(\$\$, X, r, s, a+1, b-1)$ is true, then Property $(\$\$, X, r, s, a+1, b-1)$ is true. Hence, as in 1.2, if there is an integer $u \geq 0$ with $(a+u)/r = (b-u)/s$, then Property $(\$\$, X, r, s, a, b)$ is true. In particular Property $(\$\$, X, r, r, a, a+2u)$ is true for every integer $u \geq 0$.

Proof of Theorem 0.1. We divide the proof into 7 parts which we will label (1.5),...,(1.11) to make easier their quotation.

(1.5) Since the dual of a stable vector bundle is stable and the dual of an exact sequence of vector bundles is exact, Property $(\$, X, r, s, a, b)$ is equivalent to Property $(\$, X, s, r, -b, -a)$. Hence to prove Theorem 0.1 we may assume $r \leq s$. From now on we will assume $r \leq s$.

(1.6) Let b' be the minimal integer with $a/r < b'/s$. Set $\mathbf{N}(b') := \{t \in \mathbf{Z} : t \geq b'\}$. We need to prove Property $(\$, X, r, s, a, t)$ for all integers $t \in \mathbf{N}(b')$. Set $\mathbf{B} := \{t \in \beta N(b') : (\$, X, r, s, a, t) \text{ is true}\}$. By [1], Th. 2.1, and the assumption $r \leq s$, we have $t \in \mathbf{B}$ if $a/r < t/s \leq a/r + (g-1)/s$. Since $r \leq s$ and $(b'-1)/s \leq a/r$, we have $b' \in \mathbf{B}$. We need to prove that $\mathbf{B} = \mathbf{N}(b')$. From now on we fix $b \in \mathbf{N}(b')$ and we assume $t \in \mathbf{B}$ for all integers t with $b' \leq t < b$. We need to prove that $b \in \mathbf{B}$.

(1.7) Fix vector bundles U, V on X. By Riemann–Roch we have $\dim(\mathrm{Ext}^1(V,U)) = \mathrm{rank}\,(U) \cdot \mathrm{rank}\,(V)(\mu(V) - \mu(U) + (g-1)) + h^0(X, \mathrm{Hom}(V,U))$. Notice that $h^0(X, \mathrm{Hom}(V,U))=0$ if in the Harder–Narasimhan filtration of U and V all graded semistable subquotients of V have

slope strictly less than the slope of any graded subquotient of the Harder–Narasimhan filtration of U. In particular $h^0(X, \mathrm{Hom}(V,U)) = 0$ if both U and V are semistable and $\mu(U) < \mu(V)$. Furthermore, $h^0(X, \mathrm{Hom}(V,U)) = 0$ if both U and V are semistable, $\mu(U) = \mu(V)$ and no subquotient of a Jordan– Hölder filtration of U is isomorphic to a subquotient of a Jordan - Hölder filtration of V. Assume U and V semistable with $\mu(U) = \mu(V)$; then $h^0(X, \mathrm{Hom}(V,U))$ may be bounded in terms of the number of isomorphic graded subquotients of the Jordan–Hölder filtrations of U and V.

(1.8) We want to compute the dimension of the total space of the relative Ext^1-sheaf in the sense of [4] or [8] when the bundles vary in a semi–universal family near the bundles U and V. Since $g \geq 2$ every vector bundle on X is the limit of a flat family of stable vector bundles (see [11], Prop. 2.6, or [6], Cor. 2.2). Hence to obtain an upper bound for the "dimension" of the "set" of isomorphic classes of bundles obtained from the middle bundle of all such extensions, it is sufficient to add the terms $\mathrm{rank}\,(U)^2(g-1)+1 := \dim(M(X; \mathrm{rank}\,(U), \deg(U)))$ and $\mathrm{rank}\,(V)^2(g-1)+ + 1 := \dim(M(X; \mathrm{rank}\,(V), \deg(V)))$ and then subtract $+1$. We take the following notion as upper bound for the dimension of a set, S, of vector bundles on X. We assume that all isomorphism classes of vector bundles in S appear as bundles parametrized (perhaps not one to one) by finitely many irreducible algebraic varieties, say $W(i)$, $1 \leq i \leq n$, and we take $\max_{1\leq i\leq n} \dim(W(i))$ as an upper bound for $\dim(S)$. We claim that if we vary F in $M(X; r, a)$ and G in $M(X; s, b)$ with $a/r < b/s < a/r + (g-1)$ and then take all such extensions, then this upper bound is the dimension, say by $(\$\$, X)$, of the equivalence classes of semistable vector bundles obtained in this way. Indeed, by [1], proof of Lemmas 2.3 and 2.4, i.e. quotation of step 8 of the proof of [1], Th. 2.1, if $a/r < b/s < a/r + (g-1)$, then for every non splitted extension (1), the bundles F and G are uniquely determined by E and the maps $F \to E$ and $E \to G$ are uniquely determined by E, up to a multiplicative constant.

(1.9) Let T be a saturated subbundle of E. If $h^0(X, \mathrm{Hom}(E/T, T))=0$, we have $\dim(\mathrm{Ext}^1(E/T, T)) = \mathrm{rank}\,(T) \cdot (r+s-rank(T))(\mu(E/T) - \mu(T) + + (g-1))$ by 1.7. Assume $h^0(X, \mathrm{Hom}(E/T, T)) \neq 0$ and take $f \in \in H^0(X, \mathrm{Hom}(E/T, T))$, $f \neq 0$. The map f induces $f' \in H^0(X, \mathrm{End}(E))$ with $0 < \mathrm{rank}\,(\mathrm{Im}(f')) \leq \mathrm{rank}\,(T) < r+s$. Since E is semistable, $\mu(\mathrm{Im}(f')) = = \mu(E)$ and $\mu(\mathrm{Ker}\,(f')) = \mu(E)$. Furthermore, a power of f' of order $\leq r+s$ is a projection of E onto a non–trivial direct factor which is also a factor of T. By 1.7, 1.8 and what we have just proved it follows that to obtain a contradiction to the assumption $b \notin \mathbf{B}$ it is sufficient to prove that for a general $F \in M(X; r, a)$, a general $G \in M(X; s, b)$ and a general extension (1),

the middle term E of (1) has a saturated subbundle, T, with $\mu(T) > \mu(F)$ and $r \leq \text{rank}\,(T) \leq s$.

(1.10) Now we will bound the dimension of the "set" of isomorphic classes of vector bundles equipped with a filtration with more than two terms. Let U be a vector bundle on X equipped with an increasing filtration $\{U(i)\}_{0\leq i\leq x}$ by saturated subbundles with $U(0) = 0$ and $U(x) = U$. Set $V(i) := U(i)/U(i-1)$, $1 \leq i \leq x$, $M(i) := \text{rank}\,(U(i))$, $m(i) := M(i) - -M(i-1)$, and $\varepsilon(i) := h^0(X, \text{Hom}(V(i), U(i-1))$. We want to give an upper bound, δ, for the dimension of the isomorphic classes of all such bundles U in terms of upper bounds, $\delta(i)$, for the set of isomorphic classes of all such $U(i)$, $0 \leq i \leq x$ (hence $\delta(0) = 0$ and $\delta(x) = \delta$). For every integer i with $1 \leq i \leq x$ we have the exact sequences

$$0 \to U(i-1) \to U(i) \to V(i) \to 0 \tag{6}$$

Hence by 1.7 and 1.8 we may take $\delta(i) \leq \delta(i-1) + m(i)^2(g-1) + +M(i-1)m(i)(\mu(V(i) - \mu(U(i-1)) + (g-1)) + \varepsilon(i)$. In our situation we will have $U(x)$ semistable (or even stable) and either $\varepsilon(i) = 0$ or we could drop the term $m(i)^2(g-1)$ because by 1.9 $V(i)$ would appear as one of the finitely many factors of $U(i-1)$. There are two very easy cases for the comparison of maximal dimensions of 3-step filtrations with respect to 2-step filtration and these two cases will be used in 1.11. In the first case the bundle, say E, has rank $r+s$, $0 < r \leq s$, and there is an integer ρ with $s < \rho < r+s$ such that the filtration $\{E(i)\}_{0\leq i\leq 3}$, $E(0) := \{0\}$, $E(3) := E$, has rank $(E(1)) = \rho - s$ and rank $(E(2)) = \rho$, while the 2-step filtration, say $\{E'(i)\}_{0\leq i\leq 2}$, $E'(0) := \{0\}$, $E'(2) := E$ has rank $(E'(1)) = r$; the comparison is very easy because rank $(E(2)/E(1)) =$ rank $(E'(1)/E'(0))$. In the second case we have rank $(E(1)) = r$, rank $(E(2)) = r+s-1$ and rank $(E(3)) = r+s$, i.e. the 3-step filtration is a refinement of the 2-step filtration $\{0\} \subset E(1) \subset E(3)$; here the comparison is trivial because we will need it at the end of 1.11 in a case with $\mu(E(3)/E(1)) - \mu(E(2)/E(1)) < g-1$.

(1.11) There is an integer $u > 0$ such that $(a+u)/r \geq (b-u)/s$. By Proposition 1.1 and induction on u we may assume $(a+1)/r \geq (b-1)/s$. By (1.5) we may assume $r \leq s$. Hence $(g-1)/max\{r,s\} \geq 1/s$. Hence, by [1], Th. 2.1, we conclude if $(b-1)/s \leq a/r$. Hence we may assume $a/r < (b-1)/s$ and that Property $(\$, X, r, s, a, b-1)$ is true. Since $(b-1)/s \leq (a+1)/r$, we have $(b-1)/s \leq (a+b)/(r+s)$ with equality only if $(b-1)/s = (a+1)/r$. We fix the general exact sequence (1) with data (r, s, a, b). In order to obtain a contradiction we assume that E is not stable. By Property $(\$\$, X)$ the bundle E is semistable. Hence there is a proper subbundle A of E with $\mu(A) = \mu(E)$. The bundle E/A is semistable. We take any such bundle A

with $\rho := \mathrm{rank}\,(A)$ minimal. By the minimality of ρ the bundle A is stable. By the last assertion of 1.9 we have either $0 < \rho < r$ or $s < \rho < r + s$. Call $f : A \to G$ the induced map. Set $K := \mathrm{Ker}\,(f)$, $\rho' := \mathrm{rank}\,(\mathrm{Im}(f))$ and $w := \mathrm{rank}\,(K) = \rho - \rho'$. First assume f surjective. In particular we have $\rho \geq s$ and, since (1) does not split, $\rho > s$. Hence $0 < w < r$. We have a filtration $\{E_i\}_{0 \leq i \leq 3}$ of E with $E_0 = 0$, $E_3 = E$, $E_1 = K$, $E_2 = A$ and $E_2/E_1 \cong G$. As in the last part of 1.9 and the first 3-step filtration considered in 1.10, we obtain a contradiction in the following way. The set of all bundles K depends at most on $(\rho-\rho')^2(g-1)+1$ parameters. The set of all bundles G depends on $s^2(g-1)+1$ parameters. Since A is stable we have $h^0(X, \mathrm{Hom}(G, K)) = 0$. Hence the set of all such bundles A depends at most on $(\rho-\rho')^2(g-1)+s^2(g-1)+1+ws(b/s-\mu(K)+(g-1))$ parameters. We have $\deg(K) + b = \deg(A) = \rho(a+b)/(r+s)$. Then we continue seeing E as extension of E/A by A with $\mu(E/A) = \mu(A) = (a+b)/(r+s)$. Now assume f not surjective. First assume $\rho' = s$. Since f is not surjective, we have $b'' := \deg(\mathrm{Im}(f)) \leq b - 1$. We use $\mathrm{Im}(f)$ instead of G and the integer b'' instead of b in the previous proof and again we obtain a contradiction; alternatively, use that $\mu(\mathrm{Im}(f)) \leq (b-1)/s$ and copy the proof of the case $0 < \rho' < s$ we will study now. Now assume $0 < \rho' < s$. Since G is general, we have $\mu(\mathrm{Im}(f)) \leq b/s-(g-1)(s-\rho')/s \leq b/s-(s-\rho')/s$ ([10] or [9], Th. 3.7, Remark 3.14 and references quoted there) and in particular $\mu(\mathrm{IM}(f)) \leq$ $\leq (b-1)/s-(g-2)/s$. Since A is stable, we have $\mu(\mathrm{IM}(f)) \geq (a+b)/(r+s)$ with equality only if $w = 0$. Since $(b-1)/s \leq (a+b)/(r+s)$ with equality only if $(a+1)/r = (b-1)/s$, we obtain a contradiction except if $g = 2$, $w = 0$, $\rho = \rho' = s-1$ and $(a+1)/r = (b-1)/s$; since $\rho < s$ we conclude even in this case if $r < s$; assume $g = 2$, $\rho = \rho' = s-1$, $r = s$, $(a+1)/r = (b-1)/s$; in this case the saturation of $\mathrm{Im}(f)$ induces a filtration of G and hence a 3-step filtration of E with graded subquotients of ranks r, $s-1 = r-1$ and 1; hence we conclude by the last part of 1.10. Now assume $\rho' = 0$, i.e. $f = 0$. Hence A is a subsheaf of F. We obtain $\mu(A) \leq \mu(F) = a/r$, contradiction.

REFERENCES

1. BALLICO, E., BRAMBILA-PAZ, L. and RUSSO, B. – *Exact sequences of stable vector bundles on projective curves,* Math. Nachr. 194 (1998), 5-11.
2. BALLICO, E. and RAMELLA, L. – *The restricted tangent bundle of smooth curves in Grassmannians and curves in flag varieties,* preprint.
3. BALLICO, E. and RUSSO, B. – *Exact sequences of semi–stable vector bundles on algebraic curves,* Proc. London Math. Soc (to appear).

4. BANICA, C., PUTINAR, M. and SCHUMAKER, G. – *Variation der globalen Ext in deformationen kompakter komplexer Räume,* Math. Ann. 250 (1980), 135-155.
5. BRAMBILA-PAZ, L. AND LANGE, H. – *A stratification of the moduli space of vector bundles on curves,* J. reine angew. Math. 494 (1998), 173-187.
6. HIRSCHOWITZ, A. – *Problèmes de Brill–Noether en rang supérieur,* unpublished 1984 preprint.
7. LANGE, H. – *Zur Klassifikation von Regelmanningfaltigkeiten,* Math. Ann. 262 (1983), 447-459.
8. LANGE, H. – *Universal families of extensions,* J. Algebra 83 (1983), 101-112.
9. LANGE, H. – *Some geometrical aspects of vector bundles on curves,* Aportaciones Mat. 5 (1992), 53-74.
10. MUKAI, S. and SAKAI,F. – *Maximal subbundles of vector bundles on a curve,* Manuscripta Math. 52 (1985), 251-256.
11. NARASIMHAN, M. S. and RAMANAN, S. – *Deformations of the moduli space of vector bundles over an algebraic curve,* Annals of Math. 101 (1975), 391-417.
12. RUSSO, B. and TEIXIDOR I BIGAS, M. – *On a conjecture of Lange,* J. Algebraic Geometry (to appear).
13. TEIXIDOR I BIGAS, M. – *Stable extensions of line bundles,* J. reine angew. Math. 502 (1998), 163-173.
14. TEIXIDOR I BIGAS, M. – *On Lange's conjecture,* preprint.

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