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LOCAL SEPARATING POINTS AND ARCWISE CONNECTEDNESS*

BY

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Abstract. E. D. Tymchatyn posed in the Houston Problem Book two problems relating properties of the set of all local separating points to the arcwise connectedness of a continua. We give the negative answers to his problems.

E.D. TYMCHATYN in [1], Problem 68 and Problem 67, p. 378, posed the following problems:

Problem A. *If for each subcontinuum C of the continuum X, the set of all local separating points of C is connected, then is every connected subset of X arcwise connected?*

Problem B. *Suppose X is a continuum such that, if C is a subcontinuum of X then the set of all local separating points of C is not the union of countably many closed disjoint proper subsets of C. Then, is every connected subset of X arcwise connected?*

We give the negative answers to his problems in Corollary A and Corollary B. We start with some definitions. A *continuum* means a nonempty compact connected metric space.

Definition 1. A point p of a locally compact separable metric space L is a *local separating point* of L provided there exists an open set U of

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L containing p and two points x and y of the component containing p of U such that $U \setminus \{p\}$ is the sum of two mutually separated point sets, one containing x and the other containing y.

We recall from [2], p. 137 the definition of the *closed topologist's sine curve* with it's basic properties:

Definition 2. Let S be the graph of $g(x) = \sin(1/x)$ for $0 < x \leq 1$. Let T denote the segment joining $(0, -1)$ and $(0, 1)$. The closed topologist's sine curve $\bar{S}$ is defined by the union $\bar{S} = S \cup T$ considered as a subset of the Euclidean plane with the induced topology.

The closed topologist's sine curve $\bar{S}$ is compact, being closed and bounded, as well connected. Clearly $\bar{S}$ is not locally connected. No arc in $\bar{S}$ can join the point $(0, 0)$ to $(1, \sin(1))$, so neither space $\bar{S}$ is arc connected. We see that $\bar{S}$ has two arcwise connected components, S and T.

We have some simple observations:

Proposition 1. *Any nondegenerate subcontinuum of the closed topologist's sine curve is either homeomorphic to the arc or to the closed topologist's sine curve.*

Proposition 2. *The set of local separating points of an arc is homeomorphic to the open unit interval.*

Proposition 3. *The set of local separating points of a closed topologist's sine curve is homeomorphic to the open unit interval.*

Proof. We observe that the only local separating points of the closed topologist's sine curve $\bar{S} = S \cup T$ (see Definition 2) are exactly the points of $S \setminus \{(1, \sin(1))\}$. This set is homeomorphic to the open unit interval. ∎

Now we give the negative answer to Problem A :

Corollary A. *For each subcontinuum C of the closed topologist's sine curve $\bar{S}$, the set of all local separating points of C is connected, but $\bar{S}$ is not arcwise connected.*

Proof. Due to Proposition 1, Proposition 2 and Proposition 3 the set of all local separating points of any subcontinuum of $\bar{S}$ is homeomorphic to the open unit interval, hence it is connected. ∎

We have one more simple observation:

Proposition 4. *The open unit interval J is not the union of countably many closed disjoint proper subsets of J.*

Proof. Suppose that $J = \cup F_n$ is the union of closed disjoint proper subsets F_n of J. After two steps we have at least one closed interval $[a, b] \subset (J \setminus F_1 \setminus F_2)$, where $\{a, b\} \subset F_1 \cup F_2$. In the open interval (a, b) we soon obtain another interval $[a', b']$. This one divides the interval $[a, b]$ into two pieces $[a, a']$ and $[b', b]$, and so on. Finally we get (similarly to the process of forming the Cantor third set) uncountable set outside the union $\cup F_n$. This proves the impossibility of exhausting the open unit interval by a sequence of disjoint closed proper subsets of J. ∎

Now we give the negative answer to Problem B :

Corollary B. *For each subcontinuum C of the closed topologist's sine curve $\bar{S}$, the set of all local separating points of C is not the union of countably many closed disjoint proper subsets of C, but $\bar{S}$ is not arcwise connected.*

Proof. Due to Propositions 1–3 the set of all local separating points of any subcontinuum C of $\bar{S}$ is homeomorphic to the open unit interval, and due to Proposition 4 it is not the union of countably many closed disjoint proper subsets of C. ∎

REFERENCES

1. COOK, H., INGRAM, W.T., LELEK, A. – *A list of problems known as Houston Problem Book*, in *Continua with the Houston problem book*, H. Cook, W.T. Ingram, K.T. Kuperberg, A. Lelek and P. Minc (eds), Lecture Notes in pure and applied mathematics 170, Marcel Dekker, New York (N.Y.), 1995.
2. STEEN L.A., SEEBACH J.A. Jr. – *Counterexamples in topology*, Springer-Verlag, New-York, 1987.

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