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ON SOME FIXED POINT THEOREMS FOR CONTRACTIVE MAPPING IN D-METRIC SPACES

BY

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Abstract. In this paper some fixed point theorems for contractive mappings in a D-metric space are proved which improve the recent fixed point theorems of RHOADES [6] under weaker conditions with correct proof.

1. Introduction. Recently, DHAGE [1], [2] introduced the notion of a D-metric space which is as follows. A D-metric D on a nonempty set X is a function $D : X \times X \times X \to R$ satisfying the following properties:

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, (non negativity)
(ii) $D(x, y, z) = 0$ if and only if $x = y = z$, (coincidence)
(iii) $D(x, y, z) = D(x, z, y) = ...$, (symmetry)
(iv) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$
for $x, y, z, a \in X$ (tetrahedral inequality)

The nonempty set X together with a D-metric D is called a *D-metric space* which is completely different structure from 2-*metric* spaces and is denoted by (X, D). A number of fixed point theorems have been proved in 2-metric spaces; however, HSIAO [5] showed that all such theorems are trivial in the sense that all iterations of mappings under consideration are collinear. The situation for a D-metric is quite different. The details of a D-metric space along with some specific examples are given in DHAGE [1], [2]. In a recent paper RHOADES [6] proved two fixed point theorems in a D-metric space claiming the inclusion of the fixed point theorems of DHAGE [2] along with some other limitations. But we observe that the proof of Theorem 1 and statement of Theorem 2 of RHOADES [6] along with the claim made therein concerning the limittions is not correct. So the purpose

of this paper is to improve the fixed point theorems of RHOADES [6] under weaker conditions with simple and correct proof.

2. Preliminaries. Throughout this paper, let X denote a D-metric space with D-metric D. Let $f : X \to X$, then by an *orbit* of f at a point $x \in X$ is a set $\theta(x)$ in X given by

$$\theta(x) = \{x, fx, f^2x, ...\} \tag{2.1}$$

The orbit $\theta(x)$ is said to be *bounded* if there exists a constant $k > 0$ such that $D(u, v, w) \leq k$ for all $u, v, w \in \theta(x)$ and the constant k is called the D-bound for $\theta(x)$. The D-metric space X is said to be *f-orbitally bounded* if $\theta(x)$ is bounded for each $x \in X$. Again the D-metric spaces X is called *f–orbitally complete* if every D-Cauchy sequence $\{x_n\} \subseteq \theta(x)$, $x \in X$, converges to a point in X. Finally $f : X \to X$ is called *f–orbitally continuous* if for any sequence $\{x_0\} \subseteq \theta(x)$, $x \in X$, $x_n \to x^*$ implies $fx_n \to fx^*$.

We need the following useful results of DHAGE [4] in the sequel.

Lemma 2.1. *Let $\{x_n\}$ be a bounded sequence in a D-metric space X satisfying, for some $0 \leq \alpha < 1$,*

$$D(x_n, x_{n+1}, x_m) \leq \alpha^n K \tag{2.2}$$

for all $m > n \in N$, where k is a D-bound of $\{x_n\}$. Then $\{x_n\}$ is D-Cauchy.

Proof. The proof is given in DHAGE [4]. But for the sake of completeness, we give the details of proof. From (2.2), it follows that

$$\begin{aligned} &D(x_n, x_{n+1}, x_{n+p}) \leq \alpha^n K \text{ and} \\ &D(x_n, x_{n+1}, x_{n+p+t}) \leq \alpha^n K \end{aligned} \tag{2.3}$$

for all $p, t \in N$. Now by repeated application of tetrahedral inequality, we obtain

$$\begin{aligned} &D(x_n, x_{n+1}, x_{n+p+t}) \\ &\quad \leq D(x_n, x_{n+1}, x_{n+p}) + D(x_n, x_{n+1}, x_{n+p+t}) + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\quad \leq D(x_n, x_{n+1}, x_{n+p}) + D(x_n, x_{n+1}, x_{n+p+t}) + D(x_{n+1}, x_{n+2}, x_{n+p}) \\ &\quad + D(x_{n+1}, x_{n+2}, x_{n+p+t}) + ... \\ &\quad \leq 2(\alpha^n + \alpha^{n+1} + ... + \alpha^{n+p+t})K \\ &\quad = \frac{2\alpha^n}{1-\alpha} \to 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a D-Cauchy sequence in X and the proof of the lemma is complete.

As a direct consequence of Lemma 2.1, we obtain

Lemma 2.2. *Let X be a f-orbitally bounded D-metric space satisfying, for some $0 \leq \alpha < 1$,*

$$D(f^n x, f^{n+1}x, f^m x) \leq \alpha^n K \tag{2.4}$$

for all $m > n \in N$, where K is a D-bound of $\theta(x)$, $x \in X$. Then $\{f^n x\}$ is a D-Cauchy for each $x \in X$.

3. Main results.

Theorem 3.1. *Let f be a self map of a f-orbitally bounded and f-orbitally complete D-metric space X satisfying, for some $0 \leq q < 1$,*

$$\begin{aligned} &D(fx, fy, fz) \\ &\leq q \max\{D(x,y,z), D(x,fx,z), D(y,fy,z), D(x,fy,z), D(y,fx,z)\} \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$. Then f has a unique fixed point p and f is continuous at p.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{x_n\}$ in X by

$$x_0 = x, \ x_{n+1} = fx_n, \ n \geq 0. \tag{3.2}$$

If $x_r = x_{r+1}$ for some $r \in N$, then f has a fixed point ($p = x_r$ is a fixed point of f); otherwise we assume that $x_n \neq x_{n+1}$ for each $n \in N$. We show that $\{x_n\}$ is D-Cauchy.

Now for $m > 2$, by (3.1), we obtain,

$$\begin{aligned} &D(x_1, x_2, x_m) \\ &\quad = D(fx_0, fx_1, fx_m) \\ &\quad \leq q \max\{D(x_0, x_1, x_{m+1}), D(x_0, x_1, x_{m-1}), D(x_1, x_2, x_{m-1}), \\ &\quad D(x_0, x_2, x_{m-1}), D(x_1, x_1, x_{m-1})\} \\ &\quad \leq q \max_{\substack{0 \leq a \leq 1 \\ 1 \leq b \leq 2}} D(x_a, x_b, x_{m-1}) \leq qk, \end{aligned}$$

where k is a D-bound of $\theta(x)$. Similarly, for $m > 3$,

$$D(x_2, x_3, x_m) \leq q^2 \max_{\substack{0 \leq a \leq 2 \\ 1 \leq b \leq 3}} D(x_a, x_b, x_{m-2}) \leq q^2 k$$

In general, for $m > n$,

$$D(x_n, x_{n+1}, x_m) \leq q^n \max_{\substack{0\leq a\leq n\\ 1\leq b\leq n+1}} D(x_a, x_b, x_{m-n}) \leq q^n k$$

Now an application of Lemma 2.2 yields that $\{x_n\}$ is D-Cauchy. Since X is f-orbitally complete, $\{x_n\}$ converges, say to a point p in X. Then proceeding as in the proof of Theorem 1 of RHOADES [6], it is shown that p is a unique fixed point of f.

To show f is continuous at p, let $\{y_n\}\subseteq X$ with $y_n \to p$. Then from (3.1), it follows that

$$D(fy_m, fy_n, fp) \leq q \max\{D(y_m, y_n, p), D(y_m, fy_m, p), D(y_n, fy_n, p), D(y_m, fy_n, p), D(y_n, fy_m, p)\}$$

and so

$$\lim_{m,n} D(fy_m, fy_n, fp) \leq q \max\left\{\lim_m D(p, fy_m, p), \lim_n D(p, fy_n, p)\right\} \tag{3.3}$$

But

$$\begin{aligned} D(p, fy_n, p) &= D(fp, fy_m, fp) \\ &\leq q \max\{D(p, y_m, p), D(p, fy_m, y_m), D(p, fy_m, p)\} \end{aligned}$$

and so

$$\lim_n D(p, fy_m, p) \leq q \lim_m D(p, fy_m, p)$$

As $q < 1$, $\lim_m D(p, fy_m, p) = 0$. Similarly

$$\lim_n D(p, fy_n, p) = 0.$$

From (3.3), it follows that

$$\lim_{m,n} D(fy_m, fy_n, fp) = 0$$

which implies that $fy_n \to fp$, and hence f is continuous at p (see DHAGE [3]). This completes the proof.

Corollary 3.1. *Let f be a self–map of a f-orbitally bounded and f-orbitally complete D-metric space X, m a positive integer satisfying, for some $0 \leq q < 1$*

$$D(f^m x, f^m y, f^m z) \leq q \max\{D(x, y, z), D(x, f^m x, z), D(y, f^m y, z), D(x, f^m y, z), D(y, f^m x, z)\} \tag{3.4}$$

for all $x, y, z \in X$. Then f has a unique fixed point p, f^m is continuous at p and f is f-orbitally continuous at p.

Theorem 3.2. *Lt f be a self-map or a f-orbitally bounded and f-orbitally complete D-metric space X, satisfying, for some $0 \leq q < 1$*

$$[D(fx, fy, fz)]^2 \leq q \max\{[D(x, y, z)]^2, D(x, fx, z), D(y, fy, z), D(x, fy, z), D(y, fx, z)\} \tag{3.5}$$

for all $x, y, z \in X$. Then f has a unique fixed point p and f is continuous at p.

Proof. Since $ab \leq \max\{a^2, b^2\}$ for $a \geq 0$, $b \geq 0$, the inequality (3.5) reduces to (3.1) and hence the desired conclusion follows by an application of Theorem 3.1.

Remark 3.1. the conclusion of Corollary 3.1 also remains true if the condition (3.4) replaced by

$$[D(f^m x, f^m y, f^m z)]^2 \leq q \max\{[D(x, y, z)]^2, D(x, f^m x, z), D(y, f^m y, z), D(x, f^m y, z), D(y, f^m x, z)\} \tag{3.6}$$

for all $x, y, z \in X$.

Remark 3.2. It is known that f-orbitally boundedness is weaker condition than boundedness of a metric space, and it is shown in DHAGE [4] that almost all fixed point theorems for contraction mappings in ordinary metric spaces involve this condition, though it is not specifically mentioned in the statement of such theorems. Therefore, Theorem 3.1 does not involve any extra unusual condition on a D-metric space in order to guarantee the existence of fixed point which is against the claim made in RHOADES [6].

Next we prove a fixed point theorem in tri-D-metric space which again includes Theorem of DHAGE [2] as the special case under weaker conditions.

Theorem 3.3. *Let X be a D-metric space with three D-metrics D, D_1 and D_2. Assume that the following conditions hold in X.*

(i) *$D_2(x, y, z) \leq D_1(x, y, z) \leq D(x, y, z)$ for all $x, y, z \in X$*
(ii) *X is f-orbitally bounded w.r.t. D_1,*
(iii) *X is f-orbitally complete w.r.t. D_1,*
(iv) *f is f-orbitally continuous w.r.t. D_2, and*
(v) *f satisfies (3.1) on X w.r.t. D.*

Then f has a unique fixed point p in X and f is continuous at p w.r.t. D.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{x_n\}$ is X by (3.2). Then proceeding as in the proof of Theorem 3.1 with similar arguments, it is shown that $\{x_n\}$ is D-Cauchy w.r.t. D. In view hypothesis (i), it follows that $\{x_n\}$ is also D-Cauchy w.r.t. D_1.

From hypothesis (iii), it follows that there is a point $p \in X$ such that $\lim_n X_n = p$. Now $D_2 \leq D_1$ on X^3; therefore we get

$$\lim_{m,n} D_2(x_m, x_n, p) \leq \lim_{m,n} D_1(x_m, x_n, p) = 0$$

This shows that $x_n \to p$ w.r.t. D_2. By (iv)

$$p = \lim_n x_{n+1} = \lim_n fx_n = f\left(\lim_n x_n\right) = fp$$

Finally, the uniqueness of p and the continuity of f at p w.r.t. D follow from hypothesis (v). This completes the proof.

Corollary 3.2. *Let X be a D-metric space with three D-metric D, D_1 and D_2 and $f : X \to X$. Assume that all the conditions* (i)–(v) *of Theorem* 3.3 *hold with* (v) *replaced by* (3.4). *Then f has a unique fixed point p, f^m is continuous at p w.r.t. D and f is f-orbitally continuous at p.*

The measure of noncompactness α of a bounded set A in a D-metric space X is defined by

$$\alpha(A) = \inf\left\{r > 0 : A = \bigcup_{i=1}^{n} A_i;\ \operatorname{diam}(Ai) \leq r\ i = 1, 2, ..., n\right\} \tag{3.7}$$

where $\operatorname{diam}(Ai)$ denotes the diameter of Ai w.r.t. the D-metric D. The set function α enjoys the following properties:

(α_1) $\alpha(A) = 0$ if and only if A is precompact,
(α_2) $\alpha(A) = \alpha(\overline{A})$, where $\overline{A}$ denotes the closure of A,
(α_3) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$, and
(α_4) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

Definition 3.1. A mapping $f : X \to X$ is said to be α condensing if for any bounded set A in X, $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ for $\alpha(A) > 0$.

Theorem 3.4. *Let f be a f-orbitally continuous and α-condensing self-map of a f-orbitally complete and f-orbitally bounded D-metric space X satisfying*

$$\begin{aligned} D(fx, fy, fz) < \max\{&D(x, y, z), D(x, fx, z), D(y, fy, z), \\ &D(x, fy, z), D(z, fx, z)\} \end{aligned} \tag{3.8}$$

for all $x, y, z \in X$ *with*

$$\max\{D(x,y,z), D(x,fx,z), D(y,fy,z), D(x,fy,z), D(y,fx,z)\} \neq 0.$$

Then f *has a unique fixed point* p *and* f *is continuous at* p.

Proof. First we note that if f has a fixed point, it is unique. Let $x \in X$ be arbitrary and define a sequence in X by (3.2). Denote $A = \{x_0, x_1, x_2, ..., x_n, ...\}$. We show that A is a precompact set in X. If not, then $\alpha(A) > 0$, and

$$\alpha(A) = \alpha(\{x_0\} \cup \{x_1, ..., x_n, ...\}) = \alpha(\{x_0\} \cup f(A)) = \\ = \max\{\alpha(\{x_0\}), \alpha(f(A))\} < \alpha(A),$$

which is a contradiction. So, A is precompact and $\overline{A}$ is compact, since X is f-orbitally complete. Since $\overline{A}$ is compact, both sides of (3.8) are bounded on $\overline{A}$. Now there are two cases:

Case I. Suppose that there is a point (x, y, z) in $\overline{A}$ such that the right hand side of (3.8) is zero. Then $x = fx = z$, and x is a fixed point of f, so it is unique.

Case II. Suppose that the right hand side of (3.8) is positive for all $x, y, z \in \overline{A}$. Define

$$T(x,y,z) = \\ = \frac{D(fx., fy, fz)}{\max\{D(x,y,z), F(x,fx,z), D(y,fy,z), D(x,fy,z), D(y,fx,z)\}} \tag{3.9}$$

for all $x, y, z \in \overline{A}$.

Since f is continuous on $\overline{A}$ and D is continuous function, it follows that T is continuous on $\overline{A}$. So T attains its maximum at some point (u, v, w) in $\overline{A}$. Call this value c. It is clear from (3.8) that $0 < c < 1$. Then f satisfies (2.1) with $q = c$. Now the desired conclusions follows by an appliction of Theorem 3.1. This completes the proof.

Corollary 3.3. *Let* f *be a* f*-orbitally continuous and* α*-condensing self-map of a* f*-orbitally complete and* f*-orbitally bounded* D*-metric space* X*,* m *a positive integer, satisfying*

$$D(f^m x, f^m y, f^m z) < \max\{D(x,y,z), D(x,f^m x,z), \\ D(y,f^m y,z), D(x,f^m y,z), D(y,f^m,z)\} \tag{3.10}$$

for all $x, y, z \in X$ *with*

$$\max\{D(x,y,z), D(x,f^m x,z), D(y,f^m y,z), D(x,f^m y,z), D(y,f^m x,z)\} \neq 0.$$

Then f *has a unqiue fixed point* p, f^m *is continuous at* p *and* f *is* f*-orbitally continuous at* p.

Corollary 3.4. (Theorem 2 of RHOADES [6]) *Let* f *be a continuous self-map of a compact* D*-metric space* X *satisfying* (3.8). *Then* f *has unique fixed point.*

In closing we mention that the results of this paper may be extended to a pair of mappings on a D-metric space satisfying the contractive condition of the type (3.1) or (3.8). Some of the results in this direction will be reported elsewhere.

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