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A REMARK ABOUT THE DENSITY OF THE DOMAIN OF $m-$ACCRETIVE OPERATORS

BY

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Abstract. The usual definition of m-accretivity in general Banach spaces does not imply that the m-accretive operators are densely defined with respect to the space norm. In this note the presence of a certain norm with respect to which the m-accretive operators are densely defined is established. The norm is given by means of m-accretive operators. This property of density, however, reveals the extreme topological nature of m-accretivity.

Let's consider X a Banach space. We recall that a m-accretive operator $A : D(A) \subset X \to X$ is a linear operator with

(i) *accretivity*, that is $\|x + \lambda Ax\|_X \geq \|x\|_X$ for all $x \in D(A),\ \lambda > 0$;
(ii) *surjectivity*, that is, the equation $x + \lambda Ax = f$ has solution for all pairs $(\lambda, f) \in \mathbb{R} \times X$.

The space $D(A)$ is endowed subsequently with two characteristic norms:

(a) the *graph norm* $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$ which makes $D(A)$ isomorphic and isometric with $(G(A), \|\cdot\|_{X\times X})$, where $G(A) = \{(x, Ax) : x \in D(A)\}$, via the isometry $G{:}D(A) \to G(A)$ given by $G(x) = (x, Ax)$;
(b) the *isomorphism norm* $|||x|||_{D(A)} = \|x + Ax\|_X$ which makes $D(A)$ isomorphic and isometric with $(X, \|\cdot\|_X)$ via the isometry J_1, where $J_\lambda{:}X \to D(A)$, $\lambda > 0$, is given by $J_\lambda = (I + \lambda A)^{-1} \in L(X)$, $\|J_\lambda\|_{L(X)} \leq 1$.

It follows easily that $|||x|||_{D(A)} \leq \|x\|_{D(A)} \leq 3\,|||x|||_{D(A)}$ for all $x \in D(A)$.

The *Yosida approximations* of A, namely $A_\lambda \in L(X)$, $\lambda > 0$, are given in terms of the next formula

$$A_\lambda x = \frac{x - J_\lambda x}{\lambda} = AJ_\lambda x \text{ for all } x \in X,\ \lambda > 0.$$

The operators $(A_\lambda)_{\lambda>0}$ are all m-accretive, $D(A)-$invariant (*i.e.* $J_\lambda Ax = AJ_\lambda x$ for all $x \in D(A)$) and, if $D(A)$ is assumed dense in X with respect to the space norm $\|.\|_X$, works as below

$$\|A_\lambda x - Ax\|_X \xrightarrow{\lambda\searrow 0} 0 \text{ for all } x \in D(A).$$

Proposition. *If A is m-accretive in some Banach space X, then*

$$|||x||| = \|J_1 x\|_X$$

defines a norm on X and $X \subset \overline{D(A)}^{|||\cdot|||}$.
($\overline{W}^{|||\cdot|||}$ *is the closure of the set W with respect to the norm $|||.|||$*).

Proof. We start by noticing that $J_1 J_\lambda = J_\lambda J_1$, $\lambda > 0$, and denote by X_{-1} the closure of X with respect to the norm $|||.|||$. In order to show that $D(A)$ is dense in X_{-1}, consider $x^* \in X^*_{-1}$ such that $\langle x^*, f\rangle_{X^*_{-1},X_{-1}} = 0$ for all $f \in D(A)$. Herein, by X^*_{-1} we refer to the dual space of X_{-1} endowed with the norm $\|z\|_{X^*_{-1}} = \sup\limits_{|||x|||\le 1} \langle z, x\rangle_{X^*_{-1},X_{-1}}$.

As in [1], we only have to prove that $x^* = 0$. Even more, it is enough to show that

$$\langle x^*, x\rangle_{X^*_{-1},X_{-1}} = 0 \text{ for all } x \in X.$$

For $x \in X$ set $f = J_1 x \in D(A)$. It follows easily that

$$\langle x^*, x\rangle_{X^*_{-1},X_{-1}} = \langle x^*, Af\rangle_{X^*_{-1},X_{-1}} = \langle x^*, AJ_1 x\rangle_{X^*_{-1},X_{-1}}$$

and

$$\begin{aligned} |||AJ_\lambda f - Af||| &= \|AJ_1(J_\lambda f) - AJ_1 f\|_X \le \|J_1(J_\lambda f) - J_1 f\|_{D(A)} \\ &\le 3\,|||J_1(J_\lambda f) - J_1 f|||_{D(A)} = 3\,\|J_\lambda f - f\|_X \le 3\lambda\,\|Af\|_X\,. \end{aligned}$$

Finally,

$$\left|\langle x^*, AJ_\lambda f - Af\rangle_{X^*_{-1},X_{-1}}\right| = \left|\langle x^*, Af\rangle_{X^*_{-1},X_{-1}}\right| \le \|x^*\|_{X^*_{-1}}\,|||AJ_\lambda f - Af|||\,.$$

It ends with $\langle x^*, Af\rangle_{X^*_{-1},X_{-1}} = 0$ since $\lim\limits_{\lambda\searrow 0} |||AJ_\lambda f - Af||| = 0$.

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