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# TANGENT BUNDLES SUBJECTED TO FINSLERIAN $\beta$–CHANGES

BY

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**0. Introduction.** In 1963 K. YANO and E.T. DAVIES [8] have considered the tangent bundle $T(M)$ of a differentiable manifold $M$ and define a function $L$ on $T(M)$ with the properties necessary to make $M$ into a Finsler space. Furthermore, they have considered the special case in which $L$ defines a Riemannian metric on $M$. In particular, using the lift in Sasaki's sense, they have given the following two main theorems.

**Theorem A.** *The necessary and sufficient condition in order that the tangent bundle of a Finsler space to be Kählerian is that the Finsler space shall have absolute parallelism of line elements.*

**Theorem B.** *The tangent bundle of a Riemannian manifold is Kählerian if and only if the Riemannian manifold is locally flat.*

In 1984 C. SHIBATA [5] studied some geometrical properties of tensors being invariant by $\beta$–change of the metric. The author and S. KIKUCHI [2] also investigated the geometric characters of the Finsler space with ${R_0}^i{}_{hk}=0$, that is, the theory of the Finsler space and the osculating Riemannian spaces.

Our purpose is to see how certain structures on the tangent bundle over a Finsler space are influenced by a $\beta$–change of the Finslerian functions. In particular, the osculating Riemannian spaces of a Finsler space will be discussed.

The terminology and notations are referred to well–known RUND's book [4] and MATSUMOTO's monograph [3].

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**1. Introduction. Tangent bundle and $\beta$–changes.** We shall be concerned with an $n$–dimensional $C^\infty$–paracompact manifold $M$ and its tangent bundle $T(M)$ with the projection $\pi$. Let $F^n = (M^n, L)$ be an $n$–dimensional Finsler space with a fundamental function $L(x, y)$ and a tensor $g_{ji}(x, y) = (\partial^2 L^2/\partial y^j \partial y^i)/2$. We consider a $\beta$–change of Finsler metric which is defined by $L \longrightarrow \overline{L} = f(L, \beta)$, and have another Finsler space $\overline{F}^n = (M^n, \overline{L})$ with $\overline{L} = f(L, \beta)$, where $\beta = b_i(x)y^i$ with a covector field $b_i(x)$ on $M^n$ and $f(L, \beta)$ is a positively homogeneous function of $L$ and $\beta$ of degree one. Then we can introduce in $F^n$ the Cartan connection $C\Gamma =$ $= (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ and another Cartan connection $C\overline{\Gamma} = (\overline{F}_j{}^i{}_k, \overline{N}^i{}_j, \overline{C}_j{}^i{}_k)$ is derived in $\overline{F}^n$ from a $\beta$–change of metrics.

For the later use, we show here a lemma.

**Lemma 1.** ([5]) *If the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on $F^n$, the difference tensor $D_j{}^i{}_k (:= \overline{F}_j{}^i{}_k - F_j{}^i{}_k)$ vanishes.*

From $D^i{}_j (:= \overline{N}^i{}_j - N^i{}_j) = D_j{}^i{}_k y^k = D_j{}^i{}_0$, this lemma leads us to $\overline{N}^i{}_j = N^i{}_j$. The fundamental function $L(x, y)$ gives a global 1–form $\omega = g_{ji}(x, y)y^j dx^i$, and, by exterior derivation, a global 2–form $d\omega$ of rank $2n$ with skew–symmetric coefficients $F_{BA} (A, B, \ldots = 1, 2, \ldots, 2n)$. Then we can introduce a positive definite Riemannian metric $G_{BA}$ in $T(M)$ which is permutable with $F$, and which defines, by $F_C{}^A = F_{CB}G^{BA}$, an almost complex structure $F_C{}^E F_E{}^A = -\delta_C{}^A$. Using the adapted frame we can now write for the components of tensors the following expressions :

$$G_{\gamma\beta} = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix}, \qquad G^{\beta\alpha} = \begin{pmatrix} g^{ih} & 0 \\ 0 & g^{ih} \end{pmatrix} \tag{1.1}$$

$$F_{\gamma\beta} = \begin{pmatrix} 0 & g_{ji} \\ -g_{ji} & 0 \end{pmatrix}, \qquad F_\beta{}^\alpha = \begin{pmatrix} 0 & \delta^h{}_i \\ -\delta^h{}_i & 0. \end{pmatrix} \tag{1.2}$$

The fundamental tensor $\overline{g}_{ji} = (\partial^2 \overline{L}^2/\partial y^j \partial y^i)/2$ of $\overline{F}^n$ is given by

$$\overline{g}_{ji} = p g_{ji} + a_{ji}, \tag{1.3}$$

where we put $p = f f_L/L \neq 0$,

$$\begin{aligned} a_{ji} &= p_0 b_j b_i + p_{-1}(b_j y_i + b_i y_i) + p_{-2} y_j y_i, \\ p_0 &= f f_{\beta\beta} + f_\beta{}^2, \quad p_{-1} = (f f_{L\beta} + f_L f_\beta)/L, \\ p_{-2} &= (f f_{LL} + f_L{}^2 - f f_L/L)/L^2, \end{aligned}$$

and subscripts $L, \beta$ denote partial differentiations by $L, \beta$ respectively.

From $F_{\gamma\beta}, \overline{F}_{\gamma\beta}$ and (1.3) we can easily obtain

**Proposition 1.** *Let $b_i(x)$ be parallel with respect to the Cartan connection $C\Gamma$ on $F^n$. Then $d\overline{\omega}$ defines a symplectic structure in $T(\overline{F}^n)$ if and only if $d\omega$ defines a symplectic structure in $T(F^n)$.*

**Proposition 2.** *Let $b_i(x)$ be parallel with respect to the Cartan connection $C\Gamma$ on $F^n$. Then $(\overline{F}, \overline{G})$ is an almost Hermitian structure in $T(\overline{F}^n)$ if and only if $(F, G)$ is an almost Hermitian structure in $T(F^n)$.*

**Proof.** We can easily verify that $\overline{F}_\gamma{}^\beta \overline{F}_\beta{}^\alpha = -\delta_\gamma{}^\alpha$ is equivalent to $F_\gamma{}^\beta F_\beta{}^\alpha = -\delta_\gamma{}^\alpha$. From (1.1), (1.2) and (1.3) we obtain that ${}^t\overline{F}\,\overline{G}\,\overline{F} = \overline{G}$ is equivalent to ${}^tFGF = G$. Thus $\overline{F}_\beta{}^\delta \overline{F}_\alpha{}^\gamma \overline{G}_{\delta\gamma} = \overline{G}_{\beta\alpha}$ is equivalent to $F_\beta{}^\delta F_\alpha{}^\gamma G_{\delta\gamma} = G_{\beta\alpha}$.

From Proposition 2 and the fundamental 2–forms $d\omega, d\overline{\omega}$, we immediately have

**Proposition 3.** *Let $b_i(x)$ be parallel with respect to the Cartan connection $C\Gamma$ on $F^n$. Then $(\overline{F}, \overline{G})$ is an almost Kählerian structure in $T(\overline{F}^n)$ if and only if $(F, G)$ is an almost Kählerian structure in $T(F^n)$.*

SHIBATA [5] has shown that the $(v)h$–torsion tensor with respect to the Cartan connection $C\overline{\Gamma}$ on $\overline{F}^n$ is written as

$$\overline{R}^i{}_{jk} = R^i{}_{jk} + \mathrm{a}_{(jk)}\big\{D^i{}_{j|k} - (\partial D^i{}_j/\partial y^r + p^i{}_{jr})D^r{}_k\big\}, \tag{1.4}$$

where $D^i{}_j = D_j{}^i{}_0$, the notation $\mathrm{a}_{(jk)}$ denotes the interchange of indices, $j, k$ and substraction, and $(|)$ the $h$–covariant differentiation with respect to the Cartan connection $C\Gamma$ on $F^n$. By Lemma 1, if $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on $F^n$, then (1.4) reduces to $\overline{R}^i{}_{jk} = R^i{}_{jk}$. As the vanishing of $R^i{}_{jk}$ is equivalent to the condition that the Finsler space $F^n$ has absolute parallelism of line elements (see [7]), using Theorem A, we get

**Theorem 1.1.** *Let $b_i(x)$ be parallel with respect to the Cartan connection $C\Gamma$ on $F^n$. Then the tangent bundle $T(\overline{F}^n)$ of the Finsler space $\overline{F}^n$ given by a $\beta$–change is Kählerian if and only if the tangent bundle $T(F^n)$ of the Finsler space $F^n$ is Kählerian.*

Next, we shall restrict ourselves to a Riemannian space $F^n$ with a Riemannian metric $\alpha$. Then $\overline{F}^n = (M^n, \overline{L} = f(\alpha, \beta))$ which is obtained from $F^n = (M^n, L = \alpha)$ by a $\beta$–change becomes a Finsler space with an $(\alpha, \beta)$–metric.

SHIBATA [5] also has shown that the $h$–curvature tensor of $F^n$ is writen as

$$\overline{R}_h{}^i{}_{jk}(x,y) = R_h{}^i{}_{jk}(x) + \overline{C}_h{}^i{}_m y^1 R_1{}^m{}_{jk}(x) + \mathrm{a}_{jk}\{D_j{}^i{}_{h|k} + D_r{}^i{}_j D_h{}^r{}_k + \overline{C}_h{}^i{}_m(D^m{}_{j|k} + D_s{}^m{}_j D^s{}_k)\}, \tag{1.5}$$

where (|) is the $h$–covariant differentiation with respect to the Cartan connection $C\overline{\Gamma}$ on $\overline{F}^n$. Let $b_i(x)$ be parallel with respect to the Riemannian connection on Riemannian space $F^n$.
By Lemma 1, (1.5) reduces to

$$\overline{R}_h{}^i{}_{jk}(x,y) = R_h{}^i{}_{jk}(x) + \overline{C}_h{}^i{}_m y^1 R_1{}^m{}_{jk}(x). \tag{1.6}$$

Using now Theorem A and Theorem B, we get

**Theorem 1.2.** *Let $b_i(x)$ be parallel with respect to the Riemannian connection on Riemannian space $F^n = (M^n, L = \alpha)$. Then the tangent bundle $T(\overline{F}^n)$ of the Finsler space $\overline{F}^n = (M^n, \overline{L} = f(\alpha,\beta))$ given by a $\beta$–change is Kählerian if and only if the tangent bundle $T(F^n)$ of the associated Riemannian space $F^n$ is Kählerian.*

**Proof.** Contracting (1.6) by $y^h$, we obtain $\overline{R}^i{}_{jk}(x,y) = y^h R_h{}^i{}_{jk}(x)$. It is seen that $\overline{R}^i{}_{jk}(x,y) = 0$ is equivalent to $R_h{}^i{}_{jk}(x) = 0$.

**Remark.** KIKUCHI [1] has given the conditions that the Finsler spaces with Randers metric, Kropina metric and generalized Kropina metric are locally Minkowskian. All these metrics have the form $f(\alpha,\beta)$.

**2. Tangent bundles of osculating Riemannain spaces.** A vector field $\xi^i(x)$ in a domain $D$ of the Finsler space $F^n$ is said to be stationary if it satisfies the partial differential equations

$$\xi^i{}_{|\,j} = \partial\xi^i/\partial x^j + F_j{}^i{}_k(x,\xi)\xi^j = 0, \tag{2.1}$$

and the integrability conditions of (2.1) are reduced to the form
$R_0{}^i{}_{hk}(x,y) = 0$ in $D$ [2].

We shall here suppose that our argument is restricted to $\pi^{-1}(D)$ and now consider the particular case in which the Finsler metric is osculating Riemannian. Putting ${}^*g_{ji}(x) = g_{ji}(x,\xi(x))$ and using Theorem B, the equation $R_0{}^i{}_{hk} = 0$ gives us the following theorem.

**Theorem 2.1.** *Let $R_0{}^i{}_{hk} = 0$ be satisfied in a domain $D$ of the Finsler space $F^n$. Then the tangent bundles $T({}^*F^n)$ over the osculating Riemannian spaces ${}^*F^n$ are Kählerian if and only if the osculating Riemannian spaces ${}^*F^n$ are locally flat.*

The author and KIKUCHI [2] gave the following theorem.

**Theorem C.** *The differential equations (2.1) are completely integrable and the osculating Riemannian spaces with its solutions are always locally flat if and only if $R_j{}^i{}_{hk}(x, y) = 0$ is satisfied in $D$.*

By virtue of Theorem 2.1, Theorem A and Theorem C, we obtain

**Theorem 2.2.** *Both the tangent bundle $T(F^n)$ of the Finsler space $F^n$ and the tangent bundles $T({}^*F^n)$ of its osculating Riemannian spaces ${}^*F^n$ are Kählerian if and only if $R_j{}^i{}_{hk}(x, y) = 0$ is satisfied in $D$.*

Next, we are concerning with $\beta$–changes and the stationary vector fileds.

**Lemma 2.** ([5]) *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$. Then the $h$–curvature tensor $\overline{R}_h{}^i{}_{jk}$ of $\overline{F}^n$, obtained from $F^n$ by a $\beta$– change, vanishes if and only if the $h$–curvature tensor $R_h{}^i{}_{jk}$ of $F^n$ vanishes.*

Thus, from Lemma 2 and Theorem 2.2, we have

**Theorem 2.3.** *Let $b_i(x)$ be parallel with respect to the Cartan connection $C\Gamma$ on $F^n$. Then both the tangent bundle $T(\overline{F}^n)$ of the Finsler space $\overline{F}^n$ given by a $\beta$–change and the tangent bundles $T({}^*\overline{F}^n)$ over the osculating Riemannian spaces ${}^*\overline{F}^n$ are Kählerian if and only if both the tangent bundle $T(F^n)$ of the Finsler space $F^n$ and the tangent bundles $T({}^*F^n)$ over the osculating Riemannian spaces ${}^*F^n$ are Kählerian.*

As a relation between the tangent bundle of the associated Riemannian space and the tangent bundles of the osculating Riemannian spaces, from Theorem 1.2 and Theorem 2.1, we have

**Theorem 2.4.** *Let $b_i(x)$ be parallel with respect to the Riemannian connection on Riemannian space $F^n = (M^n, L = \alpha)$. If the tangent bunde $T(F^n)$ of the Riemannian space $F^n$ which is asociated with $\overline{F}^n$ given by a $\beta$–change is Kählerian, then the tangent bundles $T({}^*\overline{F}^n$ over the osculating Riemannian spaces ${}^*\overline{F}^n$ are Kählerian.*

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