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TOTALLY UMBILICAL DEGENERATE HYPERSURFACES OF R_1^5

BY

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1. Preliminaries. Let $\tilde{M}$ be a real (2n+1)-dimensional differentiable manifold. According to [3], we say that $\tilde{M}$ is an almost contact metric manifold, if there exist a tensor field f of type (1,1), a 1-form η, a vector field ξ and a metric tensor $\tilde{g}$ so that

$$\text{(1.1)}\qquad \begin{array}{l} a)\ \ f^2 = -I + \eta \otimes \xi, \quad b)\ \ \eta(\xi) = 1, \quad c)\ \ \eta \circ f = 0, \\ d)\ \ \eta(X) = \tilde{g}(X, \xi), \quad e)\ \ \tilde{g}(fX, Y) + \tilde{g}(X, fY) = 0, \end{array}$$

for any vector field X, Y tangent to $\tilde{M}$, where I is the identity of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Throughout the paper, we suppose all manifolds to be paracompact and smooth. Denote by $F(\tilde{M})$ the algebra of differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the $F(\tilde{M})$-module of differentiable sections of a vector bundle E over $\tilde{M}$.

It is known that the almost contact metric manifold $\tilde{M}$ is a Sasakian manifold if (see [3] for the case when $\tilde{g}$ is a Riemannian metric tensor and [2] for the case when $\tilde{g}$ is a semi-Riemannian metric tensor)

$$(\tilde{\nabla}_X f)Y = -\eta(Y)X + \tilde{g}(X, Y)\xi, \quad \forall X, Y \in \Gamma(T\tilde{M}),$$

where $\tilde{\nabla}$ is the Levi-Civita connection with respect to the metric tensor $\tilde{g}$.

In this paper we assume that the $\tilde{g}$ on the considered Sasakian manifold is a semi–Riemannian metric tensor.

Let M be a hypersurface of a Sasakian manifold $\tilde{M}$. Let T_xM and $T_xM^\perp$ be the tangent and the normal space of M at $x \in M$, respectively. Then we have

$$T_xM^\perp = \{X_x \in T_x\tilde{M} \mid \tilde{g}_x(X_x, Y_x) = 0, \quad \forall\ Y_x \in T_xM\}.$$

Denote also by TM and $TM^{\perp}$ the tangent and the normal space to M, respectively.

Denote by g the induced tensor metric on M from $\tilde{g}$. We say that the hypersurface M is degenerate if the normal bundle $TM^{\perp}$ becomes a distribution of rank 1 on M, (see [1]) or equivalently $det[g] = 0$. Denote by N the vector field which spans $TM^{\perp}$, that is $TM^{\perp} = span\{N\}$, and $U = -fN$. Because M is a degenerate hypersurface of $\tilde{M}$, we have

$$\tilde{g}(U,U) = \tilde{g}(N,N) = 0. \tag{1.2}$$

A *screen distribution* (see [5]) SM on M is a complementary orthogonal distribution of $TM^{\perp}$ in TM, that is $TM = SM \perp TM^{\perp}$, where $\perp$ between vector bundles means orthogonal sum. From [1] we recall

Theorem 1.1. *Let M be a degenerate hypersurface of a Sasakian manifold $\tilde{M}$, and let SM be a screen distribution on M. Then there exist a unique vector bundle $tr(TM)$ of rank* 1 *over M, such that, for any non-zero section N of $TM^{\perp}$ on a coordinate neighbordhood of M, there exists a unique local section V of $tr(TM)$ satisfying*

$$(a)\ \tilde{g}(N,V) = 1,\ \ (b)\ \tilde{g}(V,V) = 0,\ \ (c)\ \tilde{g}(V,X) = 0, \quad \forall X \in \Gamma(SM). \tag{1.3}$$

Denote $W = -fV$ and according to [5] we can take the screen distribution SM such that $W, V \in \Gamma(SM)$.

Denote by ∇ the torsion-free connection induced from $\tilde{\nabla}$ on M. It is known that in general ∇ is not a metric connection (see [5]). The Gauss and Weingarten formulae for the degenerate hypersurface M of $\tilde{M}$ are

$$\begin{array}{ll} (a) & \tilde{\nabla}_X Y = \nabla_X Y + B(X,Y)V, \\ (b) & \tilde{\nabla}_X V = -A_V X + a(X)V, \quad \forall X, Y \in \Gamma(TM), \quad V \in tr(TM), \end{array} \tag{1.4}$$

respectively (see [5]), where B is a $F(M)$-valued bilinear form and $a(X) = \tilde{g}(\tilde{\nabla}_X V, N)$. Because $N \in \Gamma(TM)$, using the second relation of (1.2), (1.3) (a) and (1.4) (a) we get $B(X,N) = 0,\ \forall X \in \Gamma(TM)$. Denote by u the projection morphism of TM to SM. The Gauss and Weingarten formulae for the screen distribution SM of M are (see [1], [5])

$$\begin{array}{ll} (a) & \nabla_X uY = \nabla^{\star}_X uY + C(X,uY)N, \\ (b) & \nabla_X N = -A^{\star}_N X - a(X)N, \ \ \forall X, Y \in \Gamma(TM), \end{array} \tag{1.5}$$

where $\nabla^{\star}_X uY,\ A^{\star}_N X \in \Gamma(SM), \quad C(X,uY) \in F(M), \quad a(X) = g(\nabla_X V, N)$. It is easy to see that C is a bilinear form (see [2]). Next we suppose that the structure vector field ξ is tangent to M. Now it is easy to see that $\xi \in \Gamma(SM)$ (see [4]). According to [4] we say that the hypersurface M is totally contact umbilical degenerate hypersurface of $\tilde{M}$ if and only if

$$A_N^{\star}uX = \lambda(uX - \eta(X)\xi) - \eta(X)U - \omega(X)\xi, \quad \forall X \in \Gamma(TM), \tag{1.6}$$

where $\omega(X) = g(X, U), \quad X \in \Gamma(TM)$.

From [4] we recall

Proposition 1.1. *Let M be a degenerate hypersurface of a Sasakian manifold $\tilde{M}$. Then we have*

$$\begin{array}{l} B(X, uY) = g(A_N^{\star}X, uY), \quad B(X,\xi) = -\omega(X), \ \forall X, Y \in \Gamma(TM), \\ A_N^{\star}\xi = -U. \end{array} \tag{1.7}$$

Next we consider the Sasakian manifold R_s^{2n+1} (see [2]) and make the following notations: $0_{k,h}$ is the $k \times h$ null matrix, I_k is the $k \times k$ unit matrix and

$$\epsilon^{n+a} = \epsilon^a = \begin{cases} -1 & \text{for } a \in \{1, 2, \ldots, s\} \\ 1 & \text{for } a \in \{s+1, \ldots, n\} \end{cases}.$$

Then we take (x^i, y^i, z), $i = 1, ..., n$, as cartesian coordinates on R_s^{2n+1} and with respect to the natural field of frames $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z}\right\}$, $i = 1, ..., n$, define the tensor field f of type $(1,1)$ by its matrix

$$[f] = \begin{bmatrix} O_{n,n} & I_n & O_{n,1} \\ -I_n & O_{n,n} & O_{n,1} \\ O_{1,n} & \epsilon^i y^i & 0 \end{bmatrix}.$$

The structure vector field ξ and the 1-form η are defined by

$$\xi = 2\frac{\partial}{\partial z},$$

and

$$\eta = \frac{1}{2}\left\{dz - \sum_{i=1}^{n} \epsilon^i y^i dx^i\right\},$$

respectively. Finally the semi–Riemannian metric $\tilde{g}$ is defined by the matrix

$$(1.8)\qquad [\tilde{g}]=\frac{1}{4}\begin{bmatrix} -\delta_{ab}+y^a y^b & -y^a y^{b^*} & O_{s,s} & O_{s,n-s} & y^a \\ -y^a y^{b^*} & \delta_{a^*b^*}+y^{a^*} y^{b^*} & O_{n-s,s} & O_{n-s,n-s} & -y^{a^*} \\ O_{s,s} & O_{s,n-s} & -I_s & O_{s,n-s} & O_{s,1} \\ O_{n-s,s} & O_{n-s,n-s} & O_{n-s,s} & I_{n-s} & O_{n-s,1} \\ y^a & -y^{a^*} & O_{1,s} & O_{1,n-s} & 1 \end{bmatrix}$$

where $a,b\in\{1,...s\}$, and $a^*,b^*\in\{s+1,...,n\}$. Then $(f,\xi,\eta,\tilde{g})$ defines a Sasakian structure on R^{2n+1}_s (see [1]). For the sake of simplicity next we use the following notations: $y^i = x^{n+i}$, $x^{2n+1} = z$, $1 \leq i \leq n$. Now let M be a hypersurface isometrically immersed in R^{2n+1}_s, locally given by the equations

$$x^A = f^A(u^1,u^2,\dots u^{2n}),\ \ \text{rank}\left[\frac{\partial f^A}{\partial u^\alpha}\right] = 2n;\ A\in\{1,\dots,2n+1\},$$
$$\alpha\in\{1,\dots,2n\},$$

where f^A are smooth functions on a domain $D \subset R^{2n}$. Denote by $D_A = D^A$ the 2n x 2n determinant obtained from $\left[\frac{\partial f^i}{\partial u^\alpha}\right]$, $1 \leq i \leq 2n+1$, $1 \leq \alpha \leq 2n$ by deleting the A^{th} column. Then from [4] we recall

Theorem 1.2. *Let M be a hypersurface of the Sasakian manifold R^{2n+1}_s. Then M is a degenerate hypersurface tangent to the structure vector field ξ iff*

$$D^{2n+1} = 0 \quad \textit{and} \quad \sum_{\alpha=1}^{2n}\epsilon^\alpha (D^\alpha)^2 = 0.$$

2. Totally contact umbilical degenerate hypersurface of R^5_1. The purpose of this section is to determinate the necessary and suficient conditions for the existence of a totally contact umbilical degenerate hypersurface of R^5_1. Let M be a degenerate hypersurface of R^5_1 given by

$$(2.1)\qquad x^i = f^i(u^1,u^2,u^3,u^4),\ \ 1 \leq i \leq 5,$$

Because the hypersurface M is degenerate and tangent to the structure vector field ξ, from Theorem 1.2 we have

$$(2.2)\qquad D_5 = 0, (D_1)^2 + (D_3)^2 = (D_2)^2 + (D_4)^2 (= b \neq 0)$$

By using (1.2) and (1.3), we deduce

$$(2.3)\quad N = -D_1\frac{\partial}{\partial x^1} - D_2\frac{\partial}{\partial x^2} - D_3\frac{\partial}{\partial x^3} - D_4\frac{\partial}{\partial x^4} + (y^1 D_1 - y^2 D_2)\frac{\partial}{\partial x^5},$$

and using (1.3) we obtain

$$V=\frac{2}{b}\left(D_1\frac{\partial}{\partial x^1}-D_2\frac{\partial}{\partial x^2}+D_3\frac{\partial}{\partial x^3}-D_4\frac{\partial}{\partial x^4}-(y^1D_1+y^2D_2)\frac{\partial}{\partial x^5}\right), \tag{2.4}$$

By direct calculation we get

$$\begin{cases} U=D_3\dfrac{\partial}{\partial x^1}+D_4\dfrac{\partial}{\partial x^2}-D_1\dfrac{\partial}{\partial x^3}-D_2\dfrac{\partial}{\partial x^4}+(y^2D_4-y^1D_3)\dfrac{\partial}{\partial x^5}, \\ W=\dfrac{2}{b}\left(-D_3\dfrac{\partial}{\partial x^1}+D_4\dfrac{\partial}{\partial x^2}+D_1\dfrac{\partial}{\partial x^3}-D_2\dfrac{\partial}{\partial x^4}+(y^1D_3+y^2D_4)\dfrac{\partial}{\partial x^5}\right). \end{cases} \tag{2.5}$$

Then it is easy to see that the screen distribution $S(M)=span\{U,W,\xi\}$ and the tangent bundle $TM=span\{U,W,N,\xi\}$. By direct calculation we deduce that U,W,N,ξ are linearly independent and thus these vector fields can form a field of frame of TM. We shall calculate the second fundamental form B with respect to the field of frames $\{U,W,N,\xi\}$ on M. First by direct calculation, using (1.2), (1.4) (a) and the first relation of (1.3), we deduce

$$B(W,\xi)=-1,\quad B(X,N)=B(U,\xi)=B(\xi,\xi)=0,\quad \forall X\in\Gamma(TM), \tag{2.6}$$

and therefore we only have to calculate $B(W,W),B(U,U),B(U,W)$. Next, by using well known formulae for Christoffel symbols we state:

Lemma 2.1. *Let R^5_1 be the Sasakian manifold endowed with the tensor metric $\tilde g$ given by* (1.8) *for* n=2. *Then the Christoffel symbols with respect to $\tilde g$ are given by*:

$$\begin{aligned} &\tilde\Gamma^1_{13}=\tilde\Gamma^2_{14}=-\tilde\Gamma^4_{12}=-\tilde\Gamma^5_{35}=-\frac{1}{2}y^1,\ \tilde\Gamma^1_{23}=\tilde\Gamma^2_{24}=-\tilde\Gamma^3_{12}=-\tilde\Gamma^5_{45}=\frac{1}{2}y^2,\\ &\tilde\Gamma^1_{35}=\tilde\Gamma^2_{45}=-\tilde\Gamma^3_{15}=-\tilde\Gamma^4_{25}=-\frac{1}{2},\\ &\tilde\Gamma^5_{14}=\tilde\Gamma^5_{23}=-\frac{1}{2}y^1y^2,\quad \tilde\Gamma^3_{11}=y^1,\ \tilde\Gamma^4_{22}=-y^2,\\ &\tilde\Gamma^5_{13}=\frac{1}{2}(1+(y^1)^2),\quad \tilde\Gamma^5_{24}=\frac{1}{2}((y^2)^2-1), \end{aligned} \tag{2.7}$$

and $\tilde\Gamma^i_{jk}=0$ for any other triple (i,j,k).

By using Lemma 2.1 we infer

Lemma 2.2. *Let R^5_1 be the Sasakian manifold endowed with the tensor metric $\tilde g$ given by* (1.8). *Then we have*

$$\tilde\nabla_U\frac{\partial}{\partial x^1}=-\frac{y^1}{2}N+\frac{D_1}{4}\xi,\quad \tilde\nabla_U\frac{\partial}{\partial x^2}=-\frac{y^2}{2}N-\frac{D_2}{4}\xi,\quad \tilde\nabla_U\frac{\partial}{\partial x^3}=-\frac{D_3}{4}\xi,$$
$$\tilde\nabla_U\frac{\partial}{\partial x^4}=\frac{D_4}{4}\xi,\quad \tilde\nabla_U\frac{\partial}{\partial x^5}=-\frac{1}{2}N,\quad \tilde\nabla_W\frac{\partial}{\partial x^1}=-\frac{y^1}{2}V-\frac{D_1}{2b}\xi,\quad \tilde\nabla_W\frac{\partial}{\partial x^2}=$$

$$= \frac{y^2}{2}V - \frac{D_2}{2b}\xi, \quad \tilde{\nabla}_W \frac{\partial}{\partial x^3} = \frac{D_3}{2b}\xi, \quad \tilde{\nabla}_W \frac{\partial}{\partial x^4} = \frac{D_4}{2b}\xi, \quad \tilde{\nabla}_W \frac{\partial}{\partial x^5} = -\frac{1}{2}V,$$

$$\tilde{\nabla}_N \frac{\partial}{\partial x^1} = \frac{1}{2}y^1U + \frac{1}{2}D_3\xi; \quad \tilde{\nabla}_N \frac{\partial}{\partial x^2} = -\frac{1}{2}y^2U - \frac{1}{2}D_4\xi;$$

$$\tilde{\nabla}_N \frac{\partial}{\partial x^3} = \frac{1}{2}D_1\xi; \quad \tilde{\nabla}_N \frac{\partial}{\partial x^4} = -\frac{1}{2}D_3\xi; \quad \tilde{\nabla}_N \frac{\partial}{\partial x^5} = \frac{1}{2}U.$$

From Lemmas 2.1 and 2.2 we deduce

Lemma 2.3. *Let M be the hypersurface given by* (2.1). *Then we have*

$$(2.8)\begin{cases}
(a)\ \tilde{\nabla}_U U = -U(D_3)\frac{\partial}{\partial x^1} - U(D_4)\frac{\partial}{\partial x^2} + U(D_1)\frac{\partial}{\partial x^3} + \\
\qquad + U(D_2)\frac{\partial}{\partial x^4} + \frac{U(y^1D_3 - y^2D_4) + D_2D_4 - D_1D_3}{2}\xi, \\
(b)\ \tilde{\nabla}_U W = 2U\left(\frac{D_3}{b}\right)\frac{\partial}{\partial x^1} - 2U\left(\frac{D_4}{b}\right)\frac{\partial}{\partial x^2} - 2U\left(\frac{D_1}{b}\right)\frac{\partial}{\partial x^3} - \\
\qquad - 2U\left(\frac{D_2}{b}\right)\frac{\partial}{\partial x^4} - \left(U\left(\frac{y^1D_3 + y^2D_4}{b}\right) + \frac{D_2D_4 - D_1D_3}{b}\right)\xi, \\
(c)\ \tilde{\nabla}_W U = -W(D_3)\frac{\partial}{\partial x^1} - W(D_4)\frac{\partial}{\partial x^2} + W(D_1)\frac{\partial}{\partial x^3} \\
\qquad + W(D_2)\frac{\partial}{\partial x^4} + \frac{W(y^1D_3 - y^2D_4) + D_2D_4 + D_1D_3}{2}\xi, \\
(d)\ \tilde{\nabla}_W W = 2W\left(\frac{D_3}{b}\right)\frac{\partial}{\partial x^1} - 2W\left(\frac{D_4}{b}\right)\frac{\partial}{\partial x^2} - 2W\left(\frac{D_1}{b}\right)\frac{\partial}{\partial x^3} + \\
\qquad + 2W\left(\frac{D_2}{b}\right)\frac{\partial}{\partial x^4} + \left(U\left(\frac{y^1D_3 + y^2D_4}{b}\right) + \frac{D_2D_4 - D_1D_3}{b^2}\right)\xi, \\
(e)\ \tilde{\nabla}_U N = U(D_1)\frac{\partial}{\partial x^1} + U(D_2)\frac{\partial}{\partial x^2} + U(D_3)\frac{\partial}{\partial x^3} + U(D_4)\frac{\partial}{\partial x^4} \\
\qquad + \frac{U(y^2D_2 - y^1D_1) + (D_1)^2 - (D_2)^2}{2}\xi, \\
(f)\ \tilde{\nabla}_W N = W(D_1)\frac{\partial}{\partial x^1} + W(D_2)\frac{\partial}{\partial x^2} + W(D_3)\frac{\partial}{\partial x^3} + \\
\qquad + W(D_4)\frac{\partial}{\partial x^4} + \left(\frac{W(y^2D_2 - y^1D_1)}{2} + \frac{(D_3)^2 - (D_2)^2}{2b}\right)\xi, \\
(g)\ \tilde{\nabla}_N N = N(D_1)\frac{\partial}{\partial x^1} + N(D_2)\frac{\partial}{\partial x^2} + N(D_3)\frac{\partial}{\partial x^3} + \\
\qquad + N(D_4)\frac{\partial}{\partial x^4} + \frac{N(y^2D_2 - y^1D_1) + D_1D_3 - D_2D_4}{2}\xi.
\end{cases}$$

By using Lemma 2.3 and (1.4) (a) we deduce

Lemma 2.4. *Let M be a degenerate hypersurface of the Sasakian manifold R_1^5. Then we have*

$$(2.9)\quad \begin{aligned} &(a)\ B(U,U)=\tilde{g}(\tilde{\nabla}_U U,N)=\frac{1}{4}\left((D_1)^2 U\left(\frac{D_3}{D_1}\right)-(D_2)^2 U\left(\frac{D_4}{D_2}\right)\right),\\ &(b)\ B(U,W)=\tilde{g}(\tilde{\nabla}_U W,N)=\frac{1}{4}\left((D_1)^2 W\left(\frac{D_3}{D_1}\right)-(D_2)^2 W\left(\frac{D_4}{D_2}\right)\right),\\ &(c)\ B(W,W)=\tilde{g}(\tilde{\nabla}_W W,N)=-\frac{1}{4}\left((D_1)^2 W\left(\frac{D_3}{D_1}\right)+(D_2)^2 W\left(\frac{D_4}{D_2}\right)\right).\end{aligned}$$

Because M is a totally contact umbilical degenerate hypersurface of R_1^5, according to (1.6) we must find $\lambda \in F(M)$ so that

$$(2.10)\qquad A_N^* U = \lambda U; \quad A_N^* W = \lambda W - \xi,$$

because $A_N^*\xi = -U$. By using (1.5) (b) and (2.10), we get

$$(2.11)\qquad \nabla_U N = -\lambda U - a(U)N; \quad \nabla_W N = -\lambda W - a(W)N + \xi.$$

By direct calculation we infer

$$(2.12)\qquad \lambda = -\tilde{g}(\tilde{\nabla}_U N, W) = \tilde{g}(\tilde{\nabla}_U W, N) = B(U,W).$$

From (1.2), (1.7) and (2.11) we deduce

$$(2.13)\qquad g(A_N^* U - \lambda U, U) = B(U,U),$$

$$(2.14)\qquad g(A_N^* U - \lambda U, W) = B(U,W) - \lambda = 0,$$

$$(2.15)\qquad g(A_N^* U - \lambda U, \xi) = \eta(A_N^* U) = 0,$$

$$(2.16)\qquad g(A_N^* W - \lambda W + \xi, \xi) = 1 + \eta(A_N^* W) = 1 - 1 = 0,$$

$$(2.17)\qquad g(A_N^* W - \lambda W + \xi, U) = g(A_N^* W, U) - \lambda = B(U,W) - \lambda = 0,$$

$$(2.18)\qquad g(A_N^* W - \lambda W + \xi, W) = g(A_N^* W, W) = B(W,W).$$

Next from (2.10)-(2.18) we have

Theorem 2.1. *The hypersurface M of the Sasakian manifold R_1^5, given by* (2.1) *is totally contact umbilical iff U and W are asymptotically directions, that is $B(U,U) = B(W,W) = 0$.* (see [6]).

With the help of Lemma 2.4 and Theorem 2.1 we obtain

Proposition 2.1. *The hypersurface M of the Sasakian manifold R_1^5 defined by* (2.1) *is totally contact umbilical degenerate hypersurface iff*

$$
\begin{aligned}
(D_1)^2 U\left(\frac{D_3}{D_1}\right) &= (D_2)^2 U\left(\frac{D_4}{D_2}\right), \\
(D_1)^2 W\left(\frac{D_3}{D_1}\right) &+ (D_2)^2 W\left(\frac{D_4}{D_2}\right) = 0.
\end{aligned}
\tag{2.19}
$$

Next we construct an example of degenerate totally contact umbilical hypersurface of R_1^5.

Let M be the hypersurface of R_1^5 given by

$$x^3 = F(x^1, x^2, x^4) \tag{2.20}$$

with F of class C^q, $q \geq 2$. By direct calculation we have

$$D_1 = -F_1, \quad D_2 = F_2, \quad D_3 = 1, \quad D_4 = F_4, \quad D_5 = 0,$$

where we have denoted by $F_i = \dfrac{\partial F}{\partial x^i}$, i = 1, 2 ,4. We shall also denote by $F_{ij} = \dfrac{\partial^2 F}{\partial x^i x^j}$, $i, j = 1,\ 2,\ 4$.

Then it is easy to see that using (2.5)

$$
\begin{aligned}
U &= \frac{\partial}{\partial x^1} + F_4 \frac{\partial}{\partial x^2} + F_1 \frac{\partial}{\partial x^3} - F_2 \frac{\partial}{\partial x^4} + \frac{y^2 F_4 - y^1}{2}\xi, \\
W &= \frac{2}{1 + F_1^2}\left(-\frac{\partial}{\partial x^1} + F_4 \frac{\partial}{\partial x^2} - F_1 \frac{\partial}{\partial x^3} - F_2 \frac{\partial}{\partial x^4} + \frac{y^1 + y^2 F_4}{2}\xi\right).
\end{aligned}
$$

Then we infer that

$$
\begin{aligned}
(D_1)^2 U\left(\frac{D_3}{D_1}\right) - (D_2)^2 U\left(\frac{D_4}{D_2}\right) &= F_{11} - 2F_2F_4F_{24} + (F_4)^2 F_{22} \\
&+ (F_2)^2 F_{44} + 2(F_4F_{12} - F_2F_{14}),
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
(D_1)^2 W\left(\frac{D_3}{D_1}\right) + (D_2)^2 W\left(\frac{D_4}{D_2}\right) &= -F_{11} + 2F_2F_4F_{24} - (F_4)^2 F_{22} \\
&-(F_2)^2 F_{44} + 2(F_4F_{12} - F_2F_{14}),
\end{aligned}
\tag{2.22}
$$

Taking into account that M is degenerate totally contact umbilical hypersurface of R_1^5, then from Proposition 2.2, the relations (2.21), (2.22) and the second relation of (2.2), we obtain

$$(F_1)^2 + 1 = (F_2)^2 + (F_4)^2, \tag{2.23}$$

$$F_{12}F_4 = F_{14}F_2, \tag{2.24}$$

$$F_{11} + F_{22}(F_4)^2 + F_{44}(F_2)^2 - 2F_2F_4F_{24} = 0. \tag{2.25}$$

Next we want to determine all functions F so that (2.23)–(2.25) are verified. Due to (2.23) there exists the functions u and v defined on a domain $D \subseteq R^3$ so that

$$F_1 = sh\ v, \quad F_2 = cos\ u\ ch\ v, \quad F_4 = sin\ u\ ch\ v. \tag{2.26}$$

Because $F_{ij} = F_{ji}$, using (2.25), we deduce

$$\frac{\partial u}{\partial x^1} = cos\ u\frac{\partial v}{\partial x^4} - sin\ u\frac{\partial v}{\partial x^2}, \tag{2.27}$$

$$tanh\ v\frac{\partial v}{\partial x^1} = cos\ u\frac{\partial v}{\partial x^2} + sin\ u\frac{\partial v}{\partial x^4}, \tag{2.28}$$

$$tanh\ v\frac{\partial u}{\partial x^1} = cos\ u\frac{\partial u}{\partial x^2} + sin\ u\frac{\partial u}{\partial x^4}, \tag{2.29}$$

From (2.24), (2.27)–(2.29) we obtain

$$cos\ u\frac{\partial v}{\partial x^4} - sin\ u\frac{\partial v}{\partial x^2} = 0. \tag{2.30}$$

Further, from (2.27) and (2.30) we deduce $\dfrac{\partial u}{\partial x^1} = 0$. Next from (2.29) we infer

$$cos\ u\frac{\partial u}{\partial x^2} + sin\ u\frac{\partial u}{\partial x^4} = 0,$$

from which we get that the function u satisfies a relation of the form

$$x^2 sin\ u - x^4 cos\ u = h(u), \tag{2.31}$$

where h is a function of class C^q, $q \geq 2$. By using (2.28) we deduce

$$v = arctanh\left(\frac{x^1}{sinu}\frac{\partial u}{\partial x^2} + C\right), \tag{2.32}$$

with $C \in R$ and u is a smooth function of variable x^2 and x^4. Hence we may state

Proposition 2.2. *The hypersuraface M given by* (2.20) *is a degenerate hypersurface of R_1^5 if and only if the relation* (2.6) *holds. Moreover, if the functions u and v from* (2.6) *satisfy the relations* (2.31) *and* (2.32) *then M is totally contact umbilical degenerate hypersurface of R_1^5.*

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