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ON SAINT–VENANT'S PROBLEM IN MICROPOLAR VISCOELASTICITY

BY

C. GALEŞ

Summary. The present paper is concerned with Saint–Venant's problem for inhomogeneous and anisotropic cylinder in the linear theory of Cosserat viscoelasticity. The material is at rest at all times $t < 0$ and the coefficients are independent of the axial coordinate. On the basis of the solutions of some generalized plane strain problems we describe two classes of solutions to Saint–Venant's problem. These classes are used in order to obtain a solution for the relaxed Saint–Venant's problem.

1. Introduction. The continuum media with microstructure are extensively studied in literature. The memory effects are coupled with the microstructure of the material in [3]. The basic equations for the linear theory of micropolar viscoelasticity are presented in [3], [7]. In [7], IEŞAN establishes some general theorems in linear micropolar theory of viscoelastodynamics.

In the present paper, we consider the Saint–Venant's problem for a cylinder made of an inhomogeneous and anisotropic micropolar viscoelastic material. We extend the results established by CHIRIŢĂ [1]. In the first part of the paper we define the generalized plane strain for a quasi–static micropolar viscoelastic material. The existence and uniqueness of the solution for the resulting two–dimensional boundary value problem can be established by using the theory described by FICHERA [4]. Then an analysis of Saint–Venant's problem is given by means of some plane cross section solutions. We treat Saint–Venant's problem by reformulating the equilibrium equations with the axial variable playing the role of a parameter. We deduce the conditions upon the solution of Saint–Venant's problem in order to treat it as a generalized plane strain problem. Further, we point out two classes of solutions of Saint–Venant's problem that may be expressed in terms of a plane motion. In the last part of the paper, we use these classes in order to obtain a solution for the relaxed Saint–Venant's problem.

2. Saint–Venant's problem. Throughout this paper B denotes the interior of a straight cylinder of length L with the open cross–section D and the lateral boundary π. We call ∂D the boundary of the generic cross–section D. Throughout this paper a rectangular Cartesian coordinate system $0x_k$ $(k = 1, 2, 3)$ is used. The rectangular Cartesian coordinate system is chosen such that the x_3–axis is parallel to the generators of B, and the $x_1 0 x_2$–plane contains one of the terminal cross–sections. We call $D(0)$ the cross–section located at $x_3 = 0$ and $D(L)$ the cross–section which lies in the plane $x_3 = L$.

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1, 2)$, whereas Latin subscripts (unless otherwise specified) to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; superposed dots denote differentiation with respect to the time variable; where no confusion may occur, we suppress the dependence upon the spatial variables.

We assume that the body occupying B is a linearly viscoelastic micropolar material that is at rest at all times $t < 0$. Let u_i be the components of the displacement vector and φ_i the components of the microrotation vector. Then

$$e_{ij}(\mathbf{u}) = u_{j,i} + \varepsilon_{jik}\varphi_k,\ \kappa_{ij} = \varphi_{j,i} \tag{1}$$

are the kinematic characteristics of the strain associated with $\mathbf{u} = (u_i, \varphi_i)$. The components of the stress tensor and the components of the couple–stress tensor are [3]

$$\begin{aligned} S_{ij}(\mathbf{u}) &= A_{ijkl}(0)e_{kl} + B_{ijkl}(0)\kappa_{kl} + \int_0^t (\dot{A}_{ijkl}(t-z)e_{kl}(z)+ \\ &\qquad +\dot{B}_{ijkl}(t-z)\kappa_{kl}(z))dz, \\ M_{ij}(\mathbf{u}) &= B_{klij}(0)e_{kl} + C_{ijkl}(0)\kappa_{kl} + \int_0^t (\dot{B}_{klij}(t-z)e_{kl}(z)+ \\ &\qquad +\dot{C}_{ijkl}(t-z)\kappa_{kl}(z))dz. \end{aligned} \tag{2}$$

The characteristic coefficients of the material $A_{ijkl}, B_{ijkl}, C_{ijkl}$ satisfy the symmetry relations

$$A_{ijkl} = A_{klij},\ C_{ijkl} = C_{klij}. \tag{3}$$

Moreover, we assume that

$$\text{(4)}\quad A_{ijkl} = A_{ijkl}(x_1, x_2; t),\ B_{ijkl} = B_{ijkl}(x_1, x_2; t),\ C_{ijkl} = C_{ijkl}(x_1, x_2; t)$$

are smooth functions on $\overline{D}\times[0, \infty)$.

The surface tractions and surface couples acting at point $\mathbf{x}$ on the oriented surface ∂B are given by

$$s_i(\mathbf{u}) = S_{ji}(\mathbf{u})n_j,\ m_i(\mathbf{u}) = M_{ji}(\mathbf{u})n_j, \tag{5}$$

where n_j are the components of the outward unit normal to ∂B.

We call a six–dimensional vector field $\mathbf{u}$ a quasi–static equilibrium motion field for B, if for each time $t \in [0, T)$, we have $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ and is continuous with respect to t on $[0, T)$ and

$$S_{ji,j}(\mathbf{u}) = 0,\ M_{ji,j}(\mathbf{u}) + \varepsilon_{ijk}S_{jk}(\mathbf{u}) = 0 \tag{6}$$

hold on B.

Saint–Venant's problem for B consists in the determination of a quasi–static equilibrium motion field, subjected to the requirements

$$s_i(\mathbf{u}) = 0,\ m_i(\mathbf{u}) = 0, \quad \text{on } \pi, \tag{7}$$

$$\begin{aligned} s_i(\mathbf{u}) &= s_i^{(1)}, \quad m_i(\mathbf{u}) = m_i^{(1)} \quad \text{on } D(0),\\ s_i(\mathbf{u}) &= s_i^{(2)}, \quad m_i(\mathbf{u}) = m_i^{(2)} \quad \text{on } D(L), \end{aligned} \tag{8}$$

for each time $t \in [0, T)$. Here $s_i^{(1)}, m_i^{(1)}$ and $s_i^{(2)}, m_i^{(2)}$ are functions preassigned on $D(0)$ and $D(L)$, respectively, for each time $t \in [0, T)$.

The conditions of equilibrium of the considered cylinder can be written in the form

$$\begin{gathered} \int_{D(0)} s_i^{(1)} da + \int_{D(L)} s_i^{(2)} da = 0,\\ \int_{D(0)} (\varepsilon_{ijk}x_j s_k^{(1)} + m_i^{(1)}) da + \int_{D(L)} (\varepsilon_{ijk}x_j s_k^{(1)} + m_i^{(2)}) da = 0. \end{gathered} \tag{9}$$

Under suitable smoothness hypotheses on π and on the given functions $s_i^{(\alpha)}$ and $m_i^{(\alpha)}$, a solution of Saint–Venant's problem exists being continuous with respect to time on $[0, T)$ (cf. FICHERA [4]).

If we introduce the relations (1) and (2) into (5) and (6), it results that $\mathbf{u}$ satisfies the boundary value problem

$$\begin{aligned} &\mathcal{S}_i(\mathbf{u}) \equiv [A_{jikl} \otimes (u_{l,k} + \varepsilon_{lkm}\varphi_m) + B_{jikl} \otimes \varphi_{l,k}]_{,j} = 0,\\ &\mathcal{S}_{i+3}(\mathbf{u}) \equiv [B_{klji} \otimes (u_{l,k} + \varepsilon_{lkm}\varphi_m) + C_{jikl} \otimes \varphi_{l,k}]_{,j} +\\ &\qquad + \varepsilon_{ijk}[A_{jklm} \otimes (u_{m,l} + \varepsilon_{mln}\varphi_n) + B_{jklm} \otimes \varphi_{m,l}] = 0 \text{ in } B, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\mathcal{B}_i(\mathbf{u}) = [A_{\alpha ikl} \otimes (u_{l,k} + \varepsilon_{lkm}\varphi_m) + B_{\alpha ikl} \otimes \varphi_{l,k}]n_\alpha = 0,\\ &\mathcal{B}_{i+3}(\mathbf{u}) = [B_{kl\alpha i} \otimes (u_{l,k} + \varepsilon_{lkm}\varphi_m) + C_{\alpha ikl} \otimes \varphi_{l,k}]n_\alpha = 0 \text{ on } \pi \end{aligned} \tag{11}$$

and

$$\begin{aligned} &S_{3i}(\mathbf{u}) = -s_i^{(1)}, \quad M_{3i}(\mathbf{u}) = -m_i^{(1)} \quad \text{on } D(0),\\ &S_{3i}(\mathbf{u}) = s_i^{(2)}, \quad M_{3i}(\mathbf{u}) = m_i^{(2)} \quad \text{on } D(L), \end{aligned} \tag{12}$$

where we have used the notation

$$(f \otimes g)(t) = f(0)g(t) + \int_0^t \dot{f}(t-z)g(z)dz. \tag{13}$$

We have introduced the operators $[\mathcal{S}_i, \mathcal{S}_{i+3}]$ and $[\mathcal{B}_i, \mathcal{B}_{i+3}]$ with domains $C([0,T]; C^2(B) \cap C^1(\overline{B}))$ and $C([0,T]; C^1(\pi))$ respectively.

Let us denote by

$$\Lambda = \{\mathbf{u} = (u_i, \varphi_i) | \mathcal{S}_i(\mathbf{u}) = 0, \mathcal{S}_{i+3}(\mathbf{u}) = 0 \text{ in } B, \mathcal{B}_i(\mathbf{u}) = 0, \mathcal{B}_{i+3}(\mathbf{u}) = 0 \text{ on } \pi\}.$$

3. The generalized plane strain state. Following [6] we define the state of generalized plane strain of the considered cylinder to be the state in which the functions v_i and ψ_i depend only on x_1 and x_2 and t

$$v_i = v_i(x_1, x_2; t),\ \psi_i = \psi_i(x_1, x_2; t),\ (x_1, x_2) \in D,\ t \in [0, T). \tag{14}$$

The above restriction implies that the components of the stress tensor and couple–stress tensor are functions of x_1 and x_2 and t, i.e. $T_{ij} = T_{ij}(x_1, x_2, t)$, $M_{ij} = M_{ij}(x_1, x_2, t)$. Moreover, we have

$$\begin{aligned} T_{ij}(\mathbf{v}) &= A_{ij\beta l} \otimes v_{l,\beta} + A_{ijkl} \otimes (\varepsilon_{lkm}\psi_m) + B_{ij\beta l} \otimes \psi_{l,\beta},\\ N_{ij}(\mathbf{v}) &= B_{\beta lij} \otimes v_{l,\beta} + B_{klij} \otimes (\varepsilon_{lkm}\psi_m) + C_{ij\beta l} \otimes \psi_{l,\beta}. \end{aligned} \tag{15}$$

A six dimensional vector field $\mathbf{v}$ is an admissible motion provided $\mathbf{v}$ is continuous with respect to time variable on $[0,T)$ and, moreover, for each $t \in [0,T)$,

i) $\mathbf{v}$ is independent of x_3

ii) $\mathbf{v} \in C^1(\overline{D}) \cap C^2(D)$.

The generalized plane strain problem for $D \cup \partial D$ consists in finding an admissible motion $\mathbf{v}$ which satisfies the equations of equilibrium

$$T_{\alpha i,\alpha}(\mathbf{v}) + f_i = 0,\ N_{\alpha i,\alpha}(\mathbf{v}) + \varepsilon_{ijk} T_{jk}(\mathbf{v}) + l_i = 0 \text{ in } D \tag{16}$$

and the boundary conditions

$$T_{\alpha i}(\mathbf{v}) n_\alpha = p_i,\ N_{\alpha i}(\mathbf{v}) n_\alpha = q_i \text{ on } \partial D \tag{17}$$

for each $t \in [0,T)$. In the above relations $f_i(x_1,x_2;t)$ and $l_i(x_1,x_2;t)$ denote the body loads and $p_i(x_1,x_2;t)$ and $q_i(x_1,x_2;t)$ denote the surface loads.

If we introduce the relations (15) into (16) and (17) we get the boundary value problem

$$\begin{aligned} \mathcal{T}_i(\mathbf{v}) &\equiv [A_{\alpha i\beta l} \otimes v_{l,\beta} + A_{\alpha ikl} \otimes (\varepsilon_{lkm}\psi_m) + B_{\alpha i\beta l} \otimes \psi_{l,\beta}]_{,\alpha} = -f_i, \\ \mathcal{T}_{i+3}(\mathbf{v}) &\equiv [B_{\beta l\alpha i} \otimes v_{l,\beta} + B_{kl\alpha i} \otimes (\varepsilon_{lkm}\psi_m) + C_{\alpha i\beta l} \otimes \psi_{l,\beta}]_{,\alpha} + \\ &+ \varepsilon_{ijk}[A_{jk\beta l} \otimes v_{l,\beta} + A_{jkml} \otimes (\varepsilon_{lmn}\psi_n) + B_{jk\beta l} \otimes \psi_{l,\beta}] = -l_i \text{ in } D, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\mathcal{P}_i(\mathbf{v}) \equiv [A_{\alpha i\beta l} \otimes v_{l,\beta} + A_{\alpha ikl} \otimes (\varepsilon_{lkm}\psi_m) + B_{\alpha i\beta l} \otimes \psi_{l,\beta}] n_\alpha = p_i, \\ &\mathcal{P}_{i+3}(\mathbf{v}) \equiv [B_{\beta l\alpha i} \otimes v_{l,\beta} + B_{kl\alpha i} \otimes (\varepsilon_{lkm}\psi_m) + C_{\alpha i\beta l} \otimes \psi_{l,\beta}] n_\alpha = q_i \text{ on } \partial D \end{aligned} \tag{19}$$

for each $t \in [0,T)$. Here the operators $[\mathcal{T}_i, \mathcal{T}_{i+3}]$ and $[\mathcal{P}_i, \mathcal{P}_{i+3}]$ have the domains $C([0,T); C^1(\overline{D}) \cap C^2(D))$ and $C([0,T); C^1(\partial D))$ respectively.

The necessary and sufficient conditions for the existence of the solution $\mathbf{v}$ for the boundary value problem associated to $D \cup \partial D$, are given by

$$\int_D f_i da + \int_{\partial D} p_i ds = 0, \tag{20}$$

$$\int_D (\varepsilon_{3\alpha\beta} x_\alpha f_\beta + l_3) da + \int_{\partial D} (\varepsilon_{3\alpha\beta} x_\alpha p_\beta + q_3) ds = 0. \tag{21}$$

Under suitable smoothness hypotheses on the given forces a solution of the generalized strain problem exists for each $t \in [0,T)$ (cf. FICHERA [4]).

4. Analysis of Saint–Venant's problem by plane cross–section solution. Following the method developed in [1] we study the possibility to reduce the system (10) and the lateral boundary conditions (11) to a generalized plane strain problem.

Thus, in what follows we consider the system (10) and the boundary conditions (11) on the cross section $D \cup \partial D$. Therefore, we consider the plane boundary value problem

$$\mathcal{S}_i(\mathbf{u}) = 0,\ \mathcal{S}_{i+3}(\mathbf{u}) = 0 \text{ in } D \tag{22}$$

and

$$\mathcal{B}_i(\mathbf{u}) = 0,\ \mathcal{B}_{i+3}(\mathbf{u}) = 0 \text{ on } \partial D \tag{23}$$

considering $x_3 \in (0, L)$ and $t \in [0, T)$ as parameters. In this connection we propose the following question: when does $\mathbf{u} \in \Lambda$ is solution for the boundary value problem associated with the cross–section $D \cup \partial D$ of the cylinder?

The answer to the above question will be given in the terms of the vector–valued linear functionals $\mathcal{R}$ and $\mathcal{M}$, whose components are defined by

$$\mathcal{R}_i(\mathbf{u}) = \int_D S_{3i}(\mathbf{u})da,\ \mathcal{M}_i(\mathbf{u}) = \int_D (\varepsilon_{ijk}x_j S_{3k}(\mathbf{u}) + M_{3i}(\mathbf{u}))da, \tag{24}$$

and which represent the resultant force and the resultant moment about 0 of the tractions acting on the cross–section D of the cylinder. We remark that

$$\mathcal{M}_\alpha(\mathbf{u}) = \int_D (\varepsilon_{3\alpha\beta}x_\beta S_{33}(\mathbf{u}) + M_{3\alpha}(\mathbf{u}))da - \varepsilon_{3\alpha\beta}x_3\mathcal{R}_\beta(\mathbf{u}), \tag{25}$$

$$\mathcal{M}_3(\mathbf{u}) = \int_D (\varepsilon_{3\alpha\beta}x_\alpha S_{3\beta}(\mathbf{u}) + M_{33}(\mathbf{u}))da. \tag{26}$$

In order to answer the above question, we write the plane boundary value problem (22) and (23) in the form

$$\mathcal{S}_i(\mathbf{u}) = \mathcal{T}_i(\mathbf{u}) + f_i = 0,\ \mathcal{S}_{i+3}(\mathbf{u}) = \mathcal{T}_{i+3}(\mathbf{u}) + l_i = 0 \text{ in } D, \tag{27}$$

$$\mathcal{B}_i(\mathbf{u}) = \mathcal{P}_i(\mathbf{u}) - p_i = 0,\ \mathcal{B}_{i+3}(\mathbf{u}) = \mathcal{P}_{i+3}(\mathbf{u}) - q_i = 0 \text{ on } \partial D, \tag{28}$$

where

$$\begin{aligned}
&f_i = (A_{\alpha i3l}\otimes u_{l,3} + B_{\alpha i3l}\otimes\varphi_{l,3})_{,\alpha} + S_{3i,3}(\mathbf{u}),\\
&l_i = (B_{3l\alpha i}\otimes u_{l,3} + C_{\alpha i3l}\otimes\varphi_{l,3})_{,\alpha} + \varepsilon_{ijk}(A_{jk3l}\otimes u_{l,3}+\\
&\qquad\qquad +B_{jk3l}\otimes\varphi_{l,3}) + M_{3i,3}(\mathbf{u}),\\
&p_i = -(A_{\alpha i3l}\otimes u_{l,3} + B_{\alpha i3l}\otimes\varphi_{l,3})n_\alpha,\\
&q_i = -(B_{3l\alpha i}\otimes u_{l,3} + C_{\alpha i3l}\otimes\varphi_{l,3})n_\alpha.
\end{aligned} \tag{29}$$

The necessary and sufficient conditions (20) and (21) imply

$$\int_D S_{3i,3}(\mathbf{u})da = 0,\ \int_D(\varepsilon_{3\alpha\beta}x_\alpha S_{3\beta,3}(\mathbf{u}) + M_{33,3}(\mathbf{u}))da = 0. \tag{30}$$

It is easy to observe that, under the hypothesis (4), relation (2) gives

$$S_{3i,3}(\mathbf{u}) = S_{3i}(\mathbf{u}_{,3}),\ M_{3i,3}(\mathbf{u}) = M_{3i}(\mathbf{u}_{,3}), \tag{31}$$

so that the relation (30) takes the form

$$\int_D S_{3i}(\mathbf{u}_{,3})da = 0,\ \int_D(\varepsilon_{3\alpha\beta}x_\alpha S_{3\beta}(\mathbf{u}_{,3}) + M_{33}(\mathbf{u}_{,3}))da = 0. \tag{32}$$

On the other hand, relation (30) proves, by means of relations (24) and (26), that

$$(\mathcal{R}_i(\mathbf{u}))_{,3} = 0,\ (\mathcal{M}_i(\mathbf{u}))_{,3} = 0. \tag{33}$$

We are thus led to the following result.

Proposition 1. *Let $u\in\Lambda$. If relation* (33) *holds true then u can be expressed in terms of a plane motion.*

Corollary 1. *Let $u\in\Lambda$ and relation* (33) *holds true. Then*

$$(\mathcal{M}_\alpha(u))_{,3} = 0. \tag{34}$$

Proof. In view of relations (6), (7) and (25), we get

$$\begin{aligned}
(\mathcal{M}_\alpha(u))_{,3} &= \int_D(\varepsilon_{3\alpha\beta}x_\beta S_{33,3}(u) + M_{3\alpha,3}(u))da - \varepsilon_{3\alpha\beta}\mathcal{R}_\beta(u) =\\
&= -\int_D[\varepsilon_{3\alpha\beta}x_\beta S_{\rho3,\rho}(u)+M_{\rho\alpha,\rho}(u)+\varepsilon_{\alpha3\beta}S_{3\beta}(u)+\varepsilon_{\alpha\beta3}S_{\beta3}(u)]da-\\
&\quad - \varepsilon_{3\alpha\beta}\mathcal{R}_\beta(u) = -\int_{\partial D}(\varepsilon_{3\alpha\beta}x_\beta s_3(u) + m_\alpha(u))ds-\\
&\quad - \varepsilon_{\alpha3\beta}\mathcal{R}_\beta(u) - \varepsilon_{3\alpha\beta}\mathcal{R}_\beta(u) = 0.
\end{aligned} \tag{35}$$

Remark 1. Relations (33) and (35) yield

$$\int_D (\varepsilon_{3\alpha\beta} x_\beta S_{33}(u_{,33}) + M_{3\alpha}(u_{,33}))da = 0. \tag{36}$$

Relations (32) and (36) allow us to point two classes of six–dimensional vector fields, who belong to Λ, that can be expressed in terms of a plane motion.

The class C_{I}. We denote by

$$C_{\mathrm{I}} = \{u^0 = (u_i^0, \varphi_i^0) \mid u_{i,3}^0 = \alpha_i(t) + \varepsilon_{ijk}\beta_j(t)x_k,\ \varphi_{i,3}^0 = \beta_i(t)\} \tag{37}$$

If $u^0 \in C_{\mathrm{I}}$ then $u_{,3}^0$ is a rigid motion. In view of relations (32), (33), (35) and (37), we deduce that for $u^0 \in C_{\mathrm{I}}$, we have

$$\mathcal{R}_\alpha(u^0) = 0,\ (\mathcal{R}_3(u^0))_{,3} = 0,\ (\mathcal{M}_i(u^0))_{,3} = 0. \tag{38}$$

On the other hand, if $u^0 \in C_{\mathrm{I}}$, then from (37) we deduce

$$\begin{aligned} u_\alpha^0 &= -\frac{1}{2}a_\alpha(t)x_3^2 - \varepsilon_{3\alpha\beta}a_4(t)x_\beta x_3 + w_\alpha(x_1, x_2; t), \\ u_3^0 &= (a_3(t) + a_\rho(t)x_\rho)x_3 + w_3(x_1, x_2; t), \\ \varphi_\alpha^0 &= \varepsilon_{3\alpha\beta}a_\beta(t)x_3 + \chi_\alpha(x_1, x_2; t),\ \varphi_3^0 = a_4(t)x_3 + \chi_3(x_1, x_2; t), \end{aligned} \tag{39}$$

except for an additive rigid motion. Here $w = (w_i, \chi_i)$ is an arbitrary vector field independent of x_3, and we have used the notations $a_\alpha(t) = \varepsilon_{3\rho\alpha}\beta_\rho(t)$, $a_3(t) = \alpha_3(t)$, $a_4(t) = \beta_3(t)$. The corresponding stress tensor and couple stress tensor are

$$\begin{aligned} S_{ij}(u^0) &= T_{ij}(w) + A_{ij33}\otimes(a_3 + a_\rho x_\rho) - A_{ij3\alpha}\otimes(\varepsilon_{3\alpha\beta}a_4 x_\beta) + \\ &\qquad + B_{ij33}\otimes a_4 + B_{ij3\alpha}\otimes(\varepsilon_{3\alpha\beta}a_\beta), \\ M_{ij}(u^0) &= N_{ij}(w) + B_{33ij}\otimes(a_3 + a_\rho x_\rho) - B_{3\alpha ij}\otimes(\varepsilon_{3\alpha\beta}a_4 x_\beta) + \\ &\qquad + C_{ij33}\otimes a_4 + C_{ij3\alpha}\otimes(\varepsilon_{3\alpha\beta}a_\beta). \end{aligned} \tag{40}$$

Clearly $S_{ij}(u^0)$ and $M_{ij}(u^0)$ are independent of the axial coordinate. Therefore, the boundary value problem defined by (22) and (23) become

$$
\begin{aligned}
\mathcal{S}_i(u^0) \quad &= \mathcal{T}_i(w) + [A_{\alpha i33}\otimes(a_3 + a_\rho x_\rho) - A_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_4x_\rho)+\\
&+ B_{\alpha i33}\otimes a_4 + B_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_\rho)]_{,\alpha} = 0,\\
\mathcal{S}_{i+3}(u^0) \quad &= \mathcal{T}_{i+3}(w) + [B_{33\alpha i}\otimes(a_3 + a_\rho x_\rho) - B_{3\beta\alpha i}\otimes(\varepsilon_{3\beta\rho}a_4x_\rho)+\\
&+ C_{\alpha i33}\otimes a_4 + C_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_\rho)]_{,\alpha}+\\
&+ \varepsilon_{ijk}[A_{jk33}\otimes(a_3 + a_\rho x_\rho) - A_{jk3\alpha}\otimes(\varepsilon_{3\alpha\beta}a_4x_\beta)+\\
&+ B_{jk33}\otimes a_4 + B_{jk3\alpha}\otimes(\varepsilon_{3\alpha\beta}a_\beta)] = 0 \text{ in } D,
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\mathcal{B}_i(u^0) \quad &= \mathcal{P}_i(w) + [A_{\alpha i33}\otimes(a_3 + a_\rho x_\rho) - A_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_4x_\rho)+\\
&+ B_{\alpha i33}\otimes a_4 + B_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_\rho)]n_\alpha = 0,\\
\mathcal{B}_{i+3}(u^0) \quad &= \mathcal{P}_{i+3}(w) + [B_{33\alpha i}\otimes(a_3 + a_\rho x_\rho) - B_{3\beta\alpha i}\otimes(\varepsilon_{3\beta\rho}a_4x_\rho)+\\
&+ C_{\alpha i33}\otimes a_4 + C_{\alpha i3\beta}\otimes(\varepsilon_{3\beta\rho}a_\rho)]n_\alpha = 0 \text{ on } \partial D.
\end{aligned}
\tag{42}
$$

It follows from Proposition 1 and relation (38) that the necessary and sufficient conditions to solve the above problem are satisfied for any functions $a_s(t)$, $s = 1, 2, 3, 4$. Thus, w represents the solution of the plane boundary value problem defined by (41) and (42).

We denote by $w^{(j)}$ a solution of the boundary value problem (41) and (42) when $a_i = \delta_{ij}$, $a_4 = 0$, and by $w^{(4)}$ a solution of the boundary value problem (41) and (42) when $a_i = 0$ and $a_4 = 1$. Therefore, $w^{(s)}$, $s = 1, 2, 3, 4$, are characterized by the equations

$$
\mathcal{T}_i(w^{(s)}) + f_i^{(s)} = 0,\ \mathcal{T}_{i+3}(w^{(s)}) + l_i^{(s)} = 0 \ \text{ in } D
\tag{43}
$$

$$
\mathcal{P}_i(w^{(s)}) = p_i^{(s)}, \quad \mathcal{P}_{i+3}(w^{(s)}) = q_i^{(s)} \qquad \text{on } D
\tag{44}
$$

where

$$
\begin{aligned}
f_i^{(\beta)} &= (A_{\alpha i33}x_\beta + B_{\alpha i3\rho}\varepsilon_{3\rho\beta})_{,\alpha},\ f_i^{(3)} = A_{\alpha i33,\alpha},\ f_i^{(4)} = \\
&= -(A_{\alpha i3\beta}\varepsilon_{3\beta\rho}x_\rho - B_{\alpha i33})_{,\alpha},\\
l_i^{(\beta)} &= (B_{33\alpha i}x_\beta + C_{\alpha i3\rho}\varepsilon_{3\rho\beta})_{,\alpha} + \varepsilon_{ijk}(A_{jk33}x_\beta + B_{jk3\alpha}\varepsilon_{3\alpha\beta}),\\
l_i^{(3)} &= B_{33\alpha i,\alpha} + \varepsilon_{ijk}A_{jk33},\\
l_i^{(4)} &= -(B_{3\beta\alpha i}\varepsilon_{3\beta\rho}x_\rho - C_{\alpha i33})_{,\alpha} - \varepsilon_{ijk}(A_{jk3\alpha}\varepsilon_{3\alpha\beta}x_\beta - B_{jk33}),\\
p_i^{(\beta)} &= -(A_{\alpha i33}x_\beta + B_{\alpha i3\rho}\varepsilon_{3\rho\beta})n_\alpha,\ p_i^{(3)} = \\
&= -A_{\alpha i33}n_\alpha,\ p_i^{(4)} = (A_{\alpha i3\beta}\varepsilon_{3\beta\rho}x_\rho - B_{\alpha i33})n_\alpha,\\
q_i^{(\beta)} &= -(B_{33\alpha i}x_\beta + C_{\alpha i3\rho}\varepsilon_{3\rho\beta})n_\alpha,\ q_i^{(3)} = \\
&= -B_{33\alpha i}n_\alpha,\ q_i^{(4)} = (B_{3\beta\alpha i}\varepsilon_{3\beta\rho}x_\rho - C_{\alpha i33})n_\alpha.
\end{aligned}
\tag{45}
$$

Clearly, we have

$$w = \sum_{s=1}^{4} w^{(s)} \otimes a_s. \tag{46}$$

In what follows we assume that $w^{(s)}$ $(s = 1, 2, 3, 4)$ are known. Then u^0 can be written in the form

$$u^0 = \sum_{s=1}^{4} u^{(s)} \otimes a_s, \tag{47}$$

where

$$
\begin{aligned}
&u_\alpha^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + w_\alpha^{(\beta)}, && u_\alpha^{(3)} = w_\alpha^{(3)}, && u_\alpha^{(4)} = -\varepsilon_{3\alpha\beta}x_\beta x_3 + w_\alpha^{(4)},\\
&u_3^{(\beta)} = x_3 x_\beta + w_3^{(\beta)}, && u_3^{(3)} = x_3 + w_3^{(3)}, && u_3^{(4)} = w_3^{(4)},\\
&\varphi_\alpha^{(\beta)} = \varepsilon_{3\alpha\beta}x_3 + \chi_\alpha^{(\beta)}, && \varphi_\alpha^{(3)} = \chi_\alpha^{(3)}, && \varphi_\alpha^{(4)} = \chi_\alpha^{(4)},\\
&\varphi_3^{(\beta)} = \chi_3^{(\beta)}, && \varphi_3^{(3)} = \chi_3^{(3)}, && \varphi_3^{(4)} = x_3 + \chi_3^{(4)}.
\end{aligned}
\tag{48}
$$

It follows from (40) and (47) that

$$
\begin{aligned}
S_{ij}(u^0) &= \sum_{s=1}^{4} S_{ij}(u^{(s)}) \otimes a_s, \\
M_{ij}(u^0) &= \sum_{s=1}^{4} M_{ij}(u^{(s)}) \otimes a_s,
\end{aligned} \tag{49}
$$

where

$$
\begin{aligned}
S_{ij}(u^{(\beta)}) &= T_{ij}(w^{(\beta)}) + A_{ij33}x_\beta + B_{ij3\alpha}\varepsilon_{3\alpha\beta} \\
S_{ij}(u^{(3)}) &= T_{ij}(w^{(3)}) + A_{ij33}, \\
S_{ij}(u^{(4)}) &= T_{ij}(w^{(4)}) - A_{ij3\alpha}\varepsilon_{3\alpha\beta}x_\beta + B_{ij33}, \\
M_{ij}(u^{(\beta)}) &= N_{ij}(w^{(\beta)}) + B_{33ij}x_\beta + C_{ij3\alpha}\varepsilon_{3\alpha\beta}, \\
M_{ij}(u^{(3)}) &= N_{ij}(w^{(3)}) + B_{33ij}, \\
M_{ij}(u^{(4)}) &= N_{ij}(w^{(4)}) - B_{3\alpha ij}\varepsilon_{3\alpha\beta}x_\beta + C_{ij33}.
\end{aligned} \tag{50}
$$

Obviously, relations (43), (44) and (50) give

$$
\mathcal{S}_i(u^{(s)}) = 0,\ \mathcal{S}_{i+3}(u^{(s)}) = 0 \quad \text{in } D \tag{51}
$$

$$
\mathcal{B}_i(u^{(s)}) = 0,\ \mathcal{B}_{i+3}(u^{(s)}) = 0 \text{ on } \partial D \tag{52}
$$

These relations imply

$$
\begin{aligned}
\mathcal{R}_\alpha(u^{(s)}) &= \int_D S_{3\alpha}(u^{(s)})da = \int_D (S_{3\alpha}(u^{(s)}) + x_\alpha S_{\rho 3,\rho}(u^{(s)}))da = \\
&= \int_D [S_{3\alpha}(u^{(s)}) - S_{\alpha 3}(u^{(s)}) + (x_\alpha S_{\rho 3}(u^{(s)}),_\rho]da = \\
&= \int_D [\varepsilon_{3\rho\alpha}M_{\beta\rho,\beta}(u^{(s)}) + (x_\alpha S_{\rho 3}(u^{(s)})),_\rho] = 0.
\end{aligned} \tag{53}
$$

Finally, we note that for $u^0 \in C_{\mathrm{I}}$ we have

$$
\begin{aligned}
\mathcal{R}_3(u^0) &= \sum_{s=1}^{4} D_{3s} \otimes a_s,\ \mathcal{M}_\alpha(u^0) = \\
&= \sum_{s=1}^{4} \varepsilon_{3\alpha\beta} D_{\beta s} \otimes a_s,\ \mathcal{M}_3(u^0) = \sum_{s=1}^{4} D_{4s} \otimes a_s,
\end{aligned} \tag{54}
$$

where

$$
(55)\qquad \begin{aligned}
D_{\alpha s} &= \int_D (x_\alpha S_{33}(u^{(s)}) + \varepsilon_{3\rho\alpha} M_{3\rho}(u^{(s)}))da,\\
D_{3s} &= \int_D S_{33}(u^{(s)})da,\ D_{4s} = \int_D (\varepsilon_{3\alpha\beta} x_\alpha S_{3\beta}(u^{(s)}) + M_{33}(u^{(s)}))da.
\end{aligned}
$$

Let $\hat{a}(t)$ be the four–dimensional vector $(a_1(t), a_2(t), a_3(t), a_4(t))$. We shall write $u^0\{\hat{a}\}$ for the vector u^0 defined by relation (47) indicating thus its dependence on the functions $a_1(t), a_2(t), a_3(t)$ and $a_4(t)$.

In view of relations (32) and (36) we are led to introduce the following class.

The class C_{II}. We denote by C_{II} the class of six–dimensional vectors $u^* = (u_i^*, \varphi_i^*) \in \Lambda$ for which the conditions (33) hold true, and, moreover

$$
(56)\qquad u^*_{,33} \text{ is a rigid motion.}
$$

For $u^* \in C_{\mathrm{II}}$ it follows that $u^*_{,3} \in C_{\mathrm{I}}$ and, by means of the above discussion, we have

$$
(57)\qquad u^*_{,3} = u^0\{\hat{\mathbf{b}}\},
$$

and hence

$$
(58)\qquad u^* = \int_0^{x_3} u^0\{\hat{\mathbf{b}}\}dx_3 + u^0\{\hat{\mathbf{c}}\} + w^*(x_1, x_2; t),
$$

where $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are arbitrary four–dimensional vectors, depending only on the time t on $[0, T)$, and $w^* = (w_i^*, \chi_i^*)$ is an arbitrary vector field independent of x_3.

The components of the stress tensor and couple stress tensor corresponding to u^* defined by (58) have the form

$$
(59)\qquad \begin{aligned}
S_{ij}(u^*) &= \sum_{s=1}^{4} (c_s + x_3 b_s) \otimes S_{ij}(u^{(s)}) + k_{ij} + T_{ij}(w^*),\\
M_{ij}(u^*) &= \sum_{s=1}^{4} (c_s + x_3 b_s) \otimes M_{ij}(u^{(s)}) + h_{ij} + N_{ij}(w^*),
\end{aligned}
$$

where

$$
\begin{aligned}
k_{ij} &= \sum_{s=1}^{4}(A_{ij3l}\otimes w_l^{(s)}\otimes b_s + B_{ij3l}\otimes \chi_l^{(s)}\otimes b_s),\\
h_{ij} &= \sum_{s=1}^{4}(B_{3lij}\otimes w_l^{(s)}\otimes b_s + C_{ij3l}\otimes \chi_l^{(s)}\otimes b_s).
\end{aligned}
\tag{60}
$$

In view of relations (53), (33), (55) and (59), we get

$$
\sum_{s=1}^{4} D_{3s}\otimes b_s = 0,\ \sum_{s=1}^{4} D_{4s}\otimes b_s = 0.
\tag{61}
$$

Because the conditions (33) are satisfied, on the basis of relations (51) and (52), the plane boundary value problem defined by (22) and (23) reduces to

$$
\begin{aligned}
&\mathcal{T}_i(w^*) + k_{\alpha i,\alpha} + \sum_{s=1}^{4} b_s\otimes S_{3i}(u^{(s)}) = 0,\\
&\mathcal{T}_{i+3}(w^*) + h_{\alpha i,\alpha} + \varepsilon_{ijk}k_{jk} + \sum_{s=1}^{4} b_s\otimes M_{3i}(u^{(s)}) = 0 \text{ in } D,
\end{aligned}
\tag{62}
$$

and

$$
\begin{aligned}
&\mathcal{P}_i(w^*) + k_{\alpha i}n_\alpha = 0,\\
&\mathcal{P}_{i+3}(w^*) + h_{\alpha i}n_\alpha = 0 \text{ on } \partial D.
\end{aligned}
\tag{63}
$$

The necessary and sufficient conditions for the existence of a solution of this problem are satisfied on the basis of relations (53) and (61).

Thus, we have

Proposition 2. *If $u^* \in C_{\mathrm{II}}$, it has the form* (58)*, where* $\hat{\mathbf{b}}$ *satisfies the conditions* (61)*. Moreover,* w^* *can be obtained from the generalized plane strain problem defined by relations* (62) *and* (63)*.*

Remark 2. Let $u^* \in C_{\mathrm{II}}$. Then, from relations (25), (53) and (59), we have

$$
\begin{aligned}
&\mathcal{R}_\alpha(u^*)=\sum_{s=1}^{4} D_{\alpha s}\otimes b_s,\ \mathcal{R}_3(u^*)=\sum_{s=1}^{4} D_{3s}\otimes c_s+\int_D (k_{33}+T_{33}(w^*))da\\
&\mathcal{M}_\alpha(u^*)=\sum_{s=1}^{4}\varepsilon_{3\alpha\beta}D_{\beta s}\otimes c_s+\int_D[\varepsilon_{3\alpha\beta}x_\beta(k_{33}+T_{33}(w^*))+h_{3\alpha}+N_{3\alpha}(w^*)]da,\\
&\mathcal{M}_3(u^*)=\sum_{s=1}^{4} D_{4s}\otimes c_s+\int_D[\varepsilon_{3\alpha\beta}x_\alpha(k_{3\beta}+T_{3\beta}(w^*))+h_{33}+N_{33}(w^*)]da.
\end{aligned}
\tag{64}
$$

5. The relaxed Saint–Venant's problem. The relaxed Saint–Venant's problem consists in the determination of a quasi–static equilibrium motion field u that satisfies the conditions

$$s_i(u) = 0,\ m_i(u) = 0 \text{ on } \pi, \tag{65}$$

and

$$\mathcal{R}_i(u) = -R_i(t),\ \mathcal{M}_i(u) = -M_i(t) \text{ on } x_3 = 0, \tag{66}$$

where R_i and M_i are continuous functions preassigned on $[0, T)$. Similar conditions are assumed on the end located at $x_3 = L$.

In what follows we proceed to determine a solution of the relaxed Saint–Venant's problem. In view of the discussion in the previous Section, we use the decomposition of the relaxed problem into problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ characterized by

$(\mathcal{P}_1)$ (extension–bending–torsion): $R_\alpha = 0$,
$(\mathcal{P}_2)$ (flexure): $R_3 = M_i = 0$.

The solution of the problem $(\mathcal{P}_1)$. In view of the results of the previous Section, a solution of the problem $(\mathcal{P}_1)$ has the form

$$u_{\mathrm{I}} = u^0 = \sum_{s=1}^{4} u^{(s)} \otimes a_s. \tag{67}$$

Relations (54) and (66) give for determination of the unknown functions $a_s(t)$, $s = 1, 2, 3, 4$, the following system

$$\sum_{s=1}^{4} D_{\alpha s} \otimes a_s = \varepsilon_{3\alpha\beta} M_\beta,\ \sum_{s=1}^{4} D_{3s} \otimes a_s = -R_3,\ \sum_{s=1}^{4} D_{4s} \otimes a_s = -M_3. \tag{68}$$

Let us denote by $\mathcal{D}(t)$ the 4×4 matrix whose components are $D_{rs}(t)$, $(r, s = 1, 2, 3, 4)$. Let $K_{\mathrm{I}} = (M_2, -M_1, -R_3, -M_3)^\top$ and $\hat{a} = \hat{a}^\top$. Then the above system can be written in the following matrix form:

$$\mathcal{D}(0)\hat{a}(t) + \int_0^t \dot{\mathcal{D}}(t - z)\hat{a}(z)dz = K_{\mathrm{I}}(t). \tag{69}$$

Now, we observe that $S_{ij}(u^{(s)})$ and $M_{ij}(u^{(s)})$ at $t = 0$ coincide with the components of the stress and couple stress in the auxiliary strain problems from micropolar elasticity corresponding to a Cosserat material with $A_{ijkl}(0)$, $B_{ijkl}(0)$, $C_{ijkl}(0)$ characteristic coefficients.

We assume that the strain energy density corresponding to u

$$\begin{aligned} W(u) &= \frac{1}{2}A_{ijkl}(0)e_{ij}(u)e_{kl}(u) + B_{ijkl}(0)e_{ij}(u)\kappa_{kl}(u)+ \\ &+ \frac{1}{2}C_{ijkl}(0)\kappa_{ij}(u)\kappa_{kl}(u), \end{aligned} \tag{70}$$

is a positive definite quadric form in the components of the strain measures $e_{ij}(u)$ and $\kappa_{ij}(u)$.

Let us prove that the system (69) can be solved for a_1, a_2, a_3 and a_4. We show that the matrix $\mathcal{D}(0)$ is invertible. For this, we use the method developed in [5]. The strain energy $U(u)$ on $\overline{B}$ is

$$U(u) = \int_B W(u)dv. \tag{71}$$

The functional $U(\cdot)$ generates the bilinear symmetric functional

$$\begin{gathered} <u, \mathbf{v}> = \int_B [A_{ijkl}(0)e_{ij}(u)e_{kl}(v)+ \\ + B_{ijkl}(0)(e_{ij}(u)\kappa_{kl}(v) + e_{ij}(v)\kappa_{kl}(u)) + C_{ijkl}(0)\kappa_{ij}(u)\kappa_{kl}(v)]dv. \end{gathered} \tag{72}$$

In view of (67), (70) and (71), we get

$$U(u^0(0)) = \frac{1}{2}\sum_{r,s=1}^{4} <u^{(r)}(0), u^{(s)}(0)> > a_r(0)a_s(0). \tag{73}$$

Since $W(u^0(0))$ is positive definite and $u^0(0)$ is not a rigid motion, it follows that

$$\det <u^{(r)}(0), u^{(s)}(0)> \neq 0. \tag{74}$$

By (48), (50), (53) and $u^{(s)} \in \Lambda$ $(s = 1, 2, 3, 4)$,

$$\begin{aligned} <u^{(\alpha)}(0), u^{(\beta)}(0)> &= \int_{\partial B}(u_i^{(\alpha)}(0)s_i(u^{(\beta)}(0)) + \varphi_i^{(\alpha)}(0)m_i(u_i^{(\beta)}(0)))da = \\ &= -\frac{1}{2}L^2\mathcal{R}_\alpha(u^{(\beta)}(0)) + LD_{\alpha\beta}(0) = LD_{\alpha\beta}(0), \end{aligned}$$

$$
\begin{aligned}
&< u^{(\alpha)}(0), u^{(3)}(0) > = LD_{\alpha 3}(0),\\
&< u^{(\alpha)}(0), u^{(4)}(0) > = LD_{\alpha 4}(0),\\
&< u^{(3)}(0), u^{(3)}(0) > = LD_{33}(0),\\
&< u^{(3)}(0), u^{(4)}(0) > = LD_{34}(0),\\
&< u^{(4)}(0), u^{(4)}(0) > = LD_{44}(0).
\end{aligned} \tag{75}
$$

It follows from (72) and (75) that $D_{rs}(0) = D_{sr}(0)$ and

$$\det \mathcal{D}(0) \neq 0. \tag{76}$$

Therefore we deduce from (69) that

$$\hat{a}(t) + \int_0^t [\mathcal{D}(0)]^{-1} \dot{\mathcal{D}}(t-z)\hat{a}(z)dz = [\mathcal{D}(0)]^{-1} K_{\mathrm{I}}(t). \tag{77}$$

Since $R_3(t)$ and $M_i(t)$ are continuous on $[0, T)$, the Volterra integral equation (77) has one and only one solution $\hat{a}(t)$ continuous on $[0, T)$, which can be obtained by the method of successive approximations [2].

Therefore, the solution of the problem $(\mathcal{P}_1)$ is given by relation (67) where $u^{(s)}$ are defined by the relation (48), and the unknown functions $a_s(t)$ are determined by means of the system (68).

The solution of the problem $(\mathcal{P}_2)$. On the basis of the results from the previous Section, we seek a solution of the problem $(\mathcal{P}_2)$ in the class C_{II}. Therefore, we seek a solution in the form (58), i.e.

$$u_{\mathrm{II}} = u^*, \tag{78}$$

where

$$
\begin{aligned}
u_\alpha^* &= -\frac{1}{6}b_\alpha(t)x_3^3 - \frac{1}{2}c_\alpha(t)x_3^2 - \frac{1}{2}b_4(t)\varepsilon_{3\alpha\beta}x_\beta x_3^2 - c_4(t)\varepsilon_{3\alpha\beta}x_\beta x_3 + \\
&\qquad + \sum_{s=1}^{4}(c_s + x_3 b_s)\otimes w_\alpha^{(s)} + w_\alpha^*, \\
u_3^* &= \frac{1}{2}(b_\rho(t)x_\rho + b_3(t))x_3^2 + (c_\rho(t)x_\rho + c_3(t))x_3 + \\
&\qquad + \sum_{s=1}^{4}(c_s + x_3 b_s)\otimes w_3^{(s)} + w_3^*, \\
\varphi_\alpha^* &= \frac{1}{2}\varepsilon_{3\alpha\beta}b_\beta(t)x_3^2 + \varepsilon_{3\alpha\beta}c_\beta(t)x_3 + \sum_{s=1}^{4}(c_s + x_3 b_s)\otimes\chi_\alpha^{(s)} + \chi_\alpha^*, \\
\varphi_3^* &= \frac{1}{2}b_4(t)x_3^2 + c_4(t)x_3 + \sum_{s=1}^{4}(c_s + x_3 b_s)\otimes\chi_3^{(s)} + \chi_3^*,
\end{aligned} \tag{79}
$$

and the unknown functions $b_s(t)$ satisfy the conditions

$$
\sum_{s=1}^{4} D_{3s}\otimes b_s = 0, \ \sum_{s=1}^{4} D_{4s}\otimes b_s = 0. \tag{80}
$$

From relations (64) and (66), we get

$$
\sum_{s=1}^{4} D_{\alpha s}\otimes b_s = -R_\alpha, \tag{81}
$$

so that from the integral system (80) and (81) we can determine uniquely the functions $b_s(t)$, $s = 1, 2, 3, 4$. In what follows, we assume that the functions $b_s(t)$ are known. Then the vector w^* can be determined from the generalized plane strain problem defined by relations (62) and (63). Further, from the relations (64) and (66), we get for the determination of the unknown functions $c_s(t)$, $s = 1, 2, 3, 4$, the following integral system:

$$
\begin{aligned}
\sum_{s=1}^{4} D_{\alpha s}\otimes c_s &= -\int_D [x_\alpha(k_{33} + T_{33}(w^*)) + \varepsilon_{3\rho\alpha}(h_{3\rho} + N_{3\rho}(w^*))]da, \\
\sum_{s=1}^{4} D_{3s}\otimes c_s &= -\int_D (k_{33} + T_{33}(w^*))da, \\
\sum_{s=1}^{4} D_{4s}\otimes c_s &= -\int_D [\varepsilon_{3\alpha\beta}x_\alpha(k_{3\beta} + T_{3\beta}(w^*)) + h_{33} + N_{33}(w^*)]da.
\end{aligned} \tag{82}
$$

Therefore, a solution of the problem $(\mathcal{P}_2)$ has the form (79) where the unknown functions $b_s(t)$ and $c_s(t)$ are determined by means of the Volterra integral equations systems defined by relations (80) and (81), and (82) respectively; the vector field w^* is determined as a solution of the generalized plane strain problem defined by relations (62) and (63).

Finally, we note that the relaxed Saint–Venant's problem has a solution of the form

$$u = u_{\mathrm{I}} + u_{\mathrm{II}}, \tag{83}$$

where u_{I} and u_{II} are defined by relations (67) and (78), respectively.

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Department of Mathematics
University of Iaşi
11, Copou Boulevard
6600 Iaşi
ROMANIA