ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVI, s.I a, Matematică, 2000, f.1.

A THEORY OF NONSIMPLE MICROSTRETCH FLUIDS

BY

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Abstract. The paper is concerned with a theory of microstretch fluids in which the second–order velocity gradient is added to the classic set of independent constitutive variables. First, the field equations and a uniqueness result are established. Then a linearized theory is derived. A representation of Galerkin type for the solutions to the field equations is given. The effect of concentrated body loads is also studied.

1. Introduction. The theory of microfluids was introduced by ERINGEN [1] in order to study fluids whose microelements can deform independently from their centroidal motions. The theory has been the subject of an enormous number of investigations. Various reviews of the subject have been presented by ERINGEN [2], [3], ERINGEN and KAFADAR [4], ARIMAN, SILVESTER and TURK [5] and BRULIN [6]. In [7], ERINGEN has introduced the theory of microstretch fluids where the micromotions consist of the intrinsic rotations and stretch. Recently, ERINGEN [8] has extended this theory to include heat conduction and dependence of constitutive equations on the microinertia tensor. The intended applications of the theory are to suspensions in viscous fluids, bubbly fluids, blood and liquid crystals.

In various papers [6], [9], [10], the authors have advocated the proposal that a consistent grade level for a micropolar theory, containing the velocity vector $\mathbf{v}$ and the angular velocity vector ψ as independent constitutive variables, should contain field derivatives of one order higher in v_k than in ψ_k. The arguments for this are drawn from general dimensional considerations as well as from the analysis of special micro-model cases.

In the first part of this paper we present a theory of microstretch fluids which includes the velocity gradients of second order as independent constitutive variables. The theory is a generalization of the theory presented by

ERINGEN in [8]. In the context of elastic continua, TOUPIN [11], MINDLIN [12] and GREEN and RIVLIN [13] have established a theory of nonsimple media which is characterized by the inclusion of higher gradients of displacement in the basic postulates. The theory of nonsimple fluids in which velocity gradients of the first and second order are present in the constitutive equations has been developed by BLEUSTEIN and GREEN [14].

In the second part of the paper we present a uniqueness theorem within the framework of the theory of isothermal incompressible fluids which occupy bounded domains. Then we derive a linear theory of non–heat–conducting compressible fluids appropriate to small departures from an equilibrium state. We establish a representation of Galerkin type for the solutions to the field equations. Finally, we use the Galerkin representation to study the problem of concentrated body loads in the case of steady vibrations.

2. Basic Equations. Throughout this paper we consider a continuum that in the present configuration at time t occupies a regular region $B(t)$ of Euclidean three–dimensional space. We denote by ∂B the boundary of B and designate by $\mathbf{n}$ the outward unit normal of ∂B. We assume that ∂B is a smooth surface. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{u}$ has the order p, we write $u_{ij\ldots s}$ (p subscripts) for the rectangular Cartesian components of $\mathbf{u}$. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. We shall employ the Euler representation and usual summation and differentiation conventions: Latin subscripts are confined to the range (1,2,3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding spatial coordinate. In all that follows, we use a superposed dot or d/dt to denote the material derivative. The place occupied by the material point $\mathbf{X}$ in the current configuration is $\mathbf{x}$.

We present a theory of nonsimple microstretch fluids. The local form of the law of conservation of mass is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0\,, \tag{2.1}$$

where ρ is the mass density at time t, and $\mathbf{v}$ is the velocity vector field. We restrict our attention to microisotropic fluids. In this case the law of conservation of microinertia is (cf. [8])

$$\frac{\partial j}{\partial t} + j_{,s} v_s - 2j\nu = 0\,, \tag{2.2}$$

where j is the microinertia and ν is the microstretch velocity.

We postulate an energy balance in the form

$$\begin{aligned}&\int_V \rho(\dot{\varepsilon} + \dot{v}_i v_i + \xi_i \psi_i + \zeta\nu + \Gamma_{ji} v_{i,j})\, dv = \\ &= \int_V \rho(F_i v_i + M_i \psi_i + L\nu + F_{ji} v_{i,j} + S)\, dv + \\ &+ \int_{\partial V} (t_i v_i + m_i \psi_i + h\nu + \mu_{ji} v_{i,j} + q)\, da\,,\end{aligned}$$

for every part V of B and every time. Here, ε is the internal energy density, ψ_i is the microrotation rate, F_i is the body force density, M_i is the body couple density, L is the generalized body load, F_{ij} is the dipolar body force, S is the heat source density, t_i is the stress vector, m_i is the couple stress vector, h is the microstress, μ_{ji} is the dipolar surface force and q is the heat flux. Moreover, ξ_i is the microrotation spin inertia per unit mass (see ERINGEN [8])

$$\xi_i = j(\dot{\psi}_i + 2\nu\psi_i)\,, \tag{2.4}$$

ζ is the microstretch spin inertia per unit mass ([8])

$$\zeta = \frac{3}{2} j(\dot{\nu} + \nu^2 - \frac{2}{3}\psi_i\psi_i)\,, \tag{2.5}$$

and Γ_{ij} is the dipolar spin inertia per unit mass (see BLEUSTEIN and GREEN [14])

$$\Gamma_{ij} = d^2\left[(\dot{v}_j)_{,i} - v_{j,k}v_{k,i}\right]\,, \tag{2.6}$$

where d is a given constant.

We consider a second motion which differs from the given motion of the continuum by a constant rigid body translational velocity. In this case v_i is replaced by $v_i + a_i$, where a_i is an arbitrary constant. Under this transformation the functions $\rho, \varepsilon,\ \psi_i, \nu, \xi_i,\ \zeta, \Gamma_{ij}, v_{i,j}, F_i, M_i,\ L, F_{ij}, S, t_i,\ m_i, h, \mu_{ji}$ and q are not affected. Subtracting the energy balance (2.3) from that for the new motion, we obtain

$$\left[\int_V \rho\dot{v}_i dv - \int_V \rho F_i dv - \int_{\partial V} t_i da\right] a_i = 0\,,$$

for all arbitrary constants a_i. The quantities in the square brackets are independent of a_i so that we get

$$\int_V \rho\dot{v}_i dv = \int_V \rho F_i dv + \int_{\partial V} t_i da\,. \tag{2.7}$$

Let $\mathbf{n}$ be the outward unit normal of ∂V. Using the well-known method, from (2.7) we obtain

$$t_i = t_{ji} n_j\,, \tag{2/8}$$

and

$$t_{ji,j} + \rho F_i = \rho\dot{v}_i\,, \tag{2.9}$$

where t_{ij} is the stress tensor. Taking into account (2.8) and (2.9), the equation (2.3) reduces to

$$\begin{aligned}
&\int_V \rho(\dot{\varepsilon} + \xi_i\psi_i + \zeta\nu + \Gamma_{ji}v_{i,j})\,dv = \\
&= \int_V [t_{ji}v_{i,j} + \rho(M_i\psi_i + L\nu + F_{ji}v_{i,j} + S)]\,dv + \\
&+ \int_{\partial V} (m_i\psi_i + h\nu + \mu_{ji}v_{i,j} + q)\,da\,.
\end{aligned} \tag{2.10}$$

We now assume that V is a tetrahedral element bounded by a plane with arbitrary unit normal n_i, and by planes through the point x, parallel to the coordinate planes. With an argument similar to that used in obtaining (2.3), from (2.10) we find that

$$(m_i - m_{ji}n_j)\psi_i + (h - h_j n_j)\nu + (\mu_{ji} - \mu_{rji}n_r)v_{i,j} + q - q_j n_j = 0\,, \tag{2.11}$$

where m_{ji} is the couple stress tensor, h_j is the microstress vector, μ_{rji} is the hyperstress tensor and q_i is the heat flux vector. With the help of (2.11), the equation (2.10) reduces to

$$\begin{aligned}
&\rho\,[\dot{\varepsilon} + (\xi_i - M_i)\psi_i + (\zeta - L)\nu + (\Gamma_{ji} - F_{ji})v_{i,j}] = \\
&= t_{ji}v_{i,j} + \rho S + m_{ji,j}\psi_i + h_{j,j}\nu + \mu_{rji,r}v_{i,j} + \\
&+ q_{j,j} + m_{ji}\psi_{i,j} + h_j\nu_{,j} + \mu_{rji}v_{i,jr}\,.
\end{aligned} \tag{2.12}$$

Let us now consider a motion of the continua which differs from the given motion only by a superposed uniform rigid body angular velocity, the body occupying the same position at time t. Velocity in the new motion is given by $\bar{v}_i = v_i + \varepsilon_{irj} b_r x_j$ where ε_{irs} is the alternating symbol and b_i are arbitrary constants. Moreover, ψ_i is replaced by $\bar{\psi}_i = \psi_i + b_i$, and $v_{i,j}$ by $\bar{v}_{i,j} = v_{i,j} + \varepsilon_{irj} b_r$. Under this transformation the functions ρ, ε, ν, t_{ij}, S, m_{ij}, h_j, μ_{rji} and q_i are not affected. But the body loads M_i, L and F_{ij} must be accommodated by corresponding changes in rotatory accelerations (cf. ERINGEN [15]) so that

$$\xi_i - M_i = \bar{\xi}_i - \bar{M}_i\,, \qquad \zeta - L = \bar{\zeta} - \bar{L}\,, \qquad \Gamma_{ji} - F_{ji} = \bar{\Gamma}_{ji} - \bar{F}_{ji}\,.$$

Subtracting the energy equation (2.12) from the corresponding energy equation for the new motion, we obtain

$$[\rho(\xi_i - M_i) + \rho\varepsilon_{irs}(\Gamma_{rs} - F_{rs}) - \varepsilon_{irs} t_{rs} - m_{ji,j} - \varepsilon_{imn}\mu_{rmn,r}]\, b_i = 0\,,$$

for all arbitrary constants b_i. We conclude that

$$m_{ji,j} + \varepsilon_{irs}\tau_{rs} + \rho M_i = \rho\xi_i\,, \tag{2.13}$$

where we have used the notation

$$\tau_{rs} = t_{rs} + \mu_{krs,k} - \rho(\Gamma_{rs} - F_{rs})\,. \tag{2.14}$$

With the help of (2.13) and (2.14), the equation (2.12) becomes

$$\rho\dot{\varepsilon} = \tau_{ij} d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i \nu_{,i} + g\nu + q_{i,i} + \rho S\,, \tag{2.15}$$

where

$$d_{ij} = v_{j,i} + \varepsilon_{jir}\psi_r\,, \qquad \gamma_{ij} = \psi_{j,i}\,, \qquad \kappa_{ijk} = v_{k,ij}\,, \tag{2.16}$$

and

$$g = h_{i,i} + \rho L - \rho\zeta\,. \tag{2.17}$$

The entropy production inequality is

$$\int_V \rho\dot{\eta} dv - \int_V \frac{1}{\theta}\rho S\, dv - \int_{\partial V} \frac{1}{\theta} q\, da \geq 0\,,$$

where η is the entropy density and θ is the absolute temperature which is assumed to be always positive. If we apply the entropy production inequality to an arbitrary tetrahedron bounded by coordinate planes at $\mathbf{x}$ and a plane with unit normal $\mathbf{n}$, then we obtain (cf. GREEN and STEEL [16])

$$q = q_i n_i \,. \tag{2.18}$$

The local form of the entropy production inequality is

$$\rho\theta\dot{\eta} - \rho S - q_{i,i} + \frac{1}{\theta} q_i \theta_{,i} \geq 0 \,. \tag{2.19}$$

By eliminating ρS between (2.15) and (2.19) we obtain

$$\tau_{ij} d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i \nu_{,i} + g\nu - \rho\dot{\psi} - \rho\dot{\theta}\eta + \frac{1}{\theta} q_i \theta_{,i} \geq 0 \,, \tag{2.20}$$

where the Helmholtz free energy ψ is defined by

$$\psi = \varepsilon - \theta\eta \,.$$

Let us introduce the notation

$$A = (\rho, j, d_{ij}, \gamma_{ij}, \kappa_{ijk}, \nu, \nu_{,i}, \theta, \theta_{,i}) \,.$$

A heat–conducting fluid is a medium having the constitutive equations

$$\begin{aligned} &\psi = \widehat{\psi}(A)\,, \quad \tau_{ij} = \widehat{\tau}_{ij}(A)\,, \quad m_{ij} = \widehat{m}_{ij}(A)\,, \quad \mu_{ijk} = \widehat{\mu}_{ijk}(A)\,, \\ &h_i = \widehat{h}_i(A)\,, \quad g = \widehat{g}(A)\,, \quad \eta = \widehat{\eta}(A)\,, \quad q_i = \widehat{q}_i(A)\,, \\ &m_i = \widehat{m}_i(m_{rs}, n_j)\,, \quad h = \widehat{h}(h_k, n_j)\,, \quad \mu_{ji} = \widehat{\mu}_{ji}(\mu_{pqr}, n_s)\,, \end{aligned} \tag{2.21}$$

subject to the axioms of admissibility and objectivity. The axiom of objectivity requires that the constitutive functionals must be hemitropic. In what follows we restrict our attention to isotropic fluids. It follows from (2.20) and (2.21) that

$$\psi = \widetilde{\psi}(\rho, j, \theta)\,, \quad \eta = -\frac{\partial\psi}{\partial\theta}\,, \tag{2.22}$$

and

$$\tau_{ij} d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i\nu_{,i} + g\nu - \rho\frac{\partial\psi}{\partial\rho}\frac{d\rho}{dt} - \rho\frac{\partial\psi}{\partial j}\frac{dj}{dt} + \frac{1}{\theta} q_i\theta_{,i} \geq 0 \,. \tag{2.23}$$

In view of (2.1),

$$\frac{d\rho}{dt} = -\rho v_{i,i}\,, \qquad \frac{dj}{dt} = 2j\nu\,,$$

so that (2.23) becomes

$$(\tau_{ij} + \pi_{ij})d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i\nu_{,i} + (g+p)\nu + \frac{1}{\theta}q_i\theta_{,i} \geq 0\,, \tag{2.24}$$

where

$$\pi_{ij} = \rho^2\frac{\partial\psi}{\partial\rho}\delta_{ij}\,, \qquad p = -2\rho j\frac{\partial\psi}{\partial j}\,. \tag{2.25}$$

We introduce the functions s_{ij} and G by

$$\tau_{ij} = -\pi_{ij} + s_{ij}\,, \qquad g = -p + G\,. \tag{2.26}$$

The inequality (2.24) reduces to

$$s_{ij}d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i\nu_{,i} + G\nu + \frac{1}{\theta}q_i\theta_{,i} \geq 0\,. \tag{2.27}$$

If we carry (2.25) and (2.26) into (2.15) then we obtain

$$\rho\theta\dot{\eta} = s_{ij}d_{ij} + m_{ij}\gamma_{ij} + \mu_{ijk}\kappa_{ijk} + h_i\nu_{,i} + G\nu + q_{i,i} + \rho S\,. \tag{2.28}$$

Following [8], [14] we assume that s_{ij}, m_{ij}, μ_{ijk}, h_i, G and q_i are nonlinear functions of ρ, j and θ, and linear functions in the variables d_{ij}, γ_{ij}, κ_{ijk}, $\nu_{,i}$, ν and $\theta_{,i}$. In view of (2.28) we obtain

$$\begin{aligned}
s_{ij} &= (\lambda d_{rr} + \lambda_0\nu)\delta_{ij} + (\mu+\kappa)d_{ij} + \mu d_{ji}\,,\\
m_{ij} &= \varepsilon_{ijk}(\beta_0\nu_{,k} + \alpha_0\theta_{,k}) + \alpha\gamma_{rr}\delta_{ij} + \beta\gamma_{ji} + \gamma\gamma_{ij}\,,\\
\mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) +\\
&\quad + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} +\\
&\quad + \alpha_5(\kappa_{kji} + \kappa_{kij}) + \gamma_1\delta_{ij}\theta_{,k} + \gamma_2(\delta_{ik}\theta_{,j} + \delta_{jk}\theta_{,i}) +\\
&\quad + \beta_1\delta_{ij}\nu_{,k} + \beta_2(\delta_{ik}\nu_{,j} + \delta_{jk}\nu_{,i})\,,\\
h_i &= a_0\nu_{,i} + a_1\varepsilon_{irs}\gamma_{rs} + a_2\theta_{,i} + a_3\kappa_{rri} + a_4\kappa_{irr}\,,\\
G &= b_0\nu + b_1 d_{rr}\,,\\
q_i &= k\theta_{,i} + k_1\nu_{,i} + k_2\varepsilon_{irs}\gamma_{rs} + k_3\kappa_{rri} + k_4\kappa_{irr}\,,
\end{aligned} \tag{2.29}$$

where the constitutive coefficients depend on ρ, j and θ.

The inequality (2.27) is a restriction on the constitutive coefficients. Using standard techniques of the theory of quadratic forms we can derive the necessary and sufficient conditions for (2.29) to be satisfied for all independent constitutive variables (see [8], [14]).

In view of (2.18), the equation (2.11) becomes

$$(m_i - m_{ji}n_j)\psi_i + (h - h_j n_j)\nu + (\mu_{ji} - \mu_{rji}n_r)v_{i,j} = 0\,. \tag{2.30}$$

For a given motion, ψ, ν and $v_{i,j}$ in (2.30) may be chosen arbitrarily so that, on the basis of the constitutive equations (2.21), (2.24), (2.25), (2.26) and (2.29) we find that

$$m_i = m_{ji}n_j\,, \qquad h = h_j n_j\,, \qquad \mu_{ji} = \mu_{rji}n_r\,. \tag{2.29}$$

The total rate of the work over the surface ∂B is given by

$$\begin{aligned} W &= \int_{\partial B} (t_i v_i + m_i \psi_i + \mu_{ji} v_{i,j} + h\nu) da = \\ &= \int_{\partial B} (t_{ji} v_i + m_{ji}\psi_i + \mu_{jrs} v_{s,r} + h_j \nu) n_j\, da\,. \end{aligned} \tag{2.30}$$

Following TOUPIN [11] and MINDLIN [12], we can write

$$W = \int_{\partial B} (P_i v_i + R_i D v_i + m_i \psi_i + h\nu) da\,, \tag{2.33}$$

where

$$\begin{aligned} P_i = \;& [\tau_{ji} - \mu_{kji,k} + \rho(\Gamma_{ji} - F_{ji})]n_j - 2n_j D_r \mu_{jri} - \\ & - n_r n_s D\mu_{rsi} + (b_{rs} - b_{mm} n_r n_s)\mu_{rsi}\,, \\ R_i = \;& \mu_{rsi} n_r n_s\,. \end{aligned} \tag{2.34}$$

Here D_i is the surface gradient, b_{ij} is the second fundamental form of the surface ∂B and $Df = f_{,i}n_i$.

3. Field Equations. In view of (2.14) the equations of motion (2.9) become

$$\tau_{ji,j} - \mu_{kji,kj} + \rho F_i - (\rho F_{ji})_{,j} = \rho \dot{v}_i - (\rho \Gamma_{ji})_{,j}\,. \tag{3.1}$$

By using (2.4) the equations (2.13) can be written in the form

$$m_{ji,j} + \varepsilon_{irs}\tau_{rs} + \rho M_i = \rho j(\dot{\psi}_i + 2\nu\psi_i)\,. \tag{3.2}$$

From (2.5) and (2.17) we obtain the equation

$$h_{i,i} - g + \rho L = \frac{3}{2}\rho j(\dot{\nu} + \nu^2 - \frac{2}{3}\psi_i\psi_i)\,. \tag{3.3}$$

The basic equations of the theory consist of the equations (2.1), (2.2), (3.1)–(3.3), (2.28), (2.22), (2.25), (2.26), (2.29) and (2.16) for the determination of the functions ρ, v_i, j, ψ_i, ν and θ.

In what follows we assume that the constitutive coefficients from (2.29) are constants. By substituting (2.16), (2.29), (2.26) into (3.1)–(3.3) we obtain the field equations in the form

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0\,,\\
&\frac{\partial j}{\partial t} + j_{,i}v_i - 2\nu j = 0\,,\\
&\kappa_2(1 - l_2\Delta)\Delta v_k + [\kappa_1 - \kappa_2 - (\kappa_1 l_1 - \kappa_2 l_2)\Delta]v_{j,jk} +\\
&\quad + \kappa\varepsilon_{kjs}\psi_{s,j} - (\gamma_1 + 2\gamma_2)\Delta\theta_{,k} + [\lambda_0 - (\beta_1 + 2\beta_2)\Delta]\nu_{,k} + p_k +\\
&\quad + \rho F_k - (\rho F_{jk})_{,j} = \rho\dot{v}_k - (\rho\Gamma_{jk})_{,j}\,,\\
&\gamma\Delta\psi_i + (\alpha + \beta)\psi_{s,si} + \kappa\varepsilon_{irs}v_{s,r} - 2\kappa\psi_i + \rho M_i = \rho j(\dot{\psi}_i + 2\nu\psi_i)\,,\\
&a_0\Delta\nu + a_2\Delta\theta - [b_1 - (a_3 + a_4)\Delta]v_{s,s}\\
&\quad - b_0\nu + p + \rho L = \frac{3}{2}\rho j(\dot{\nu} + \nu^2 - \frac{2}{3}\psi_i\psi_i)\,,\\
&k\Delta\theta + k_1\Delta\nu + (k_3 + k_4)\Delta v_{j,j} + s_{ij}(v_{j,i} + \varepsilon_{jik}\psi_k) +\\
&\qquad + m_{ij}\psi_{j,i} + \mu_{ijk}v_{k,ij} + h_i\nu_{,i} + G\nu - \rho\theta\dot{\eta} = -\rho S\,,
\end{aligned} \tag{3.4}$$

where Δ is the Laplacian and

$$\begin{aligned}
&\kappa_1 = \lambda + 2\mu + \kappa\,, \quad \kappa_2 = \mu + \kappa\,, \quad l_2 = 2(\alpha_3 + \alpha_4)\kappa_2^{-1}\,,\\
&l_1 = 2\kappa_1^{-1}\sum_{i=1}^{5}\alpha_i\,, \quad p_k = -\pi_{jk,j}\,.
\end{aligned} \tag{3.5}$$

For an incompressible fluid we have

$$v_{i,i} = 0\,, \tag{3.6}$$

and the constitutive equations for non-heat-conducting fluids become

$$\begin{aligned}
\tau_{ij} &= -\Pi\delta_{ij} + s_{ij}\,,\\
s_{ij} &= \lambda_0\nu\delta_{ij} + (\mu + \kappa)d_{ij} + \mu d_{ji}\,,\\
m_{ij} &= \varepsilon_{ijk}\beta_0\nu_{,k} + \alpha\gamma_{rr}\delta_{ij} + \beta\gamma_{ji} + \gamma\gamma_{ij}\,,\\
\mu_{ijk} &= -\Pi_i\delta_{jk} - \Pi_j\delta_{ik} + p_{ijk}\,,\\
p_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + \kappa_{rrj}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} +\\
&\quad + \alpha_5(\kappa_{kji} + \kappa_{kij}) + \beta_1\delta_{ij}\nu_{,k} + \beta_2(\delta_{ik}\nu_{,j} + \delta_{jk}\nu_{,i})\,,\\
h_i &= a_0\nu_{,i} + a_1\varepsilon_{irs}\gamma_{rs} + a_3\kappa_{rri}\,,\\
g &= -\pi^* + G\,,\\
G &= b_0\nu\,, \qquad q_i = 0\,,
\end{aligned} \tag{3.7}$$

where

$$\psi = \psi^*(j, \theta)\,, \qquad \pi^* = -2\rho_0 j \frac{\partial \psi}{\partial j}\,, \tag{3.9}$$

ρ_0 is the constant density of incompressible fluid, and Π and Π_i are arbitrary functions to be determined in the course of solution of each particular problem.

4. A Uniqueness Result. In what follows we assume that the motion takes place under isothermal conditions. We consider incompressible fluids with the constant density ρ_0 on $B\times(0, t_1)$ and assume that $\psi = cj/2\rho_0$, where c is a constant. The constitutive equations (3.7) lead to

$$\begin{aligned} \tau_{ij} &= -\Pi\delta_{ij} + s_{ij}\,, \\ \mu_{ijk} &= -\Pi_i\delta_{jk} - \Pi_j\delta_{ik} + p_{ijk}\,, \\ g &= -cj + G\,, \end{aligned} \tag{4.1}$$

where s_{ij}, p_{ijk} and G are given by (3.7). The Clausius–Duhem inequality reduces to

$$s_{ij}d_{ij} + m_{ij}\gamma_{ij} + p_{ijk}\kappa_{ijk} + h_i\nu_{,i} + G\nu \geq 0\,. \tag{4.2}$$

In what follows we use the approximation (cf. ERINGEN [7])

$$\xi_i = j\dot{\psi}_i\,, \qquad \zeta = \frac{3}{2}j\dot{\nu}\,. \tag{4.3}$$

The basic equations are given by

$$\begin{aligned} & v_{i,i} = 0\,, \\ & \frac{\partial j}{\partial t} + j_{,i}v_i - 2\nu j = 0\,, \\ & \kappa_2(1 - l_2\Delta)\Delta v_i + \kappa\varepsilon_{ijs}\psi_{s,j} + [\lambda_0 - (\beta_1 + 2\beta_2)\Delta]\nu_{,i} - P_{,i} + \\ & \qquad + f_i = \rho_0(1 - d^2\Delta)\left(\frac{\partial v_i}{\partial t} + v_{i,s}v_s\right) \\ & \qquad + \rho_0 d^2(v_{i,s}\Delta v_s + v_{i,sm}v_{s,m})\,, \\ & \gamma\Delta\psi_i + (\alpha + \beta)\psi_{s,si} + \kappa\varepsilon_{irs}v_{s,r} - 2\kappa\psi_i + g_i \\ & \qquad = \rho_0 j\left(\frac{\partial \psi_i}{\partial t} + \psi_{i,s}v_s\right)\,, \\ & a_0\Delta\nu - b_0\nu - cj + l = \frac{3}{2}\rho_0 j\left(\frac{\partial \nu}{\partial t} + \nu_{,s}v_s\right)\,, \end{aligned} \tag{4.4}$$

where

$$P = \Pi - 2\Pi_{i,i}\,, \quad f_i = \rho_0 F_i - \rho_0 F_{ji,j}\,,$$
$$g_i = \rho_0 M_i\,, \quad l = \rho_0 L\,.$$

We suppose that f_i, g_i and l are prescribed continuous functions. We assume that the fluid occupies the bounded domain $B(t)$, $t \in I$.

To the field equations we add the initial conditions

$$\begin{aligned} &v_i(\mathbf{x},0) = v_i^0(\mathbf{x})\,, \quad j(\mathbf{x},0) = j^0(\mathbf{x})\,, \\ &\psi_i(\mathbf{x},0) = \psi_i^0(\mathbf{x})\,, \quad \nu(\mathbf{x},0) = \nu^0(\mathbf{x})\,, \quad \mathbf{x} \in \overline{B}(0)\,, \end{aligned} \tag{4.5}$$

and the boundary conditions (cf. [8], [14])

$$v_i = 0\,, \quad Dv_i = 0\,, \quad \psi_i = 0\,, \quad \nu = 0 \quad \text{on} \quad \partial B \times I\,, \tag{4.6}$$

where v_i^0, ψ_i^0, j^0 and ν^0 are prescribed continuous functions. The conditions on the boundary are related with the strict adherence (see [4], p. 62). The boundary conditions for nonsimple continuous media have been discussed by RAJAGOPAL, MASSOUDI and EKMANN [17] and RAJAGOPAL and TAO [18].

By an admissible process on $B \times I$ we mean an ordered array of functions (v_i, ψ_i, ν, j, P) with the properties: (i) v_i are of class $C^{4,1}$ on $B \times I$; (ii) v_i are of class $C^{1,0}$ on $\partial B \times I$; (iii) ψ_i and ν are of class $C^{2,1}$ on $B \times I$; (iv) ψ_i and ν are of class C^0 on $\partial B \times I$; (v) j is of class C^1 on $B \times I$; (vi) j is of class C^0 on $\partial B \times I$; (vii) P is of class $C^{1,0}$ on $B \times I$. By a solution of the boundary-initial-value problem we mean an admissible process that satisfies the equations (4.4), the initial conditions (4.5) and the boundary conditions (4.6).

We note that by (4.1), $(4.4)_1$, (2.32) and (2.33) we obtain

$$\begin{aligned} &\int_B [v_i(\tau_{ji,j} - \mu_{kji,kj}) + \psi_i(m_{ji,j} + \varepsilon_{irs}\tau_{rs}) + \nu(h_{i,i} - g)]\, dv = \\ &= \int_{\partial B} [v_i(\tau_{ji} - \mu_{kji,k}) + \psi_i m_{ji} + \nu h_j]\, n_j da - \int_B [(\tau_{ji} - \mu_{kji,k})v_{i,j} + \\ &\quad + \varepsilon_{irs}\psi_i\tau_{rs} + m_{ji}\psi_{i,j} + h_i\nu_{,i} + g\nu]\, dv = \end{aligned} \tag{4.6}$$

$$\begin{aligned} &= \int_{\partial B} \{v_i\,[(\tau_{ji} - \mu_{kji,k})n_j - n_j D_r\mu_{jri} + \\ &+ (b_{rs} - b_{mm}n_r n_s)\mu_{rsi}] + \mu_{rsi}n_r n_s Dv_i + m_{ji}n_j\psi_i + h_j n_j\nu\}\, da - \\ &- \int_B (s_{ij}d_{ij} + m_{ij}\gamma_{ij} + p_{ijk}\kappa_{ijk} + h_i\nu_{,i} + G\nu)\, dv + \int_B cj\nu dv\,. \end{aligned} \tag{4.7}$$

Theorem 4.1. *Assume that*

(i) ρ_0 *is strictly positive;*
(ii) *the entropy inequality* (4.2) *holds.*

Then for any two solutions

$$(v_i^{(1)}, \psi_i^{(1)}, \nu^{(1)}, j^{(1)}, P^{(1)}) \text{ and } (v_i^{(2)}, \psi_i^{(2)}, \nu^{(2)}, j^{(2)}, P^{(2)})$$

of the boundary–initial–value problem (4.4)–(4.6) *we have*

$$\begin{aligned} &v_i^{(1)} = v_i^{(2)}, \quad \psi_i^{(1)} = \psi_i^{(2)}, \quad \nu^{(1)} = \nu^{(2)}, \\ &j^{(1)} = j^{(2)}, \quad P^{(1)} = P^{(2)} + F, \end{aligned}$$

where F is an arbitrary function of time.

Proof. Suppose that the functions $v_i^{(\alpha)}$, $\psi_i^{(\alpha)}$, $\nu^{(\alpha)}$, $j^{(\alpha)}$, $P^{(\alpha)}$, $d_{ij}^{(\alpha)}$, $\gamma_{ij}^{(\alpha)}$, $\kappa_{ijk}^{(\alpha)}$, $\tau_{ij}^{(\alpha)}$, $m_{ij}^{(\alpha)}$, $\mu_{ijk}^{(\alpha)}$, $h_i^{(\alpha)}$, $G^{(\alpha)}$ satisfy the equations

$$v_{i,i}^{(\alpha)} = 0, \qquad \frac{\partial j^{(\alpha)}}{\partial t} + j_{,i}^{(\alpha)} v_i^{(\alpha)} = 2\nu^{(\alpha)} j^{(\alpha)}, \tag{4.8}$$

$$\begin{aligned} &\tau_{ji,j}^{(\alpha)} - \mu_{kji,kj}^{(\alpha)} + f_i = \rho_0(\dot{v}_i^{(\alpha)} - \Gamma_{ji,j}^{(\alpha)}), \\ &m_{ji,j}^{(\alpha)} + \varepsilon_{irs}\tau_{rs}^{(\alpha)} + g_i = \rho_0 j^{(\alpha)} \dot{\psi}_i^{(\alpha)}, \\ &h_{i,i}^{(\alpha)} - g^{(\alpha)} + l = \frac{3}{2}\rho_0 j^{(\alpha)} \dot{\nu}^{(\alpha)}, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \tau_{ij}^{(\alpha)} &= -\Pi^{(\alpha)}\delta_{ij} + s_{ij}^{(\alpha)}, \quad \mu_{ijk}^{(\alpha)} = -\Pi_i^{(\alpha)}\delta_{jk} - \Pi_j \delta_{ik}^{(\alpha)} + p_{ijk}^{(\alpha)}, \\ g^{(\alpha)} &= -cj^{(\alpha)} + G^{(\alpha)}, \\ s_{ij}^{(\alpha)} &= \lambda_0 \nu^{(\alpha)}\delta_{ij} + (\mu+\kappa) d_{ij}^{(\alpha)} + \mu d_{ji}^{(\alpha)}, \\ m_{ij}^{(\alpha)} &= \varepsilon_{ijk}\beta_0 \nu_{,k}^{(\alpha)} + \alpha\gamma_{rr}^{(\alpha)}\delta_{ij} + \beta\gamma_{ji}^{(\alpha)} + \gamma\gamma_{ij}^{(\alpha)}, \\ p_{ijk}^{(\alpha)} &= \frac{1}{2}\alpha_1(\kappa_{rri}^{(\alpha)}\delta_{jk} + \kappa_{rrj}^{(\alpha)}\delta_{ik}) + 2\alpha_3\kappa_{rrk}^{(\alpha)}\delta_{ij} + 2\alpha_4\kappa_{ijk}^{(\alpha)} + \\ &\quad + \alpha_5(\kappa_{kji}^{(\alpha)} + \kappa_{kij}^{(\alpha)}) + \beta_1\delta_{ij}\nu_{,k}^{(\alpha)} + \beta_2(\delta_{ik}\nu_{,j}^{(\alpha)} + \delta_{jk}\nu_{,i}^{(\alpha)}), \\ h_i^{(\alpha)} &= a_0\nu_{,i}^{(\alpha)} + a_1\varepsilon_{irs}\gamma_{rs}^{(\alpha)} + a_3\kappa_{rri}^{(\alpha)}, \qquad G^{(\alpha)} = b_0\nu^{(\alpha)}, \\ d_{ij}^{(\alpha)} &= v_{j,i}^{(\alpha)} + \varepsilon_{jir}\psi_r^{(\alpha)}, \qquad \gamma_{ij}^{(\alpha)} = \psi_{j,i}^{(\alpha)}, \qquad \kappa_{ijk}^{(\alpha)} = v_{k,ij}^{(\alpha)}, \end{aligned} \tag{4.10}$$

where

$$\Gamma_{ij}^{(\alpha)} = d^2\left[(\dot{v}_j^{(\alpha)})_{,i} - v_{j,k}^{(\alpha)} v_{k,i}^{(\alpha)}\right].$$

We introduce the notations

$$
\begin{array}{lll}
u_i = v_i^{(2)} - v_i^{(1)}\,, & w_i = \psi_i^{(2)} - \psi_i^{(1)}\,, & \varphi = \nu^{(2)} - \nu^{(1)}\,,\\
s = j^{(2)} - j^{(1)}\,, & T_{ij} = \tau_{ij}^{(2)} - \tau_{ij}^{(1)}\,, & \\
S_{ij} = s_{ij}^{(2)} - s_{ij}^{(1)}\,, & M_{ij} = m_{ij}^{(2)} - m_{ij}^{(1)}\,, & \\
N_{ijk} = \mu_{ijk}^{(2)} - \mu_{ijk}^{(1)}\,, & P_{ijk} = p_{ijk}^{(2)} - p_{ijk}^{(1)}\,, & \\
Q_i = h_i^{(2)} - h_i^{(1)}\,, & \Gamma = g^{(2)} - g^{(1)}\,, & S = G^{(2)} - G^{(1)}\,,\\
\Lambda = \Pi^{(2)} - \Pi^{(1)}\,, & \Lambda_i = \Pi_i^{(2)} - \Pi_i^{(1)}\,. &
\end{array}
\tag{4.11}
$$

The functions u_i, w_i, φ and s satisfy the initial conditions

$$
u_i(\mathbf{x},0) = 0,\ w_i(\mathbf{x},0) = 0,\ \varphi(\mathbf{x},0) = 0,\ s(\mathbf{x},0) = 0,\ \mathbf{x}\in\overline{B}(0),
\tag{4.12}
$$

and the boundary conditions

$$
u_i = 0\,,\quad Du_i = 0\,,\quad w_i = 0\,,\quad \varphi = 0 \qquad \text{on} \quad \partial B \times I\,.
\tag{4.13}
$$

In view of $(4.8)_1$ we have

$$
u_{i,i} = 0 \quad \text{on} \quad B \times I\,.
\tag{4.14}
$$

By $(4.8)_2$ and (4.11) we obtain

$$
\frac{\partial s}{\partial t} + s_{,i}u_i + s_{,i}v_i^{(1)} + j_{,i}^{(1)}u_i = 2s\nu^{(2)} + 2j^{(1)}\varphi\,,
\tag{4.15}
$$

on $B \times I$. It follows from (4.10) and (4.11) that

$$
\begin{array}{ll}
T_{ij} & = -\Lambda\delta_{ij} + S_{ij}\,, \qquad N_{ijk} = -\Lambda_i\delta_{jk} - \Lambda_j\delta_{ik} + P_{ijk}\,,\\
\Gamma & = -cs + S\,,\\
S_{ij} & = \lambda_0\varphi\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji}\,,\\
M_{ij} & = \beta_0\varepsilon_{ijk}\varphi_{,k} + \alpha\eta_{rr}\delta_{ij} + \beta\eta_{ji} + \gamma\eta_{ij}\,,\\
P_{ijk} & = \dfrac{1}{2}\alpha_1(\nu_{rri}\delta_{jk} + \nu_{rrj}\delta_{ik}) + 2\alpha_3\nu_{rrk}\delta_{ij} + 2\alpha_4\nu_{ijk} +\\
& \qquad + \alpha_5(\nu_{kji} + \nu_{kij}) + \beta_1\delta_{ij}\varphi_{,k} + \beta_2(\delta_{ik}\varphi_{,j} + \delta_{jk}\varphi_{,i})\,,\\
Q_i & = a_0\varphi_{,i} + a_1\varepsilon_{irs}\eta_{rs} + a_3\nu_{rri}\,, \qquad S = b_0\varphi\,,
\end{array}
\tag{4.16}
$$

where

$$
e_{ij} = u_{j,i} + \varepsilon_{jir}w_r\,, \qquad \eta_{ij} = w_{j,i}\,, \qquad \nu_{ijk} = u_{k,ij}\,.
\tag{4.17}
$$

The equations (4.9) imply that

$$
\begin{aligned}
T_{ji,j} - N_{kji,kj} &= \rho_0 \left[\frac{\partial u_i}{\partial t} + (v_i^{(1)} + u_i)_{,s} u_s + u_{i,k} v_k^{(1)} \right] - \rho_0 d^2 Q_{ji,j} \,, \\
M_{ji,j} + \varepsilon_{irs} T_{rs} &= \rho_0 j^{(2)} \left(\frac{\partial w_i}{\partial t} + w_{i,s} v_s^{(2)} + \psi_{i,s}^{(1)} u_s \right) + \rho_0 s V_i \,, \\
Q_{i,i} - \Gamma &= \frac{3}{2} \rho_0 j^{(2)} \left(\frac{\partial \varphi}{\partial t} + \varphi_{,s} v_s^{(2)} + \nu_{,s}^{(1)} u_s \right) + \rho_0 s W \,,
\end{aligned}
\tag{4.18}
$$

where

$$
\begin{aligned}
Q_{ji} &= \frac{\partial^2 u_i}{\partial t \partial x_j} + u_{i,sj}(u_s + v_s^{(1)}) + v_{i,sj}^{(1)} u_s \,, \\
V_i &= \frac{\partial \psi_i^{(1)}}{\partial t} + \psi_{i,s}^{(1)} v_s^{(1)} \,, \qquad W = \frac{3}{2} \left(\frac{\partial \nu^{(1)}}{\partial t} + \nu_{,s}^{(1)} v_s^{(1)} \right) .
\end{aligned}
\tag{4.19}
$$

It follows from (4.14) and (4.18) that

$$
\begin{aligned}
(T_{ji,j} - N_{kji,kj}) u_i = \rho_0 &\left[\frac{1}{2} \frac{\partial}{\partial t} (\mathbf{u}^2) + \frac{1}{2} (\mathbf{u}^2 u_k)_{,k} + \right. \\
&\left. + v_{i,k}^{(1)} u_i u_k + \frac{1}{2} (\mathbf{u}^2 v_k^{(1)})_{,k} \right] - \rho_0 d^2 (Q_{ji} u_i)_{,j} + \\
&+ \rho_0 d^2 \left[\frac{\partial}{\partial t} (u_{i,j}) + u_{i,sj} u_s + v_{i,sj}^{(1)} u_s + u_{i,sj} v_s^{(1)} \right] u_{i,j} \,, \\
(M_{ji,j} + \varepsilon_{irs} T_{rs}) w_i = &\frac{1}{2} \rho_0 j^{(2)} \frac{\partial}{\partial t} (\mathbf{w}^2) + \frac{1}{2} \rho_0 (j^{(2)} \mathbf{w}^2 v_s^{(2)})_{,s} + \\
&+ \rho j^{(2)} \psi_{i,s}^{(1)} u_s w_i + \rho_0 V_i w_i s - \frac{1}{2} \rho_0 j_{,k}^{(2)} \mathbf{w}^2 v_k^{(2)} \,, \\
(Q_{i,i} - \Gamma) \varphi = &\frac{3}{4} \rho_0 j^{(2)} \frac{\partial}{\partial t} (\varphi^2) + \frac{3}{4} \rho_0 (j^{(2)} \varphi^2 v_s^{(2)})_{,s} + \\
&+ \frac{3}{2} \rho_0 j^{(2)} \nu_{,s}^{(1)} u_s \varphi + \rho_0 s \varphi W - \frac{3}{4} \rho_0 j_{,k}^{(2)} \varphi^2 v_k^{(2)} \,.
\end{aligned}
\tag{4.20}
$$

Clearly,

$$
\begin{aligned}
u_{i,sj} u_s u_{i,j} &= \frac{1}{2} (u_{i,j} u_{i,j} u_s)_{,s} \,, \\
u_{i,sj} v_s^{(1)} u_{i,j} &= \frac{1}{2} (u_{i,j} u_{i,j} v_s^{(1)})_{,s} \,, \\
v_{i,js}^{(1)} u_s u_{i,j} &= (v_{i,js}^{(1)} u_s u_i)_{,j} - u_j u_i \Delta v_{i,j}^{(1)} - (v_{i,j}^{(1)} u_{s,j} u_i)_{,s} + \\
&\quad + v_{i,j}^{(1)} u_{s,j} u_{i,s} \,.
\end{aligned}
\tag{4.21}
$$

In view of (4.7), (4.20), (4.21) and the boundary conditions (4.13) we obtain

$$
\begin{aligned}
&-\int_B (S_{ij}e_{ij} + M_{ij}\eta_{ij} + P_{ijk}\nu_{ijk} + Q_i\varphi_{,i} + \varphi S)\,dv = \\
&= \frac{1}{2}\int_B \rho_0 \left[\frac{\partial}{\partial t}(\mathbf{u}^2 + d^2 u_{i,j}u_{i,j}) + j^{(2)}\frac{\partial}{\partial t}(\mathbf{w}^2) + \frac{3}{2}j^{(2)}\frac{\partial}{\partial t}(\varphi^2)\right] dv + \\
&+ \int_B \Big\{\rho_0 \Big[v^{(1)}_{i,j}(u_iu_j + d^2u_{s,j}u_{i,s}) - d^2u_iu_j\Delta v^{(1)}_{i,j} + \\
&+ j^{(2)}\psi^{(1)}_{i,s}w_iu_s + \frac{3}{2}j^{(2)}\nu^{(1)}_{,s}u_s\varphi + V_iw_is + s\varphi W\Big] + cs\varphi\Big\}\,dv - \\
&-\frac{1}{4}\int_B \rho_0(2\mathbf{w}^2 + 3\varphi^2)j^{(2)}_{,s}v^{(2)}_s\,dv\,.
\end{aligned}
\tag{4.22}
$$

By (4.8), (4.14) and (4.15),

$$
\frac{1}{2}\frac{\partial}{\partial t}(s^2) = 2\nu^{(2)}s^2 + 2j^{(1)}\varphi s - (\frac{1}{2}s^2)_{,j}v^{(2)}_j - j^{(1)}_{,k}u_ks\,. \tag{4.23}
$$

In view of (4.12), (4.22), (4.23) and the entropy inequality we find that

$$
\begin{aligned}
&\frac{1}{2}\int_B\rho_0\left[\frac{\partial}{\partial t}(\mathbf{u}^2+d^2u_{i,j}u_{i,j})+j^{(2)}\frac{\partial}{\partial t}(\mathbf{w}^2)+\frac{3}{2}j^{(2)}\frac{\partial}{\partial t}(\varphi^2) + \frac{\partial}{\partial t}(s^2)\right] \leq \\
&\leq -\int_B \rho_0 \Big[v^{(1)}_{i,j}(u_iu_j + d^2u_{s,j}u_{i,s}) - d^2u_iu_j\Delta v^{(1)}_{i,j} + \\
&+ j^{(2)}\psi^{(1)}_{i,s}w_iu_s + \frac{3}{2}j^{(2)}\nu^{(1)}_{,s}u_s\varphi + j^{(1)}_{,k}u_ks + V_iw_is + \\
&+ (W+c\rho_0^{-1}-2j^{(1)})s\varphi - 2\nu^{(2)}s^2\Big]\,dv + \frac{1}{4}\int_B\rho_0(2\mathbf{w}^2+3\varphi^2)j^{(2)}_{,k}v^{(2)}_k\,dv.
\end{aligned}
\tag{4.24}
$$

By the arithmetic-geometric mean inequality

$$
\begin{aligned}
-\psi^{(1)}_{i,s}w_iu_s &\leq \frac{1}{2}(\mathbf{w}^2 + \psi^{(1)}_{i,s}\psi^{(1)}_{i,j}u_su_j)\,, \\
-\nu^{(1)}_{,k}u_k\varphi &\leq \frac{1}{2}(\mathbf{u}^2 + \nu^{(1)}_{,k}\nu^{(1)}_{,k}\varphi^2)\,, \\
-V_iw_is &\leq \frac{1}{2}(\mathbf{w}^2 + V_iV_is^2)\,, \\
-j^{(1)}_{,k}u_ks &\leq \frac{1}{2}(\mathbf{u}^2 + j^{(1)}_{,p}j^{(1)}_{,p}s^2)\,, \\
-(W + c\rho_0^{-1} - 2j^{(1)})\varphi s &\leq \frac{1}{2}\left[\varphi^2 + (W + c\rho_0^{-1} - 2j^{(1)})^2s^2\right]\,.
\end{aligned}
\tag{4.25}
$$

Since $(v_i^{(\alpha)}, \psi_i^{(\alpha)}, \nu^{(\alpha)}, j^{(\alpha)})$, $(\alpha = 1, 2)$ are admissible processes on $B \times I$, the functions $v_{i,s}^{(\alpha)}$, $\Delta v_{i,j}^{(1)}$, $\psi_{i,s}^{(\alpha)}$, $\nu_{,s}^{(\alpha)}$, $j_{,s}^{(\alpha)}$, V_i, W, $j^{(\alpha)}$ and $\nu^{(\alpha)}$ are bounded on $B \times I$. If we define the function E on I by

$$E = \frac{1}{2} \int_B \rho_0 (u_i u_i + d^2 u_{i,j} u_{i,j} + w_i w_i + \varphi^2 + s^2) \, dv$$

then from (4.24) and (4.25) we conclude that there exists a positive constant m such that

$$\frac{dE}{dt} \leq mE \, .$$

By integration on $(0, \tau)$ and recalling that $E(0) = 0$, we find $E(\tau) \exp(-m\tau) \leq 0$. Thus we conclude that $E = 0$ on $B \times I$. This fact implies that $\mathbf{u} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$, $\varphi = 0$ and $s = 0$ on $B \times I$. From (4.16) and (4.17) we find $S_{ij} = 0$, $N_{ijk} = 0$ so that from $(4.18)_1$ we obtain $(P^{(1)} - P^{(2)})_{,i} = 0$. This completes the proof.

The weighted energy method of GALDI and RIONERO [19] can be used to establish a uniqueness result for unbounded domains.

5. Concentrated Body Loads. In this section we consider non-heat-conducting compressible fluids and assume that the free energy depends on the density ρ and the microinertia j. First we derive the linear theory appropiate to small departures from an equilibrium state and establish a representation of Galerkin type for the solutions to the field equations. Representations of this type for micropolar fluids have been presented in [20]. Then we use the Galerkin representation to study the problem of concentrated body loads in the case of steady vibrations.

We assume that there exists an equilibrium state of the fluid in which the density and the microinertia have the uniform values ρ_0 and j_0, repectively. We introduce the notations

$$\varphi = \rho - \rho_0 \, , \qquad \xi = j - j_0 \, . \tag{5.1}$$

We assume that φ, ξ, v_i, ψ_i and ν are small, i.e. $\varphi = \varepsilon\varphi'$, $\xi = \varepsilon\xi'$, $v_i = \varepsilon v_i'$, $\psi_i = \varepsilon\psi_i'$, $\nu = \varepsilon\nu'$ where ε is a constant small enough for squares and higher powers to be neglected, and φ', ξ', v_i', ψ_i', ν' are independent of ε.

To the second order, the free energy is taken in the form

$$\psi = \psi_0 + \pi_0 \varphi + p_0 \xi + \frac{1}{2} a' \varphi^2 + \frac{1}{2} b' \xi^2 + c' \varphi \xi \, , \tag{5.2}$$

where ψ_0, π_0, p_0, a', b' and c' are prescribed constants. Without loss of generality we assume that $\pi_0 = 0$, $p_0 = 0$. It follows from (2.25), (3.5) and (5.2) that

$$p_i = -(a\varphi + b\xi)_{,i}\,, \qquad p = c_1\varphi + d_1\xi\,, \tag{5.3}$$

where

$$a = \rho_0^2 a'\,, \quad b = \rho_0^2 c'\,, \quad c_1 = -2j_0\rho_0 c'\,, \quad d_1 = -2j_0\rho_0 b'\,.$$

From (3.4) we obtain the following equations for the functions v_i, ψ_i, ν, φ and ξ

$$\begin{aligned}
&\kappa_2(1-l_2\Delta)\Delta v_i + [\kappa_1 - \kappa_2 - (\kappa_1 l_1 - \kappa_2 l_2)\Delta]v_{j,ji} + \kappa\varepsilon_{ijs}\psi_{s,j} + \\
&\qquad + [\lambda_0 - (\beta_1 + 2\beta_2)\Delta]\nu_{,i} - a\varphi_{,i} - b\xi_{,i} + f_i = \rho_0(1-d^2\Delta)\frac{\partial v_i}{\partial t}\,,\\
&\gamma\Delta\psi_i + (\alpha+\beta)\psi_{s,si} + \kappa\varepsilon_{irs}v_{s,r} - 2\kappa\psi_i + g_i = I\frac{\partial \psi_i}{\partial t}\,,\\
&a_0\Delta\nu - [b_1 - (a_3 + a_4)\Delta]v_{s,s} - b_0\nu + c_1\varphi + d_1\xi + l = J\frac{\partial \nu}{\partial t}\,,\\
&\frac{\partial\varphi}{\partial t} - \rho_0 v_{i,i} = 0\,, \qquad \frac{\partial\xi}{\partial t} - 2j_0\nu = 0\,,
\end{aligned} \tag{5.4}$$

where $I = \rho_0 j_0$, $J = 3I/2$. The system (5.4) can be written in the form

$$\begin{aligned}
&D_1\mathbf{v} + T_1\operatorname{grad}\operatorname{div}\mathbf{v} + \kappa\operatorname{curl}\psi + T_2\operatorname{grad}\nu - a\operatorname{grad}\varphi - b\operatorname{grad}\xi = -\mathbf{f},\\
&D_2\psi + (\alpha+\beta)\operatorname{grad}\operatorname{div}\psi + \kappa\operatorname{curl}\mathbf{v} = -\mathbf{g},\\
&D_3\nu - T_3\operatorname{div}\mathbf{v} + c_1\varphi + d_1\xi = -l,\\
&\frac{\partial\varphi}{\partial t} - \rho_0\operatorname{div}\mathbf{v} = 0, \qquad \frac{\partial\xi}{\partial t} - 2j_0\nu = 0,
\end{aligned} \tag{5.5}$$

where we have used the notations

$$\begin{aligned}
D_1 &= \kappa_2(1-l_2\Delta)\Delta - \rho_0(1-d^2\Delta)\frac{\partial}{\partial t}\,,\\
D_2 &= \gamma\Delta - I\frac{\partial}{\partial t} - 2\kappa\,, \qquad D_3 = a_0\Delta - J\frac{\partial}{\partial t} - b_0\,,\\
T_1 &= \kappa_1 - \kappa_2 - (\kappa_1 l_1 - \kappa_2 l_2)\Delta\,, \quad T_2 = \lambda_0 - (\beta_1 + 2\beta_2)\Delta\,,\\
T_3 &= b_1 - (a_3 + a_4)\Delta\,.
\end{aligned} \tag{5.6}$$

In the linearized theory a superposed dot will be used to denote the partial derivative with respect to the time.

In what follows we establish a representation of Galerkin type for the solutions of the field equations (5.5). We introduce the notations

$$
\begin{aligned}
\mathcal{P} &= (D_1 + T_1\Delta)\frac{\partial}{\partial t} - a\rho_0\Delta\,,\\
\mathcal{M}_1 &= c_1\rho_0 - T_3\frac{\partial}{\partial t}\,, \qquad \mathcal{M}_2 = D_3\frac{\partial}{\partial t} + 2j_0 d_1\,,\\
Z &= T_2\frac{\partial}{\partial t} - 2j_0 b\,, \qquad \mathcal{N} = \mathcal{P}\mathcal{M}_2 - Z\Delta\mathcal{M}_1\,,\\
\mathcal{L} &= (T_1D_2 - \kappa^2)\mathcal{M}_2\frac{\partial}{\partial t} - D_2(a\rho_0\mathcal{M}_2 + Z\mathcal{M}_1)\,,\\
\Omega &= D_2 + (\alpha+\beta)\Delta\,, \quad Q = c_1D_1 - (aT_3 - c_1T_1)\Delta\,,\\
\mathcal{H} &= T_3Z\Delta + (D_1 + T_1\Delta)\mathcal{M}_2\,.
\end{aligned} \tag{5.7}
$$

Theorem 5.1. *Let*

$$
\begin{aligned}
&\mathbf{v} = (-\mathcal{N}D_2 + \mathcal{L}\operatorname{grad}\operatorname{div})\mathbf{U} + \kappa\Omega\operatorname{curl}\mathbf{W} + Z\operatorname{grad}\dot{u} -\\
&\quad -(a\mathcal{M}_2 + c_1Z)\operatorname{grad} v - (b\mathcal{M}_2 + d_1Z)\operatorname{grad} w\,,\\
&\psi = \kappa\mathcal{N}\operatorname{curl}\mathbf{U} - \left\{D_1\Omega - [(\alpha+\beta)D_1 - \kappa^2]\operatorname{grad}\operatorname{div}\right\}\mathbf{W}\,,\\
&\nu = (D_1D_2 + \kappa^2\Delta)\mathcal{M}_1\operatorname{div}\dot{\mathbf{U}} - \mathcal{P}\dot{u} + Q\dot{v} + (d_1\mathcal{P} + b\Delta\mathcal{M}_1)w\,,\\
&\varphi = -\rho_0(D_1D_2 + \kappa^2\Delta)\mathcal{M}_2\operatorname{div}\mathbf{U} + \rho_0\Delta Zu - \mathcal{H}v - \rho_0(bD_3 + d_1T_2)\Delta w,\\
&\xi = 2j_0(D_1D_2 + \kappa^2\Delta)\mathcal{M}_1\operatorname{div}\mathbf{U} - 2j_0\mathcal{P}u + 2j_0Qv + (T_2\Delta\mathcal{M}_1 - D_3\mathcal{P})w,
\end{aligned} \tag{5.8}
$$

where $\mathbf{U}$*,* $\mathbf{W}$*,* u*,* v *and* w *satisfy the equations*

$$
\begin{aligned}
&(D_1D_2 + \kappa^2\Delta)\mathcal{N}\mathbf{U} = \mathbf{f}\,,\\
&(D_1D_2 + \kappa^2\Delta)\Omega\mathbf{W} = \mathbf{g}\,,\\
&\mathcal{N}u = l\,, \quad \mathcal{N}v = 0\,, \quad \mathcal{N}w = 0\,.
\end{aligned} \tag{5.9}
$$

Then v_i*,* ψ_i*,* ν*,* φ *and* ξ *satisfy the equations* (5.5)*.*

Proof. It follows from (5.5) and (5.8) that

$$
\begin{aligned}
&D_1\mathbf{v} + T_1\operatorname{grad}\operatorname{div}\mathbf{v} + \kappa\operatorname{curl}\psi + T_2\operatorname{grad}\nu - a\operatorname{grad}\varphi -\\
&-b\operatorname{grad}\xi = -(D_1D_2 + \kappa^2\Delta)\mathcal{N}\mathbf{U} + \{D_1\mathcal{L} + T_1(\mathcal{L}\Delta - \mathcal{N}D_2) +\\
&+\kappa^2\mathcal{N} + T_2(D_1D_2 + \kappa^2\Delta)\mathcal{M}_1\frac{\partial}{\partial t} + a\rho_0(D_1D_2 + \kappa^2\Delta)\mathcal{M}_2 -\\
&-2j_0b(D_1D_2 + \kappa^2\Delta)\mathcal{M}_1\}\operatorname{grad}\operatorname{div}\mathbf{U} + (2j_0b\mathcal{P} - a\rho_0\Delta Z -\\
&-T_2\mathcal{P}\frac{\partial}{\partial t} + T_1Z\Delta\frac{\partial}{\partial t} + D_1Z\frac{\partial}{\partial t})\operatorname{grad} u + [a\mathcal{H} - 2j_0bQ +\\
&+T_2Q\frac{\partial}{\partial t} - (T_1\Delta + D_1)(a\mathcal{M}_2 + c_1Z)]\operatorname{grad} v +\\
&+[-(b\mathcal{M}_2 + d_1Z)(D_1 + T_1\Delta) + T_2(d_1\mathcal{P} + b\Delta\mathcal{M}_1) +\\
&+a\rho_0(bD_3 + d_1T_2)\Delta - b(T_2\Delta\mathcal{M}_1 - D_3\mathcal{P})]\operatorname{grad} w =\\
&= -(D_1D_2 + \kappa^2\Delta)\mathcal{N}\mathbf{U}\,.
\end{aligned} \tag{5.10}
$$

Similarly we obtain

$$
(5.11)\qquad
\begin{aligned}
&D_2\psi + (\alpha+\beta)\operatorname{grad}\operatorname{div}\psi + \kappa\operatorname{curl}\mathbf{v} = -(D_1D_2 + \kappa^2\Delta)\Omega\mathbf{W}\,,\\
&D_3\nu - T_3\operatorname{div}\mathbf{v} + c_1\varphi + d_1\xi = -\mathcal{N}u\,,\\
&\frac{\partial\varphi}{\partial t} - \rho_0\operatorname{div}\mathbf{v} = -\mathcal{N}u\,,\\
&\frac{\partial\xi}{\partial t} - 2j_0\nu = -\mathcal{N}w\,.
\end{aligned}
$$

In view of (5.9), from (5.10) and (5.11) we obtain the desired result.

In the case of steady motions of micropolar fluids the Galerkin representations were presented in [21].

We now consider a fluid which occupies the entire three-dimensional Euclidean space. We use the representation (5.8) to study the problem of concentrated body loads in the case of steady vibrations. We assume that

$$
\begin{aligned}
&\mathbf{f} = \operatorname{Re}[\mathbf{f}^*(\mathbf{x})\exp(-i\omega t)]\,,\qquad \mathbf{g} = \operatorname{Re}[\mathbf{g}^*(\mathbf{x})\exp(-i\omega t)]\,,\\
&l = \operatorname{Re}[l^*(\mathbf{x})\exp(-i\omega t)]\,,
\end{aligned}
$$

where ω is the frequency of vibration and $i = (-1)^{1/2}$. If we take

$$
(5.12)\qquad
\begin{aligned}
\mathbf{v} &= \operatorname{Re}[\mathbf{v}^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\qquad \psi = \operatorname{Re}[\psi^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\\
\nu &= \operatorname{Re}[\nu^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\qquad \varphi = \operatorname{Re}[\varphi^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\\
\xi &= \operatorname{Re}[\xi^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,
\end{aligned}
$$

then the field equations reduce to a differential system for the amplitudes $\mathbf{v}^*$, ψ^*, ν^*, φ^* and ξ^*. We denote

$$
(5.13)\qquad
\begin{aligned}
\mathbf{U} &= \operatorname{Re}[\mathbf{U}^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\qquad \mathbf{W} = \operatorname{Re}[\mathbf{W}^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\\
u &= \operatorname{Re}[u^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\qquad v = \operatorname{Re}[v^*(\mathbf{x};\omega)\exp(-i\omega t)]\,,\\
w &= \operatorname{Re}[w^*(\mathbf{x};\omega)\exp(-i\omega t)]\,.
\end{aligned}
$$

Then, from (5.8) we obtain

$$
(5.14)\qquad
\begin{aligned}
\mathbf{v}^* &= (-\mathcal{N}_1A_2 + \mathcal{L}_1\operatorname{grad}\operatorname{div})\mathbf{U}^* + \kappa\Omega_1\operatorname{curl}\mathbf{W}^* - i\omega Z_1\operatorname{grad}u^* -\\
&\qquad -(aM_2 + c_1Z_1)\operatorname{grad}v^* - (bM_2 + d_1Z_1)\operatorname{grad}w^*\,,\\
\psi^* &= \kappa\mathcal{N}_1\operatorname{curl}\mathbf{U}^* - \{A_1\Omega_1 - [(\alpha+\beta)A_1 - \kappa^2]\operatorname{grad}\operatorname{div}\}\mathbf{W}^*\,,\\
\nu^* &= -(A_1A_2 + \kappa^2\Delta)M_1 i\omega\operatorname{div}\mathbf{U}^* + i\omega\mathcal{P}_1u^* - i\omega Q_1v^* +\\
&\qquad +(d_1\mathcal{P} + b\Delta M_1)w^*\,,\\
\varphi^* &= -\rho_0(A_1A_2 + \kappa^2\Delta)M_2\operatorname{div}\mathbf{U}^* + \rho_0\Delta Z_1u^* - \mathcal{H}_1v^* -\\
&\qquad -\rho_0(bA_3 + d_1T_2)\Delta w^*\,,\\
\xi^* &= 2j_0(A_1A_2 + \kappa^2\Delta)M_1\operatorname{div}\mathbf{U}^* - 2j_0\mathcal{P}_1u^* + 2j_0Q_1v^* +\\
&\qquad +(T_2M_1\Delta - A_3\mathcal{P}_1)w^*\,,
\end{aligned}
$$

where we have used the notations

$$(5.15)\qquad \begin{aligned}
A_1 &= \kappa_2(1-l_2\Delta)\Delta + i\omega\rho_0(1-d^2\Delta)\,,\\
A_2 &= \gamma\Delta + i\omega I - 2\kappa\,,\quad A_3 = a_0\Delta + i\omega J - b_0\,,\\
\mathcal{P}_1 &= -i\omega(A_1+T_1\Delta) - a\rho_0\Delta\,,\\
M_1 &= c_1\rho_0 + i\omega T_3\,,\quad M_2 = 2j_0d_1 - i\omega A_3\,,\\
Z_1 &= -i\omega T_2 - 2j_0 b\,,\quad \mathcal{N}_1 = M_2\mathcal{P}_1 - Z_1M_1\Delta\,,\\
\mathcal{L}_1 &= -i\omega(T_1A_2-\kappa^2)M_2 - (a\rho_0M_2 + Z_1M_1)A_2\,,\\
\Omega_1 &= A_2 + (\alpha+\beta)\Delta\,,\quad Q_1 = c_1A_1 - (aT_3 - c_1T_1)\Delta\,,\\
\mathcal{H}_1 &= T_3Z_1\Delta + (A_1+T_1\Delta)M_2\,.
\end{aligned}$$

The functions $\mathbf{U}^*$, $\mathbf{W}^*$, u^*, v^* and w^* satisfy the equations

$$(5.16)\qquad \begin{aligned}
&(A_1A_2+\kappa^2\Delta)\mathcal{N}_1\mathbf{U}^* = \mathbf{f}^*\,,\\
&(A_1A_2+\kappa^2\Delta)\Omega_1\mathbf{W}^* = \mathbf{g}^*\,,\\
&\mathcal{N}_1u^* = l^*\,,\qquad \mathcal{N}_1v^* = 0\,,\quad \mathcal{N}_1w^* = 0\,.
\end{aligned}$$

Let us assume that $\mathbf{f}^* = \mathbf{0}$, $\mathbf{g}^* = \mathbf{0}$, $l^* = \delta(\mathbf{x}-\mathbf{y})$ where δ is the Dirac delta and $\mathbf{y}$ is a fixed point. In view of (5.16) we take $U_i^* = 0$, $W_i^* = 0$, $u^* = \chi(x;y)$, $v^* = 0$, $w^* = 0$, where χ satisfies the equation

$$(5.17)\qquad \mathcal{N}_1\chi = \delta\,.$$

From (5.14) we obtain the amplitudes

$$(5.18)\qquad \begin{aligned}
\mathbf{v}^* &= -i\omega Z_1 \operatorname{grad}\chi\,,\quad \psi^* = \mathbf{0}\,,\\
\nu^* &= i\omega\mathcal{P}_1\chi\,,\quad \varphi^* = \rho_0\Delta Z_1\chi\,,\quad \xi^* = -2j_0\mathcal{P}_1\chi\,.
\end{aligned}$$

In view of (5.6) and (5.15) we obtain

$$\mathcal{N}_1 = \tau\omega^2\Delta\Delta\Delta + i\omega(s_2 - \zeta_1e_3 - \zeta_3e_4)\Delta\Delta + (s_3 - \zeta_1\zeta_3)\Delta + \rho_0\zeta_2\omega^2\,,$$

where we have used the notations

$$\begin{aligned}
e_1 &= \kappa_1 - \kappa_2\,,\quad e_2 = \kappa_1l_1 - \kappa_2l_2\,,\quad e_3 = \beta_1 + 2\beta_2\,,\quad e_4 = a_3 + a_4\,,\\
\tau &= \kappa_1l_1a_1 - e_3e_4\,,\quad \zeta_1 = c_1\rho_0 + i\omega b_1\,,\quad \zeta_2 = J\omega^2 + 2j_0d_1 + i\omega b_0\,,\\
\zeta_3 &= 2j_0b + i\omega\lambda_0\,,\quad s_1 = -\rho_0\omega^2d^2 - i\omega\kappa_1\,,\quad s_2 = \zeta_2\kappa_1l_1 - a_0(s_1 - a\rho_0)\,,\\
s_3 &= (s_1 - a\rho_0)\zeta_2 - i\omega^3\rho_0a_0\,.
\end{aligned}$$

Clearly, we can write

$$(5.19)\qquad \mathcal{N}_1 = \tau\omega^2(\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)\,,$$

where σ_1^2, σ_2^2 and σ_3^2 are the roots of the equation

$$\tau\omega^2 y^3 - i\omega(s_2 - \zeta_1 e_3 - \zeta_3 e_4)y^2 + (s_3 - \zeta_1\zeta_3)y - \rho_0\zeta_2\omega^2 = 0\,. \tag{5.20}$$

In what follows we denote by σ_1, σ_2 and σ_3 the roots with nonnegative real parts. We assume that $\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq \sigma_1$, and $\tau \neq 0$. Let us consider the equation

$$(\Delta + \sigma_1^2)(\Delta + \sigma_2^2)(\Delta + \sigma_3^2)\Phi = F\,. \tag{5.21}$$

If the functions Q_j satisfy the equations

$$(\Delta + \sigma_j^2)Q_j = F\,, \qquad (\text{no sum; } j = 1,2,3)\,, \tag{5.22}$$

then the function Φ can be expressed as

$$\Phi = \sum_{j=1}^{3} B_j Q_j\,, \tag{5.23}$$

where

$$\begin{aligned} &B_1^{-1} = (\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \sigma_3^2)\,, \quad B_2^{-1} = (\sigma_2^2 - \sigma_1^2)(\sigma_2^2 - \sigma_3^2)\,, \\ &B_3^{-1} = (\sigma_3^2 - \sigma_1^2)(\sigma_3^2 - \sigma_2^2)\,. \end{aligned} \tag{5.24}$$

If we take $F = \delta/(\tau\omega^2)$, then from (5.22) we get

$$Q_j = -\frac{1}{4\pi\tau\omega^2 r}\exp(i\sigma_j r)\,, \tag{5.25}$$

where $r = |\mathbf{x} - \mathbf{y}|$. Thus, from (5.17), (5.19), (5.23) and (5.25) we conclude that

$$\chi = -\frac{1}{4\pi\tau\omega^2 r}\sum_{j=1}^{3} B_j \exp(i\sigma_j r)\,. \tag{5.26}$$

If we substitute the function χ from (5.26) into (5.18) then we obtain the amplitudes corresponding to the considered concentrated load.

We now assume that $g_i^* = m\delta_{ij}$, (j fixed), $f_i^* = 0$ and $l^* = 0$. If we take $U_i^* = 0$, $W_i^* = \Psi\delta_{ij}$, $u^* = 0$, $v^* = 0$, $w^* = 0$ then the equations (5.16) are satisfied if Ψ is a solution of the equation

$$(A_1 A_2 + \kappa^2\Delta)\Omega_1\Psi = m\,. \tag{5.27}$$

We introduce the notations

$$\begin{aligned}\eta_1 &= -2\kappa + i\omega I\,, \quad \eta_2 = (\gamma - \eta_1 l_2)\kappa_2 - i\omega\rho_0\gamma d^2\,,\\ \eta_3 &= \kappa^2 + \eta_1\kappa_2 + i\omega\rho_0(\gamma - \eta_1 d^2)\,.\end{aligned}$$

In what follows we denote by τ_j^2 the roots of the equation

$$\kappa_2 l_2\gamma z^3 + \eta_2 z^2 - \eta_3 z + i\omega\rho_0\eta_1 = 0\,.$$

Let

$$\tau_4^2 = \eta_1/(\alpha + \beta + \gamma)\,.$$

The equation (5.27) can be written in the form

$$(\Delta + \tau_1^2)(\Delta + \tau_2^2)(\Delta + \tau_3^2)(\Delta + \tau_4^2)\Psi = -\frac{1}{\gamma\kappa_2 l_2}m\,. \tag{5.28}$$

We denote by τ_s $(s = 1, 2, 3, 4)$ the roots with positive real parts and assume that $\tau_1 \neq \tau_2 \neq \tau_3 \neq \tau_4 \neq \tau_1$, $\tau_2 \neq \tau_4$, $\tau_1 \neq \tau_3$.

If the functions Y_r $(r = 1, 2, 3, 4)$ satisfy the equations

$$(\Delta + \tau_s^2)Y_s = -\frac{1}{\gamma\kappa_2 l_2}m\,, \quad (\text{no sum; } s = 1, 2, 3, 4)\,,$$

then the function Ψ can be expressed as

$$\Psi = \sum_{s=1}^{4} q_s Y_s\,,$$

where

$$q_k^{-1} = \prod_{j=1 (j\neq k)}^{4} (\tau_j^2 - \tau_k^2)\,, \qquad (k = 1, 2, 3, 4)\,.$$

If $m = \delta(\mathbf{x} - \mathbf{y})$ then we find that $\Psi = E$, where

$$E = \frac{1}{4\pi\gamma\kappa_2 l_2 r}\sum_{s=1}^{4} q_s \exp(i\tau_s r)\,. \tag{5.29}$$

It follows from (5.14) that

$$\begin{aligned}v_k^* &= \kappa\Omega_1\varepsilon_{ksj}E_{,s}\,,\\ \psi_k^* &= [(\alpha + \beta)A_1 - \kappa^2]E_{,kj} - \delta_{kj}A_1\Omega_1 E\,,\\ \nu^* &= 0\,, \qquad \varphi^* = 0\,, \qquad \xi^* = 0\,.\end{aligned}$$

In a similar way we can establish the solutions corresponding to a concentrated body force.

In the context of the nonpolar theory of fluids the fundamental solutions have been established in various papers (see e.g. [22], [23]). The problem of concentrated loads in the theory of micropolar fluids has been studied in [20], [24].

Acknowledgement. The first author was supported by the Spanish Ministry of Education and Culture as a visiting professor to the Centre de Recerca Matemàtica. The second author was supported by the project PB96–0497 of the DGES of the Spanish Ministry of Education and Culture.

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Received: 16.XII.1999

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