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ON A THEORY OF POROUS THERMOELASTIC SHELLS

BY

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Abstract. This paper is concerned with the theory of elastic materials with voids [1]. In the first part of the paper we establish a nonlinear theory of thermoelastic shells with voids. We consider thin shells modelled as Cosserat surfaces. Then, we derive the equations of the linear theory and establish a uniqueness theorem.

1. Introduction. NUNZIATO and COWIN [1] have presented a nonlinear theory for the behaviour of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field.

This representation introduces an additional degree of kinematical freedom. The theory is expected to find applications in the treatment of the mechanics of granular materials and manufactured porous bodies. The linear theory of elastic materials with voids has been derived by COWIN and NUNZIATO and COWIN [2]. The linear theory of elastic materials with voids is one of the simple extensions of the classical theory of elasticity for the treatment of materials with microstructure (see CAPRIZ and PODIO–GUIDUGLI [3]). There has been much written in recent years on the subject of the theory of elastic materials with voids.

The present paper is devoted to the theory of thermoelastic shells with voids. We consider thin shells modelled as Cosserat surfaces. For the history of the problem and the detailed analysis of the theory of shells and plates described as Cosserat surfaces we refer to the work of NAGHDI [4]. Following the approach of [4] we derive a nonlinear theory of thermoelastic shells with voids. After presenting some preliminary kinematical results in Section 2, we derive the basic field equations in Section 3 and the constitutive equa-

tions in the nonlinear case in Section 4. Next, in the context of the linear theory, we prove in Section 6 that the uniqueness theorem presented in [4] still holds for porous shells. Moreover, we establish a uniqueness theorem under less restrictive conditions, by dropping any definiteness assumption on the constitutive coefficients.

2. Preliminaries. By definition, a Cosserat surface consists of a surface embedded in a Euclidean 3–space together with a single deformable director assigned to every point of the surface (cf. [4]).

Let a body $\mathcal{C}$ be a Cosserat surface with its particles identified by the curvilinear material coordinates θ^α $(\alpha = 1, 2)$ and let $\mathcal{S}$ and $\mathtt{s}$ refer to the surface $\mathcal{C}$ in the reference and deformed configuration, respectively. Let $\mathbf{r} = \mathbf{r}(\theta^\alpha, t)$ denote the position vector of the points of $\mathtt{s}$ and let $\mathbf{d} = \mathbf{d}(\theta^\alpha, t)$ be the deformable director assigned to every point of $\mathtt{s}$. Then, the motion of a Cosserat surface is defined by

$$\mathbf{r} = \mathbf{r}(\theta^\alpha, t),\ \mathbf{d} = \mathbf{d}(\theta^\alpha, t),\ (\theta^\alpha) \in \Sigma,\ t \in I, \tag{2.1}$$

where Σ is a domain in the Euclidean 2-space and $I = [0, \infty)$. Also, we denote the base vectors of $\mathtt{s}$ by $\mathbf{a}_\alpha$ and by $\mathbf{a}_3$ the unit normal to $\mathtt{s}$

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha},\ \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0,\ \mathbf{a}_3 \cdot \mathbf{a}_3 = 1,\ (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) > 0. \tag{2.2}$$

We consider that

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{d}) > 0. \tag{2.3}$$

We denote by $a_{\alpha\beta}$ and $b_{\alpha\beta}$ the first and the second fundamental forms of the surface $\mathtt{s}$, respectively. We recall the formulae

$$\begin{gathered} a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta,\ b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} \\ \mathbf{a}_{\alpha|\beta} = b_{\alpha\beta}\mathbf{a}_3,\ b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta}. \end{gathered} \tag{2.4}$$

where a comma stands for partial differentiation with respect to the surface coordinates θ^α and a vertical bar denotes covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$. Throughout this paper we employ the usual indicial notation: Greek indices take the values $\{1, 2\}$, while Latin indices the values $\{1, 2, 3\}$. The sumation convention is also used.

Henceforth we identify the reference surface $\mathcal{S}$ with the initial surface and denote the initial values, at time $t = 0$, of the above vector fields by capital letters

$$\begin{gathered} \mathbf{R} = \mathbf{R}(\theta^\alpha) = \mathbf{r}(\theta^\alpha, 0),\ \mathbf{D} = \mathbf{D}(\theta^\alpha) = \mathbf{d}(\theta^\alpha, 0), \\ \mathbf{A}_\alpha = \mathbf{a}_\alpha(\theta^\alpha, 0),\ \mathbf{A}_3 = \mathbf{a}_3(\theta^\alpha, 0),\ (\theta^\alpha) \in \Sigma. \end{gathered} \tag{2.5}$$

Formulae of the type (2.4) also hold for the reference configuration $\mathcal{S}$.

In what follows ρ is the mass density at time t and ρ_{o} is the initial mass density. In the theory of materials with voids the mass density ρ has the decomposition (see [1], [5])

$$\rho = \nu\gamma, \tag{2.6}$$

where γ is the density field of the matrix material and ν is the volume fraction field ($0 < \nu \leq 1$). In the reference configuration, relation (2.6) can be written as

$$\rho_{\mathrm{o}} = \nu_{\mathrm{o}}\gamma_{\mathrm{o}}. \tag{2.7}$$

In the remaining of this section we shall recall some general kinematical results which can be found in [4]. Let $\mathbf{v}$ and $\mathbf{w}$ denote the velocity of a point of $\mathtt{s}$ and the director velocity at time t

$$\mathbf{v} = \dot{\mathbf{r}}(\theta^\alpha, t),\ \mathbf{w} = \dot{\mathbf{d}}(\theta^\alpha, t), \tag{2.8}$$

where a superposed dot stands for the material derivative with respect to t, holding θ^α fixed. Then

$$\dot{\mathbf{a}}_\alpha = \mathbf{v}_{|\alpha} = (v_{\lambda|\alpha} - b_{\alpha\lambda}v_3)\mathbf{a}^\alpha + (v_{3,\alpha} + b^\lambda_\alpha v_\lambda)\mathbf{a}^3, \quad \dot{\mathbf{a}}_3 = -(v_{3,\alpha} + b^\lambda_\alpha v_\lambda)\mathbf{a}^\alpha.$$

If we introduce the notations $\dot{\mathbf{a}}_i = c_{ki}\mathbf{a}^k$ and $\eta_{ki} = c_{(ki)}$, then

$$2\eta_{\alpha\beta} = \dot{a}_{\alpha\beta} = v_{\alpha|\beta} + v_{\beta|\alpha} - 2b_{\alpha\beta}v_3,\ \eta_{\alpha 3} = \eta_{33} = 0. \tag{2.9}$$

The gradient of the director $\mathbf{d}$ can be written as

$$\begin{gathered}\mathbf{d}_{,\alpha} = \lambda_{i\alpha}\mathbf{a}^i,\ \lambda_{i\alpha} = \mathbf{a}_i \cdot \mathbf{d}_{,\alpha} \\ \lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\alpha\beta}d_3,\ \lambda_{3\alpha} = d_{3,\alpha} + b^\beta_\alpha d_\beta.\end{gathered} \tag{2.10}$$

From the above formulae we obtain the gradient of the director velocity

$$\mathbf{w}_{,\alpha} = \dot{\mathbf{d}}_{,\alpha} = \dot{\lambda}_{i\alpha}\mathbf{a}^i + \lambda_{i\alpha}\dot{\mathbf{a}}^i. \tag{2.11}$$

It is often convenient to employ the relative kinematic measures

$$e_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}),\ \kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha},\ \gamma_i = d_i - D_i. \tag{2.12}$$

3. Basic field equations. We postulate the basic principles which are assumed to hold for every motion of a Cosserat surface with voids and for each part $\mathcal{P}_o$ of the reference configuration.

Balance of energy is assumed in the form [1]

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{P}_o}\rho_o[\varepsilon+\frac{1}{2}(\mathbf{v}\cdot\mathbf{v}+\alpha\mathbf{w}\cdot\mathbf{w}+k\dot{\nu}\dot{\nu})]d\sigma=\\ &=\int_{\mathcal{P}_o}\rho_o(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\mathbf{w}+p\dot{\nu}+r)d\sigma+\int_{\partial\mathcal{P}_o}(\mathbf{N}\cdot\mathbf{v}+\mathbf{M}\cdot\mathbf{w}+h\dot{\nu}-q)ds.\end{aligned}\tag{3.1}$$

We have denoted by ε, r and q the specific internal energy, the specific heat supply and the heat flux, respectively. The vector fields per unit mass $\mathbf{f}$ and $\mathbf{l}$ stand for the assigned force and the assigned director couple (see [4], Section 8), while the vector fields per unit length of $\partial\mathcal{P}_o$, $\mathbf{N}$ and $\mathbf{M}$ are the contact force and the contact director couple. The scalar fields p and h are the (external) equilibrated body force and the equilibrated stress, respectively (see [1], [5]). The inertia coefficients α and k are assumed to be prescribed and independent of time.

Principle of director momentum is expressed by

$$\frac{d}{dt}\int_{\mathcal{P}_o}\rho_o\alpha\mathbf{w}d\sigma=\int_{\mathcal{P}_o}(\rho_o\mathbf{l}-\mathbf{m})d\sigma+\int_{\partial\mathcal{P}_o}\mathbf{M}ds,\tag{3.2}$$

where $\mathbf{m}$ is the intrinsic director couple.

Balance of equilibrated force ([1],[5]) can be written in the form

$$\frac{d}{dt}\int_{\mathcal{P}_o}\rho_o k\dot{\nu}d\sigma=\int_{\mathcal{P}_o}(\rho_o p-g)d\sigma+\int_{\partial\mathcal{P}_o}hds.\tag{3.3}$$

Here, the scalar field per unit area g stands for the intrinsic equilibrated body force.

We postulate the following entropy inequality

$$\frac{d}{dt}\int_{\mathcal{P}_o}\rho_o\eta d\sigma\geq\int_{\mathcal{P}_o}\rho_o\frac{r}{\theta}d\sigma+\int_{\partial\mathcal{P}_o}\frac{q}{\theta}ds,\tag{3.4}$$

where η denotes the specific entropy and θ is the absolute temperature field. We mention that all the above field quantities act in the present configuration but are measured per unit length, per unit area or per unit mass of the material surface of $\mathcal{C}$ in the reference configuration.

We also postulate the invariance conditions under superposed rigid body motions which require that all the above fields be objective. More

precisely (see [4], Section 8), if two motions of a Cosserat surface $\{\mathbf{r}, \mathbf{d}\}$ and $\{\mathbf{r}^*, \mathbf{d}^*\}$ differs only by a superposed rigid body motion, that is

$$\mathbf{r}^*(\theta^\alpha, t') = \mathbf{r}^*_{\text{o}}(t') + \mathbf{Q}(t)[\mathbf{r}(\theta^\alpha, t) - \mathbf{r}_{\text{o}}(t)],\ \mathbf{d}^*(\theta^\alpha, t') = \mathbf{Q}(t)\mathbf{d}(\theta^\alpha, t)$$

where $t' = t + a'$, a' constant, and $\mathbf{Q}(t)$ is a proper orthogonal tensor–valued function, then the scalar fields $\rho, \nu, \varepsilon, \eta, \theta, q, r, h, g$ and p remain unchanged, while the vector fields obey the transformation laws

$$\mathbf{N}^* = \mathbf{Q}(t)\mathbf{N},\quad \mathbf{M}^* = \mathbf{Q}(t)\mathbf{M},\quad \mathbf{m}^* = \mathbf{Q}(t)\mathbf{m},$$

$$(\mathbf{f} - \dot{\mathbf{v}})^* = \mathbf{Q}(t)(\mathbf{f} - \dot{\mathbf{v}}),\quad (\mathbf{l} - \alpha\dot{\mathbf{w}})^* = \mathbf{Q}(t)(\mathbf{l} - \alpha\dot{\mathbf{w}}).$$

By using the balance of energy (3.1) for some motions which differ by superposed rigid body motions and imposing the above invariance conditions we obtain, in the same way as in [4], the principle of linear momentum and the principle of moment of momentum. Thus, the principle of linear momentum is given by

$$\frac{d}{dt}\int_{\mathcal{P}_{\text{o}}} \rho_{\text{o}}\mathbf{v}\,d\sigma = \int_{\mathcal{P}_{\text{o}}} \rho_{\text{o}}\mathbf{f}\,d\sigma + \int_{\partial\mathcal{P}_{\text{o}}} \mathbf{N}\,ds \tag{3.5}$$

and the principle of moment of momentum is expressed in the form

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{P}_{\text{o}}} \rho_{\text{o}}(\mathbf{r}\times\mathbf{v} + \alpha\mathbf{d}\times\mathbf{w})\,d\sigma = \\ &= \int_{\mathcal{P}_{\text{o}}} \rho_{\text{o}}(\mathbf{r}\times\mathbf{f} + \mathbf{d}\times\mathbf{l})\,d\sigma + \int_{\partial\mathcal{P}_{\text{o}}} (\mathbf{r}\times\mathbf{N} + \mathbf{d}\times\mathbf{M})\,ds. \end{aligned} \tag{3.6}$$

By application of the above principles to elementary curvilinear triangles on $\mathcal{S}$ (see [4], Section 9) we obtain for the fields $\mathbf{N}, \mathbf{M}, h$ and q the expressions

$$\mathbf{N} = \mathbf{N}^\alpha n_\alpha,\ \mathbf{M} = \mathbf{M}^\alpha n_\alpha,\ h = h^\alpha n_\alpha,\ q = q^\alpha n_\alpha, \tag{3.7}$$

where $\mathbf{N}^\alpha, \mathbf{M}^\alpha, h^\alpha$ and q^α transform as contravariant surface vectors and $\mathbf{n} = n_\alpha \mathbf{A}^\alpha$ is the outward unit normal to a curve on $\mathcal{S}$, lying in the surface.

Making use of an analogue of the divergence theorem for a surface (see [6], p.39) we derive, under suitable regularity assumptions, the local equations in vector form corresponding to the above principles

– linear momentum principle

$$\mathbf{N}^{\alpha}{}_{|\alpha} + \rho_{\mathrm{o}}\mathbf{f} = \rho_{\mathrm{o}}\dot{\mathbf{v}} \tag{3.8}$$

– director momentum principle

$$\mathbf{M}^{\alpha}{}_{|\alpha} + \rho_{\mathrm{o}}\mathbf{l} - \mathbf{m} = \rho_{\mathrm{o}}\alpha\dot{\mathbf{w}} \tag{3.9}$$

– moment of momentum principle

$$\mathbf{a}_{\alpha}\times\mathbf{N}^{\alpha} + \mathbf{d}\times\mathbf{m} + \mathbf{d}_{,\alpha}\times\mathbf{M}^{\alpha} = \mathbf{0}, \tag{3.10}$$

– balance of energy

$$\rho_{\mathrm{o}}r - q^{\alpha}{}_{|\alpha} - \rho_{\mathrm{o}}\dot{\varepsilon} + \mathbf{N}^{\alpha}\cdot\mathbf{v}_{,\alpha} + \mathbf{m}\cdot\mathbf{w} + \mathbf{M}^{\alpha}\cdot\mathbf{w}_{,\alpha} + g\dot{\nu} + h^{\alpha}\dot{\nu}_{,\alpha} = 0, \tag{3.11}$$

– balance of equilibrated force

$$h^{\alpha}{}_{|\alpha} + \rho_{\mathrm{o}}p - g = \rho_{\mathrm{o}}k\ddot{\nu}, \tag{3.12}$$

– entropy inequality

$$\rho_{\mathrm{o}}\theta\dot{\eta} - \rho_{\mathrm{o}}r + q^{\alpha}{}_{|\alpha} - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geq 0. \tag{3.13}$$

If we denote

$$P = \mathbf{N}^{\alpha}\cdot\mathbf{v}_{,\alpha} + \mathbf{m}\cdot\mathbf{w} + \mathbf{M}^{\alpha}\cdot\mathbf{w}_{,\alpha} + g\dot{\nu} + h^{\alpha}\dot{\nu}_{,\alpha} \tag{3.14}$$

and introduce the Helmholtz free energy function

$$\psi = \varepsilon - \eta\theta, \tag{3.15}$$

then the energy equation (3.11) takes the form

$$\rho_{\mathrm{o}}r - q^{\alpha}{}_{|\alpha} - \rho_{\mathrm{o}}(\dot{\eta}\theta + \eta\dot{\theta}) - \rho_{\mathrm{o}}\dot{\psi} + P = 0. \tag{3.16}$$

The entropy inequality (3.13) becomes

$$-\rho_{\mathrm{o}}\dot{\psi} - \rho_{\mathrm{o}}\eta\dot{\theta} + P - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geq 0. \tag{3.17}$$

In the remaining of this section we shall deduce the basic field equations written in tensor components. If we refer $\mathbf{N}^\alpha, \mathbf{M}^\alpha, \mathbf{m}$ to the base vectors $\mathbf{a}_i$, we obtain

$$\mathbf{N}^\alpha = N^{\alpha i}\mathbf{a}_i,\ \mathbf{M}^\alpha = M^{\alpha i}\mathbf{a}_i,\ \mathbf{m} = m^i\mathbf{a}_i, \tag{3.18}$$

where $N^{\alpha\beta}, M^{\alpha\gamma}$, as well as $N^{\alpha 3}, M^{\alpha 3}$ and m^α are surface tensors under the transformation of surface coordinates.

If we denote

$$\mathbf{f} = f^i\mathbf{a}_i,\ \mathbf{l} = l^i\mathbf{a}_i,\ \dot{\mathbf{v}} = c^i\mathbf{a}_i,\ \dot{\mathbf{w}} = e^i\mathbf{a}_i,$$

then the principle of linear momentum can be written in the form

$$N^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha N^{\alpha 3} + \rho_\mathrm{o} f^\beta = \rho_\mathrm{o} c^\beta,\ N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta}N^{\alpha\beta} + \rho_\mathrm{o} f^3 = \rho_\mathrm{o} c^3. \tag{3.19}$$

The principle of director momentum can be expressed as

$$\begin{aligned} M^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha M^{\alpha 3} - m^\beta + \rho_\mathrm{o} l^\beta &= \rho_\mathrm{o}\alpha e^\beta, \\ M^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta}M^{\alpha\beta} - m^3 + \rho_\mathrm{o} l^3 &= \rho_\mathrm{o}\alpha e^3. \end{aligned} \tag{3.20}$$

Moreover, the principle of moment of momentum becomes

$$\begin{aligned} \varepsilon_{\beta\alpha}[N^{\alpha\beta} + m^\beta d^\alpha + M^{\gamma\beta}\lambda^\alpha{}_\gamma] &= 0, \\ N^{\alpha 3} + (m^3 d^\alpha - m^\alpha d^3) + M^{\gamma 3}\lambda^\alpha{}_\gamma - M^{\gamma\alpha}\lambda^3{}_\gamma &= 0. \end{aligned} \tag{3.21}$$

Since the remaining of the basic field equations are scalar equations, it will suffice to express P in terms of tensor components. We find that

$$P = N'^{\alpha\beta}\eta_{\alpha\beta} + m^i\dot{d}_i + M^{\alpha i}\dot{\lambda}_{i\alpha} + g\dot{\nu} + h^\alpha\dot{\nu}_{,\alpha}, \tag{3.22}$$

where $N'^{\alpha\beta}$ is defined by

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} - m^\alpha d^\beta - M^{\gamma\alpha}\lambda^\beta{}_\gamma. \tag{3.23}$$

The symmetry of $N'^{\alpha\beta}$ in (3.23) is equivalent with the principle of moment of momentum $(3.21)_1$.

4. Constitutive equations. We postulate the following constitutive equations for a thermoelastic Cosserat shell with voids

$$
\begin{aligned}
&\psi = \tilde{\psi}(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&\eta = \tilde{\eta}(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&\mathbf{N}^\alpha = \tilde{\mathbf{N}}^\alpha(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&\mathbf{m} = \tilde{\mathbf{m}}(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&\mathbf{M}^\alpha = \tilde{\mathbf{M}}^\alpha(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&g = \tilde{g}(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&h^\alpha = \tilde{h}^\alpha(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),\\
&q^\alpha = \tilde{q}^\alpha(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}).
\end{aligned}
\tag{4.1}
$$

The above constitutive equations are assumed to hold at each particle θ^μ and for all times, but we have omitted the argument θ^μ for brevity. By imposing that the response functions (4.1) satisfy the entropy inequality (3.17) for all arbitrary values of $\dot{\mathbf{a}}_\alpha, \dot{\mathbf{d}}, \dot{\mathbf{d}}_{,\gamma}, \dot{\nu}, \dot{\nu}_{,\beta}, \dot{\theta}$ and $\dot{\theta}_{,\alpha}$ we obtain, in the classical manner, that the constitutive assumptions (4.1) reduce to

$$
\begin{gathered}
\psi = \tilde{\psi}(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta)\\
\eta = -\frac{\partial\tilde{\psi}}{\partial\theta}, \quad \mathbf{N}^\alpha = \rho_\mathrm{o}\frac{\partial\tilde{\psi}}{\partial\mathbf{a}_\alpha}, \quad \mathbf{m} = \rho_\mathrm{o}\frac{\partial\tilde{\psi}}{\partial\mathbf{d}},\\
\mathbf{M}^\alpha = \rho_\mathrm{o}\frac{\partial\tilde{\psi}}{\partial\mathbf{d}_{,\alpha}}, \quad g = \rho_\mathrm{o}\frac{\partial\tilde{\psi}}{\partial\nu}, \quad h^\alpha = \rho_\mathrm{o}\frac{\partial\tilde{\psi}}{\partial\nu_{,\alpha}},\\
q^\alpha = \tilde{q}^\alpha(\mathbf{a}_\alpha, \mathbf{d}, \mathbf{d}_{,\gamma}, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}),
\end{gathered}
\tag{4.2}
$$

and

$$
-\mathbf{q}^\alpha\theta_{,\alpha} \geq 0. \tag{4.3}
$$

By similar arguments as in [4, Section 13], we find that the constitutive equation $(4.2)_3$ can be replaced by

$$
\begin{aligned}
N'^{\alpha\beta} = N'^{\beta\alpha} = \frac{1}{2}\rho_\mathrm{o}\Bigg[&\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{a}_\alpha} - d^\alpha\frac{\partial\tilde{\psi}}{\partial\mathbf{d}} - \lambda^\alpha{}_\gamma\frac{\partial\tilde{\psi}}{\partial\mathbf{d}_{,\gamma}}\right)\mathbf{a}^\beta +\\
&+\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{a}_\beta} - d^\beta\frac{\partial\tilde{\psi}}{\partial\mathbf{d}} - \lambda^\beta{}_\gamma\frac{\partial\tilde{\psi}}{\partial\mathbf{d}_{,\gamma}}\right)\mathbf{a}^\alpha\Bigg].
\end{aligned}
\tag{4.4}
$$

Another reduction of the constitutive equations (4.2) arise from the fact that the functions $\psi, q^\alpha, \mathbf{N}^\alpha, \mathbf{m}, ...$ are objective. Using the Cauchy's

representation theorem for isotropic functions (see [4]) we obtain that $\widetilde{\psi}$ may be expressed as a different function of the set of variables

$$\mathcal{V} = \{a_{\alpha\beta}, d_\alpha, \lambda_{\beta\alpha}, \sigma, \sigma_\alpha\}, \tag{4.5}$$

where

$$\sigma = (d^3)^2, \ \sigma_\alpha = d^3\lambda_{3\alpha}. \tag{4.6}$$

Thus, we obtain

$$\begin{gathered}
\psi = \widehat{\psi}(a_{\alpha\beta}, d_\alpha, \lambda_{\beta\alpha}, \sigma, \sigma_\alpha, \nu, \nu_{,\beta}, \theta), \\
\eta = -\frac{\partial\widehat{\psi}}{\partial\theta}, \quad N^{'\alpha\beta} = N^{'\beta\alpha} = 2\rho_{\mathrm{o}}\frac{\partial\widehat{\psi}}{\partial a_{\alpha\beta}}, \\
m^\alpha = \rho_{\mathrm{o}}\frac{\partial\widehat{\psi}}{\partial d_\gamma}, \quad m^3 = \rho_{\mathrm{o}}\left(2d^3\frac{\partial\widehat{\psi}}{\partial\sigma} + \lambda^3{}_\alpha\frac{\partial\widehat{\psi}}{\partial\sigma_\alpha}\right), \\
M^{\alpha\beta} = \rho_{\mathrm{o}}\frac{\partial\widehat{\psi}}{\partial\lambda_{\beta\alpha}}, \quad M^{\alpha 3} = \rho_{\mathrm{o}}d^3\frac{\partial\widehat{\psi}}{\partial\sigma_\alpha}, \\
g = \rho_{\mathrm{o}}\frac{\partial\widehat{\psi}}{\partial\nu}, \quad h^\alpha = \rho_{\mathrm{o}}\frac{\partial\widehat{\psi}}{\partial\nu_{,\alpha}}, \\
q^\alpha = \hat{q}^\alpha(a_{\gamma\delta}, d_\gamma, \lambda_{\delta\gamma}, \sigma, \sigma_\gamma, \nu, \nu_{,\delta}, \theta, \theta_{,\gamma}).
\end{gathered} \tag{4.7}$$

In the above relations $\dfrac{\partial\widehat{\psi}}{\partial a_{\alpha\beta}}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial\widehat{\psi}}{\partial a_{\alpha\beta}} + \frac{\partial\widehat{\psi}}{\partial a_{\beta\alpha}}\right). \tag{4.8}$$

We can see from (4.7) that the constitutive equations determine only the symmetric part of $N^{\alpha\beta}$ or equivalently only $N^{'\alpha\beta}$. The skew–symmetric part of $N^{\alpha\beta}$ as well as $N^{\alpha 3}$ can be found from the moment of momentum principle (3.21), which is therefore regarded as a restriction on the constitutive equations.

An alternative form of the constitutive equations can be found in terms of relative kinematic measures such as those in (2.12). If we denote by

$$s = (d^3)^2 - (D^3)^2, \ s_\alpha = d^3\lambda_{3\alpha} - D^3\Lambda_{3\alpha}, \tag{4.9}$$

then a response function such as $\widehat{\psi}$ will depend on $\nu, \nu_{,\beta}, \theta$, the set of variables

$$\mathcal{U} = \{e_{\alpha\beta}, \gamma_\alpha, \kappa_{\beta\alpha}, s, s_\alpha\} \tag{4.10}$$

and the reference values

$$\mathcal{U}_R = \{A_{\alpha\beta}, D_\alpha, \Lambda_{\beta\alpha}, (D^3)^2, D^3\Lambda_{3\alpha}\}. \tag{4.11}$$

Thus, the set of constitutive assumptions (4.7) may be written in the form

$$\begin{gathered}
\psi = \psi(e_{\alpha\beta}, \gamma_\alpha, \kappa_{\beta\alpha}, s, s_\alpha, \nu, \nu_{,\beta}, \theta; \mathcal{U}_R), \\
\eta = -\frac{\partial\psi}{\partial\theta}, \quad N^{'\alpha\beta} = N^{'\beta\alpha} = \rho_\mathrm{o}\frac{\partial\psi}{\partial e_{\alpha\beta}}, \\
m^\alpha = \rho_\mathrm{o}\frac{\partial\psi}{\partial\gamma_\alpha}, \quad m^3 = \rho_\mathrm{o}\left(2d^3\frac{\partial\psi}{\partial s} + \lambda^3{}_\alpha\frac{\partial\psi}{\partial s_\alpha}\right), \\
M^{\alpha\gamma} = \rho_\mathrm{o}\frac{\partial\psi}{\partial\kappa_{\gamma\alpha}}, \quad M^{\alpha 3} = \rho_\mathrm{o}d^3\frac{\partial\psi}{\partial s_\alpha}, \\
g = \rho_\mathrm{o}\frac{\partial\psi}{\partial\nu}, \quad h^\alpha = \rho_\mathrm{o}\frac{\partial\psi}{\partial\nu_{,\alpha}}, \\
q^\alpha = q^\alpha(e_{\alpha\beta}, \gamma_\alpha, \kappa_{\beta\alpha}, s, s_\alpha, \nu, \nu_{,\beta}, \theta, \theta_{,\alpha}; \mathcal{U}_R).
\end{gathered} \tag{4.12}$$

Finally, we recapitulate the equations which govern the behaviour of an elastic Cosserat surface with voids. The basic field equations of the nonlinear theory consist in the equations of motion (3.19)–(3.21), the energy equation (3.16) with P given by (3.22), the equation of equilibrated force (3.12) and the constitutive equations (4.12). The heat flux q^α must satisfy the entropy inequality (4.3). To the field equations we must adjoin boundary conditions and initial conditions.

5. The linear theory. We assume that

$$\mathbf{r} = \mathbf{R} + \varepsilon\mathbf{u}, \ \mathbf{d} = \mathbf{D} + \varepsilon\delta, \ \nu = \nu_\mathrm{o} + \varepsilon\varphi, \ \theta = \theta_\mathrm{o} + \varepsilon T, \tag{5.1}$$

where ε is a small non–dimensional parameter, ν_o is the volume fraction field in the reference configuration and θ_o is the constant absolute temperature in the reference configuration. Clearly,

$$\mathbf{u} = u^i\mathbf{A}_i, \ \delta = \delta^i\mathbf{A}_i, \quad \mathbf{v} = \varepsilon\dot{\mathbf{u}}, \ \mathbf{w} = \varepsilon\dot{\delta}. \tag{5.2}$$

In the remaining of this paper we shall deal with Cosserat shells with voids whose reference director coincides with the unit normal to the reference configuration, i.e.

$$\mathbf{D} = \mathbf{A}_3. \tag{5.3}$$

According to [4] the above relation characterizes the shells with constant thickness in the reference configuration. From (5.3) and (2.10) we obtain in this case

$$D_\alpha = 0,\ D_3 = 1,\ \Lambda_{\beta\alpha} = -B_{\alpha\beta},\ \Lambda_{3\alpha} = 0. \tag{5.4}$$

Following the classical linearization procedures (also described in [4, Section 6]) and using (5.4) we finally get the appropriate expressions for the relative kinematical measures $e_{\alpha\beta}, \kappa_{i\alpha}$ and γ_i in terms of u_i, δ_i and their derivatives

$$\begin{aligned}
e_{\alpha\beta} &= \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}u_3,\\
\gamma_\alpha &= \delta_\alpha + u_{3,\alpha} + B^\gamma_\alpha u_\gamma, \qquad \gamma_3 = \delta_3,\\
\kappa_{\beta\alpha} &= \delta_{\beta|\alpha} - B_{\alpha\beta}\delta_3 - B^\gamma_\alpha u_{\gamma|\beta} + B^\gamma_\alpha B_{\beta\gamma}u_3,\\
\kappa_{3\alpha} &= \delta_{3,\alpha} + B^\gamma_\alpha \delta_\gamma + B^\gamma_\alpha u_{3,\gamma} + B^\beta_\alpha B^\gamma_\beta u_\gamma,
\end{aligned} \tag{5.5}$$

where all terms are of $O(\varepsilon)$ and the covariant differentiation is made with respect to the metric tensor $A_{\alpha\beta}$.

In order to deduce the linearized version of the field equations we must introduce additional assumptions. We assume that $\mathbf{N}^\alpha, \mathbf{M}^\alpha, \mathbf{m}, g$ and h^α are all zero in the reference configuration. We further assume that $\mathbf{N}^\alpha, \mathbf{M}^\alpha, \mathbf{m}, g, h^\alpha$ as well as their derivatives with respect to the surface coordinates, are of $O(\varepsilon)$ and that $(\eta - \eta_\mathrm{o})/\eta_\mathrm{o}$ is of $O(\varepsilon)$, where η_o denotes the standard entropy in the reference configuration. Thus, we now regard $u_i, \delta_i, N^{\alpha i}, M^{\alpha i}, m^i, g, h^\alpha, \varphi$ and T, as well as η (measured from his reference value) as infinitesimal quantities of $O(\varepsilon)$. By denoting this time

$$f^i = \mathbf{f}\cdot\mathbf{A}^i,\ l^i = \mathbf{l}\cdot\mathbf{A}^i,$$

then we can write the linearized version of the equations of motion (3.19)–(3.21) in the form

$$\begin{aligned}
N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha N^{\alpha 3} + \rho_\mathrm{o} f^\beta &= \rho_\mathrm{o}\ddot{u}^\beta,\\
N^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta}N^{\alpha\beta} + \rho_\mathrm{o} f^3 &= \rho_\mathrm{o}\ddot{u}^3
\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&M^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha}M^{\alpha 3} - m^{\beta} + \rho_{\mathrm{o}} l^{\beta} = \rho_{\mathrm{o}}\alpha\ddot{\delta}^{\beta} \\
&M^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta}M^{\alpha\beta} - m^{3} + \rho_{\mathrm{o}} l^{3} = \rho_{\mathrm{o}}\alpha\ddot{\delta}^{3}
\end{aligned} \tag{5.7}
$$

$$
\varepsilon_{\alpha\beta}[N^{\alpha\beta} - B^{\alpha}_{\gamma}M^{\gamma\beta}] = 0,\ N^{\alpha 3} - m^{\alpha} - B^{\alpha}_{\gamma}M^{\gamma 3} = 0. \tag{5.8}
$$

By virtue of (5.4), relation (3.23) becomes

$$
N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + B^{\beta}_{\gamma}M^{\gamma\alpha} \tag{5.9}
$$

Upon linearization, the energy equation (3.16) takes the form

$$
q^{\alpha}{}_{|\alpha} + \rho_{\mathrm{o}}\theta_{\mathrm{o}}\dot{\eta} - \rho_{\mathrm{o}} r = 0, \tag{5.10}
$$

while the equation of equilibrated force (3.12) remains the same

$$
h^{\alpha}{}_{|\alpha} - g + \rho_{\mathrm{o}} p = \rho_{\mathrm{o}} k\ddot{\varphi}. \tag{5.11}
$$

To obtain the corresponding constitutive equations, we remark that the variables s and s_α in (4.9) become in the linear theory and in the hypothesis (5.3)

$$
s = 2\gamma_3,\ s_\alpha = \kappa_{3\alpha},
$$

while the set of variables $\mathcal{U}_R$ from (4.11) reduces to $\{A_{\alpha\beta}, -B_{\alpha\beta}\}$. Thus, the linear constitutive assumptions take the simpler form:

$$
\begin{aligned}
&\psi = \psi(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \varphi, \varphi_{,\beta}, T; A_{\alpha\beta}, B_{\alpha\beta}, \nu_{\mathrm{o}}, \theta_{\mathrm{o}}), \\
&\eta = -\frac{\partial\psi}{\partial T},\quad N'^{\alpha\beta} = N'^{\beta\alpha} = \rho_{\mathrm{o}}\frac{\partial\psi}{\partial e_{\alpha\beta}},\quad m^{i} = \rho_{\mathrm{o}}\frac{\partial\psi}{\partial\gamma_i}, \\
&M^{\alpha i} = \rho_{\mathrm{o}}\frac{\partial\psi}{\partial\kappa_{i\alpha}},\quad g = \rho_{\mathrm{o}}\frac{\partial\psi}{\partial\varphi},\quad h^{\alpha} = \rho_{\mathrm{o}}\frac{\partial\psi}{\partial\varphi_{,\alpha}},
\end{aligned} \tag{5.12}
$$

$$
q^{\alpha} = q^{\alpha}(e_{\alpha\beta}, \gamma_i, \kappa_{i\alpha}, \varphi, \varphi_{,\beta}, T, T_{,\alpha}; A_{\alpha\beta}, B_{\alpha\beta}, \nu_{\mathrm{o}}, \theta_{\mathrm{o}}). \tag{5.13}
$$

We remind that the symmetry in $(5.12)_3$ is equivalent with the equation of moment of momentum $(5.8)_1$, while $(5.8)_2$ can be regarded as a constitutive relation for $N^{\alpha 3}$.

According to the linearization procedures and the assumptions made, the response function ψ in (5.12) is now a quadratic function of the appropriate variables, while the response function q^α in (5.13) is a linear function of its arguments.

Since the response function q^α must satisfy the entropy inequality

$$-q^\alpha T_{,\alpha} \geq 0, \tag{5.14}$$

it follows, by classical arguments, that q^α must be linear in the temperature gradient and can depend in addition only on the reference values $\{A_{\alpha\beta}, B_{\alpha\beta}, \nu_{\mathrm{o}}, \theta_{\mathrm{o}}\}$. Thus, (5.13) is replaced by

$$q^\alpha = -k^{\alpha\beta} T_{,\beta}, \tag{5.15}$$

where the conductivity tensor $k^{\alpha\beta}$ is positive semi–definite.

In the context of the linear theory, the response function ψ from (5.12) can be expressed in the form

$$\begin{aligned}
\rho_{\mathrm{o}}\psi = {} & \frac{1}{2}\,{}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + \frac{1}{2}\,{}_2C^{\alpha\beta\gamma\delta} \kappa_{\alpha\beta}\kappa_{\gamma\delta} + {}_3C^{\alpha\beta\gamma\delta} e_{\alpha\beta}\kappa_{\gamma\delta} + \\
& + {}_1C^{\alpha\beta\gamma} \kappa_{3\alpha}\kappa_{\beta\gamma} + {}_2C^{\alpha\beta\gamma} e_{\alpha\beta}\gamma_\gamma + {}_3C^{\alpha\beta\gamma} e_{\alpha\beta}\kappa_{3\gamma} + \\
& + {}_4C^{\alpha\beta\gamma}\gamma_\alpha\kappa_{\beta\gamma} + {}_5C^{\alpha\beta\gamma} e_{\alpha\beta}\varphi_{,\gamma} + {}_6C^{\alpha\beta\gamma}\kappa_{\alpha\beta}\varphi_{,\gamma} + \\
& + \frac{1}{2}\,{}_1C^{\alpha\beta}\gamma_\alpha\gamma_\beta + \frac{1}{2}\,{}_2C^{\alpha\beta}\kappa_{3\alpha}\kappa_{3\beta} + {}_3C^{\alpha\beta}\gamma_\alpha\kappa_{3\beta} + {}_4C^{\alpha\beta} e_{\alpha\beta}\gamma_3 + \\
& + {}_5C^{\alpha\beta}\kappa_{\alpha\beta}\gamma_3 + {}_4C'^{\alpha\beta} e_{\alpha\beta} T + {}_5C'^{\alpha\beta}\kappa_{\alpha\beta} T + {}_6C^{\alpha\beta}\gamma_\alpha\varphi_{,\beta} + \\
& + {}_7C^{\alpha\beta}\kappa_{3\alpha}\varphi_{,\beta} + \frac{1}{2}\,{}_8C^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + {}_9C^{\alpha\beta} e_{\alpha\beta}\varphi + {}_{10}C^{\alpha\beta}\kappa_{\alpha\beta}\varphi + \\
& + {}_1C^{\alpha}\gamma_\alpha\gamma_3 + {}_2C^{\alpha}\kappa_{3\alpha}\gamma_3 + {}_1C'^{\alpha}\gamma_\alpha T + {}_2C'^{\alpha}\kappa_{3\alpha} T + {}_3C^{\alpha}\varphi_{,\alpha}\gamma_3 + \\
& + {}_3C'^{\alpha}\varphi_{,\alpha} T + {}_4C^{\alpha}\varphi_{,\alpha}\varphi + {}_5C^{\alpha}\gamma_\alpha\varphi + {}_6C^{\alpha}\kappa_{3\alpha}\varphi + \frac{1}{2}\,{}_1C(\gamma_3)^2 + \\
& + \frac{1}{2}\,{}_2C\varphi^2 + {}_3C\gamma_3\varphi + {}_3C'\gamma_3 T + {}_4C'\varphi T - \frac{1}{2}aT^2,
\end{aligned} \tag{5.16}$$

where the coefficients depend on the reference values $\{A_{\alpha\beta}, B_{\alpha\beta}, \nu_{\mathrm{o}}, \theta_{\mathrm{o}}\}$ and some of them satisfy certain symmetry conditions

$$\begin{aligned}
& {}_1C^{\alpha\beta\gamma\delta} = {}_1C^{\gamma\delta\alpha\beta},\ {}_2C^{\alpha\beta\gamma\delta} = {}_2C^{\gamma\delta\alpha\beta}, \\
& {}_1C^{\alpha\beta} = {}_1C^{\beta\alpha},\ {}_2C^{\alpha\beta} = {}_2C^{\beta\alpha},\ {}_8C^{\alpha\beta} = {}_8C^{\beta\alpha}.
\end{aligned} \tag{5.17}$$

From (5.12) we can readily calculate explicit relations for $\eta, N'^{\alpha\beta}, m^i, M^{\alpha i}, g$ and h^α, but we do not record these formulae.

6. Uniqueness. We consider the linear theory of Cosserat shells with voids for which (5.3) holds. The fundamental system of field equations consists of the equations of motion (5.6)–(5.8), the energy equation (5.10), the equation of equilibrated force (5.11) and the constitutive equations (5.16), $(5.12)_2$–$(5.12)_7$ and (5.15). We suppose that the constitutive coefficients ${}_nC^{\alpha\beta\dots}$ satisfy the symmetry relations (5.17) and that the conductivity tensor $k^{\alpha\beta}$ is symmetric.

Let $\mathcal{S}$ be the surface of the Euclidean 3-space occupied by a Cosserat shell in the initial reference configuration and let C_i $(i = \overline{1,8})$ be subsets of $\partial\mathcal{S}$ (assumed to be a piecewise smooth curve) such that

$$\overline{C}_1 \cup C_2 = \overline{C}_3 \cup C_4 = \overline{C}_5 \cup C_6 = \overline{C}_7 \cup C_8 = \partial\mathcal{S},$$
$$C_1 \cap C_2 = C_3 \cap C_4 = C_5 \cap C_6 = C_7 \cap C_8 = \emptyset.$$

We consider the mixed boundary conditions:

$$\begin{array}{ll} u_i = \widetilde{u}_i \text{ on } \overline{C}_1\times I, & N^{\alpha i}n_\alpha = \widetilde{N}^i \text{ on } C_2\times I, \\ \delta_i = \widetilde{\delta}_i \text{ on } \overline{C}_3\times I, & M^{\alpha i}n_\alpha = \widetilde{M}^i \text{ on } C_4\times I, \\ \varphi = \widetilde{\varphi} \text{ on } \overline{C}_5\times I, & h^\alpha n_\alpha = \widetilde{h} \text{ on } C_6\times I \\ T = \widetilde{T} \text{ on } \overline{C}_7\times I, & q^\alpha n_\alpha = \widetilde{q} \text{ on } C_8\times I, \end{array} \tag{6.1}$$

where $I = [0,\infty)$ and the functions in the right–hand sides of (6.1) are prescribed. To the system of field equations we adjoin the boundary conditions (6.1) and the initial conditions

$$\begin{array}{ll} u_i(\theta^\alpha,0) = u_{0i}(\theta^\alpha), & \dot{u}_i(\theta^\alpha,0) = v_{0i}(\theta^\alpha), \\ \delta_i(\theta^\alpha,0) = \delta_{0i}(\theta^\alpha), & \dot{\delta}_i(\theta^\alpha,0) = w_{0i}(\theta^\alpha), \\ \varphi(\theta^\alpha,0) = \varphi_0(\theta^\alpha), & \dot{\varphi}(\theta^\alpha,0) = \mu_0(\theta^\alpha), \\ \eta(\theta^\alpha,0) = \eta_o(\theta^\alpha), & (\theta^\alpha) \in \Sigma, \end{array} \tag{6.2}$$

where the functions in the right–hand sides of (6.2) are given and assumed to be continuous on $\mathcal{S}$. We also suppose that $\widetilde{u}_i, \widetilde{\delta}_i, \widetilde{\varphi}$ and $\widetilde{T}$ are continuous, while $\widetilde{N}_i, \widetilde{M}_i, \widetilde{q}$ and $\widetilde{h}$ are continuous in time and piecewise regular on the appropriate domains.

To avoid repeated regularity assumptions, we assume that ${}_nC^{\alpha\beta\dots}$ and $k^{\alpha\beta}$ are continuously differentiable on $\overline{\mathcal{S}}$, ρ_o and a are continuous on $\overline{\mathcal{S}}$ and f^i, l^i, p and r are continuous on $\overline{\mathcal{S}}\times I$. A solution of the mixed problem is a set

of functions $\{u_i, \delta_i, \varphi, T\}$ satisfying the system of equations, the boundary conditions (6.1) and the initial conditions (6.2) such that they are of class C^1 on $\overline{\mathcal{S}}\times I$ and of class C^2 on $\mathcal{S}\times I$.

In deducing the following theorems we shall use arguments analogue to those presented in [7]. We denote the kinetic energy and the internal energy of the shell by $K(t)$ and $U(t)$, respectively

$$K(t)=\int_{\mathcal{S}}\rho_{\mathrm{o}}(\mathbf{v}^2+\alpha\mathbf{w}^2+k\dot{\varphi}^2)d\sigma,\quad U(t)=\int_{\mathcal{S}}\rho_{\mathrm{o}}\varepsilon d\sigma,\ t\in I. \tag{6.3}$$

Theorem 1. *Let*

$$R(t)=\int_{\mathcal{S}}\rho_{\mathrm{o}}\left(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\mathbf{w}+p\dot{\varphi}+\frac{1}{\theta_{\mathrm{o}}}Tr\right)d\sigma+\int_{\partial\mathcal{S}}(\mathbf{N}\cdot\mathbf{v}+\mathbf{M}\cdot\mathbf{w}+ +h\dot{\varphi}-\frac{1}{\theta_{\mathrm{o}}}Tq)ds, \tag{6.4}$$

for $t\in I$. Then

$$\frac{d}{dt}[K(t)+U(t)]=R(t)-\frac{1}{\theta_{\mathrm{o}}}\int_{\mathcal{S}}k^{\alpha\beta}T_{,\alpha}T_{,\beta}d\sigma. \tag{6.5}$$

Proof. Using (3.7), the analogue of the divergence theorem and then the equations (3.8), (3.9), (5.10) and (3.12) we can write

$$\begin{aligned}
&\int_{\partial\mathcal{S}}\left(\mathbf{N}\cdot\mathbf{v}+\mathbf{M}\cdot\mathbf{w}+h\dot{\varphi}-\frac{1}{\theta_{\mathrm{o}}}Tq\right)ds=\\
&\int_{\mathcal{S}}\left(\mathbf{N}^{\alpha}\cdot\mathbf{v}+\mathbf{M}^{\alpha}\cdot\mathbf{w}+h^{\alpha}\dot{\varphi}-\frac{1}{\theta_{\mathrm{o}}}Tq^{\alpha}\right)_{|\alpha}d\sigma=\\
&=\int_{\mathcal{S}}\left(\mathbf{N}^{\alpha}{}_{|\alpha}\cdot\mathbf{v}+\mathbf{M}^{\alpha}{}_{|\alpha}\cdot\mathbf{w}+h^{\alpha}{}_{|\alpha}\dot{\varphi}-\frac{1}{\theta_{\mathrm{o}}}Tq^{\alpha}{}_{|\alpha}-\frac{1}{\theta_{\mathrm{o}}}q^{\alpha}T_{,\alpha}\right)d\sigma+\\
&+\int_{\mathcal{S}}(\mathbf{N}^{\alpha}\cdot\mathbf{v}_{,\alpha}+\mathbf{M}^{\alpha}\cdot\mathbf{w}_{,\alpha}+h^{\alpha}\dot{\varphi}_{,\alpha})\,d\sigma=\int_{\mathcal{S}}\rho_{\mathrm{o}}(\mathbf{v}\cdot\dot{\mathbf{v}}+\alpha\mathbf{w}\cdot\dot{\mathbf{w}}+k\dot{\varphi}\ddot{\varphi})d\sigma-\\
&-\int_{\mathcal{S}}\rho_{\mathrm{o}}\left[(\mathbf{f}\cdot\mathbf{v}+\mathbf{l}\cdot\mathbf{w}+p\dot{\varphi})+\frac{1}{\theta_{\mathrm{o}}}T(-\theta_{\mathrm{o}}\dot{\eta}+r)\right]d\sigma-\frac{1}{\theta_{\mathrm{o}}}\int_{\mathcal{S}}q^{\alpha}T_{,\alpha}d\sigma+\\
&+\int_{\mathcal{S}}[\mathbf{N}^{\alpha}\cdot\mathbf{v}_{,\alpha}+\mathbf{M}^{\alpha}\cdot\mathbf{w}_{,\alpha}+\mathbf{m}\cdot\mathbf{w}+g\dot{\varphi}+h^{\alpha}\dot{\varphi}_{,\alpha}]d\sigma.
\end{aligned}$$

Taking into account the notation (3.14) we get

$$\frac{d}{dt}K(t)+\int_{\mathcal{S}}(P+\rho_{\mathrm{o}}T\dot{\eta})d\sigma=R(t)-\frac{1}{\theta_{\mathrm{o}}}\int_{\mathcal{S}}k^{\alpha\beta}T_{,\alpha}T_{,\beta}d\sigma. \tag{6.6}$$

From the cosntitutive equations in the form (4.2) we see that

$$\rho_o\dot{\psi} = P - \rho_o\eta\dot{T},$$

and then

$$P + \rho_o T\dot{\eta} = \rho_o\dot{\psi} + \rho_o(\eta\dot{T} + T\dot{\eta}) = \rho_o\dot{\varepsilon},$$

which shows that (6.6) reduces to (6.5) and the proof is complete. ∎

Remark 1. By means of Theorem 1, we can prove in the same way as in [4, Section 26] a uniqueness theorem for thermoelastic shells with voids in the hypotheses that ψ is nonnegative, $a > 0$, $\alpha > 0$ and $k > 0$.

Next, we present a counterpart of Brun's theorem in the isothermal theory of elasticity (see [8, p.217]).

For any $\alpha, \beta \in I$ we define

$$G(\alpha,\beta) =$$

$$\begin{aligned}(6.7)\qquad &= \int_{\mathcal{S}} \rho_o \left(\mathbf{f}(\alpha)\cdot\mathbf{v}(\beta) + \mathbf{l}(\alpha)\cdot\mathbf{w}(\beta) + p(\alpha)\dot{\varphi}(\beta) - \frac{1}{\theta_o} r(\alpha)T(\beta)\right) d\sigma + \\ &+ \int_{\partial\mathcal{S}} \left(\mathbf{N}(\alpha)\cdot\mathbf{v}(\beta) + \mathbf{M}(\alpha)\cdot\mathbf{w}(\beta) + h(\alpha)\dot{\varphi}(\beta) + \frac{1}{\theta_o} q(\alpha)T(\beta)\right) ds.\end{aligned}$$

Theorem 2. *For every $t \in I$ the following relation hold*

$$\begin{aligned}(6.8)\qquad U(t) - K(t) &= \frac{1}{2}\int_0^t [G(t+\tau, t-\tau) - G(t-\tau, t+\tau)]d\tau + \\ &+ \frac{1}{2}\int_{\mathcal{S}} [N'^{\alpha\beta}(0)e_{\alpha\beta}(2t) + M^{\alpha i}(0)\kappa_{i\alpha}(2t) + m^i(0)\gamma_i(2t) + \\ &+ h^\alpha(0)\varphi_{,\alpha}(2t) + g(0)\varphi(2t) + \rho_o\eta(2t)\theta(0)]d\sigma - \\ &- \frac{1}{2}\int_{\mathcal{S}} \rho_o[\mathbf{v}(0)\cdot\mathbf{v}(2t) + \alpha\mathbf{w}(0)\cdot\mathbf{w}(2t) + k\dot{\varphi}(0)\dot{\varphi}(2t)]d\sigma.\end{aligned}$$

Proof. Let us consider the function E defined on I by

$$\begin{aligned}E(\tau) &= N'^{\alpha\beta}(t-\tau)e_{\alpha\beta}(t+\tau) + M^{\alpha i}(t-\tau)\kappa_{i\alpha}(t+\tau) + \\ &+ m^i(t-\tau)\gamma_i(t+\tau) + h^\alpha(t-\tau)\varphi_{,\alpha}(t+\tau) + \\ &+ g(t-\tau)\varphi(t+\tau) + \rho_o T(t-\tau)\eta(t+\tau), \quad \tau \in [0,t].\end{aligned}$$

Clearly,

$$
\begin{gathered}
\dot{E}(\tau) \equiv \frac{dE}{d\tau} = [N'^{\alpha\beta}(t-\tau)\dot{e}_{\alpha\beta}(t+\tau)+ \\
+M^{\alpha i}(t-\tau)\dot{\kappa}_{i\alpha}(t+\tau) + m^i(t-\tau)\dot{\gamma}_i(t+\tau)+ \\
+h^\alpha(t-\tau)\dot{\varphi}_{,\alpha}(t+\tau) + g(t-\tau)\dot{\varphi}(t+\tau) - \rho_o\dot{\eta}(t-\tau)T(t+\tau)]- \\
-[\dot{N}'^{\alpha\beta}(t-\tau)e_{\alpha\beta}(t+\tau) + \dot{M}^{\alpha i}(t-\tau)\kappa_{i\alpha}(t+\tau)+ \\
+\dot{m}^i(t-\tau)\gamma_i(t+\tau) + \dot{g}(t-\tau)\varphi(t+\tau)+ \\
+\dot{h}(t-\tau)\varphi_{,\alpha}(t+\tau) - \rho_o\dot{\eta}(t+\tau)T(t-\tau)]+ \\
+[\rho_o\dot{\eta}(t-\tau)T(t+\tau) - \rho_o\eta(t+\tau)\dot{T}(t-\tau)].
\end{gathered} \tag{6.9}
$$

Taking into account that $\rho_o\psi$ is a quadratic function as in (5.16) and that $N'^{\alpha\beta} = \rho_o\dfrac{\partial\psi}{\partial e_{\alpha\beta}}$, etc. (relations (5.12)), one can see that some terms of the second brackets of (6.9) will cancel others from the third brackets, while the remaining terms can be regrouped to give

$$
\begin{gathered}
\dot{E}(\tau) = [N'^{\alpha\beta}(t-\tau)\dot{e}_{\alpha\beta}(t+\tau) + M^{\alpha i}(t-\tau)\dot{\kappa}_{i\alpha}(t+\tau)+ \\
+m^i(t-\tau)\dot{\gamma}_i(t+\tau) + g(t-\tau)\dot{\varphi}(t+\tau)+ \\
+h^\alpha(t-\tau)\dot{\varphi}_{,\alpha}(t+\tau) - \rho_o\dot{\eta}(t-\tau)T(t+\tau)]- \\
-[N'^{\alpha\beta}(t+\tau)\dot{e}_{\alpha\beta}(t-\tau) + M^{\alpha i}(t+\tau)\dot{\kappa}_{i\alpha}(t-\tau)+ \\
+m^i(t+\tau)\dot{\gamma}_i(t-\tau) + g(t+\tau)\dot{\varphi}(t-\tau)+ \\
+h^\alpha(t+\tau)\dot{\varphi}_{,\alpha}(t-\tau) - \rho_o\dot{\eta}(t+\tau)T(t-\tau)].
\end{gathered} \tag{6.10}
$$

The expression in the first brackets from the above relation can be put into another form. Indeed, by replacing $N'^{\alpha\beta}$ by (5.9) and $e_{\alpha\beta}, \kappa_{i\alpha}, \gamma_i$ from (5.5), we obtain

$$
\begin{gathered}
[(N^{\alpha\beta} + B^\beta_\gamma M^{\gamma\alpha})(t-\tau)(\dot{u}_{\beta|\alpha} - B_{\alpha\beta}\dot{u}_3)(t+\tau)+ \\
+M^{\alpha\beta}(t-\tau)(\dot{\delta}_{\beta|\alpha} + B^\gamma_\alpha B_{\beta\gamma}\dot{u}_3 - B_{\alpha\beta}\dot{\delta}_3 - B^\gamma_\alpha\dot{u}_{\gamma|\beta})(t+\tau)+ \\
+m^\alpha(t-\tau)(\dot{\delta}_\alpha + \dot{u}_{3,\alpha} + B^\gamma_\alpha\dot{u}_\gamma)(t+\tau) + m^3(t-\tau)\dot{\delta}_3(t+\tau)+ \\
+M^{\alpha 3}(t-\tau)(\dot{\delta}_{3,\alpha} + B^\gamma_\alpha\dot{\delta}_\alpha + B^\gamma_\alpha u_{3,\alpha} + B^\beta_\alpha B^\gamma_\beta u_\gamma)(t+\tau)+ \\
+g(t-\tau)\dot{\varphi}(t+\tau) + h^\alpha(t-\tau)\dot{\varphi}_{,\alpha}(t+\tau) - \rho_o\dot{\eta}(t-\tau)T(t+\tau)].
\end{gathered} \tag{6.11}
$$

If we use now formulae of the type

$$
\begin{gathered}
N^{\alpha\beta}(t-\tau)\dot{u}_{\beta|\alpha}(t+\tau) = \\
= [N^{\alpha\beta}(t-\tau)\dot{u}_\beta(t+\tau)]_{|\alpha} - N^{\alpha\beta}{}_{|\alpha}(t-\tau)\dot{u}_\beta(t+\tau), \\
\rho_o\dot{\eta}(t-\tau)T(t+\tau) = \frac{1}{\theta_o}\rho_o r(t-\tau)T(t+\tau) - \frac{1}{\theta_o}[q^\alpha(t-\tau)T(t+\tau)]_{|\alpha}+ \\
+\frac{1}{\theta_o}q^\alpha(t-\tau)T_{,\alpha}(t+\tau)
\end{gathered}
$$

and then replace $N^{\alpha i}{}_{|\alpha}, M^{\alpha i}{}_{|\alpha}, N^{\alpha 3}$ and $h^{\alpha}{}_{|\alpha}$ from the equations of motion (5.6), (5.7), $(5.8)_2$ and the equation of equilibrated force (5.11), respectively, we reduce the expression in the first brackets of (6.10) to the form

$$
\begin{aligned}
&[N^{\alpha i}(t-\tau)\dot{u}_i(t+\tau)+M^{\alpha i}(t-\tau)\dot{\delta}_i(t+\tau)+\\
&+h^{\alpha}(t-\tau)\dot{\varphi}(t+\tau)+\frac{1}{\theta_{\mathrm{o}}}q^{\alpha}(t-\tau)T(t+\tau)]_{|\alpha}+\\
&+\rho_{\mathrm{o}}[f^i(t-\tau)\dot{u}_i(t+\tau)+\ell^i(t-\tau)\dot{\delta}_i(t+\tau)+\\
&+p(t-\tau)\dot{\varphi}(t+\tau)-\frac{1}{\theta_{\mathrm{o}}}r(t-\tau)T(t+\tau)]-\\
&-\rho_{\mathrm{o}}\left[\ddot{u}^i(t-\tau)\dot{u}_i(t+\tau)+\alpha\ddot{\delta}^i(t-\tau)\dot{\delta}_i(t+\tau)+k\ddot{\varphi}(t-\tau)\dot{\varphi}(t+\tau)\right]+\\
&+\frac{1}{\theta_{\mathrm{o}}}k^{\alpha\beta}T_{,\beta}(t-\tau)T_{,\alpha}(t+\tau),
\end{aligned}
\tag{6.12}
$$

since all the remaining terms in (6.11) will cancel each other, as can be proved by a thorough calculation.

A similar form can be obtained for the second brackets of (6.10), except that we have to change the roles of $(t-\tau)$ and $(t+\tau)$.

Next, by integrating the expression of $\dot{E}(\tau)$ (written with the help of (6.12)) over $\mathcal{S}$ and making use of the divergence theorem, we get

$$
\begin{aligned}
&\int_{\mathcal{S}}\dot{E}(\tau)d\sigma=G(t-\tau,t+\tau)-G(t+\tau,t-\tau)+\\
&+\int_{\mathcal{S}}\frac{d}{d\tau}\rho_{\mathrm{o}}[\mathbf{v}(t-\tau)\cdot\mathbf{v}(t+\tau)+\alpha\mathbf{w}(t-\tau)\cdot\mathbf{w}(t+\tau)+k\dot{\varphi}(t-\tau)\dot{\varphi}(t+\tau)]d\sigma.
\end{aligned}
$$

If we integrate this relation from 0 to t and take into account that $E(0)=2\rho_{\mathrm{o}}\varepsilon$, we finally arrive to the desired result (6.8). ∎

Remark 2. By using relations (6.5) (integrated from 0 to t) and (6.8) we can readily deduce separate formulae for $U(t)$ and $K(t)$.

We are now able to prove the following uniqueness theorem

Theorem 3. *Assume that:*

(i) $k^{\alpha\beta}$ *is symmetric and positive semi–definite;*

(ii) *the constitutive coefficients satisfy the symmetry relations* (5.17)*;*

(iii) $\rho_{\mathrm{o}}>0$;

(iv) $\alpha>0,\ k>0$ *and* $a>0$.

Then the mixed problem under consideration has at most one solution.

Proof. Suppose that there are two solutions. Then their difference, denoted by $\{u_i, \delta_i, \varphi, T\}$, is a solution of the mixed problem which corresponds to null data. Thus, we conclude from (6.5) and (6.8) that

$$2K(t) + \frac{1}{\theta_o}\int_0^t\int_{\mathcal{S}} k^{\alpha\beta}T_{,\alpha}T_{,\beta}d\sigma\, d\tau = 0.$$

Since $k^{\alpha\beta}$ is positive semi–definite and $K(t) \geq 0$ by definition (see (6.3)), it follows that

$$\int_0^t\int_{\mathcal{S}} k^{\alpha\beta}T_{,\alpha}T_{,\beta}d\sigma\, d\tau = 0, \qquad \text{and} \tag{6.13}$$

$$\dot{\mathbf{u}} = \mathbf{0},\ \dot{\delta} = \mathbf{0},\ \dot{\varphi} = 0 \text{ on } \mathcal{S}\times I. \tag{6.14}$$

The last relations and the fact that $\mathbf{u}, \delta$ and φ vanish at $t = 0$ yield

$$u_i = 0,\ \delta_i = 0,\ \varphi = 0 \text{ on } \mathcal{S}\times I. \tag{6.15}$$

From (6.5) and (6.8) we also obtain

$$2U(t) + \frac{1}{\theta_o}\int_0^t\int_{\mathcal{S}} k^{\alpha\beta}T_{,\alpha}T_{,\beta}d\sigma\, d\tau = 0,$$

which, in view of (6.13) and (6.15), reduces to

$$\int_{\mathcal{S}} aT^2 d\sigma = 0.$$

Since $a > 0$ we conclude that $T = 0$ on $\mathcal{S}\times I$ and the proof is complete. ∎

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