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THE HOMOGENEOUS LIFT OF A RIEMANNIAN METRIC

BY

RADU MIRON

Abstract. Noticing that the Sasaki lift of a Riemannian metric g defined on the differentiable manifold M is not homogeneous on the fibres of tangent bundle TM we introduce in (2.1) a new lift G which is 0-homogeneous. G determines on $\widetilde{TM}:=TM\backslash\{0\}$ a Riemannian metric, which depends only on the metric g. Some geometrical properties of this Riemannian space are studied. The natural almost complex structure F which preserves the property of homogeneity is defined by (3.1). We prove that (G,F) is a conformal almost Kählerian structure.

Introduction. The importance of the Sasaki lift $\overset{0}{G}$, (1.7), [9] of a Riemannian metric g in the geometry of the Riemannian spaces (M,g) is well known. The tensor field $\overset{0}{G}$ determines a Riemannian structure on $\widetilde{TM} = TM\backslash\{0\}$. But $\overset{0}{G}$ is not homogeneous on the fibres of the tangent bundle TM. Therefore, we cannot study some global properties of the Riemannian space $(\widetilde{TM},\overset{0}{G})$. For instance we can not approch for this space a theorem of Gauss–Bonnet type.

In this paper, by means of (2.1), we define a new kind of lift G to TM of the Riemannian metric g. Thus G determines on $\widetilde{TM}$ a Riemannian structure, which is 0-homogeneous on the fibres of TM and depends only on g. Some geometrical properties of G are studied: the metrical N-linear connection, Levi–Civita connection, etc. Introducing the natural almost complex structure F, by (3.1), which depends only on g and preserves the property of homogeneity we get an almost Hermitian structure (G,F) which is very particular. Namely, the pair (G,F) is a conformal almost Kählerian structure. It determines the geometrical model of the Riemannian space (M,g) with respect to the homogeneous lift G.

1. The Sasaki Lift. Let (M, g) be a Riemannian space, M being a real n-dimensional manifold and (TM, π, M) its tangent bundle. On a domain $U \subset M$ of a local chart g has the components $g_{ij}(x)$, $(i, j, k, ... = 1, ..., n)$. Then on the domain of chart $\pi^{-1}(U) \subset TM$ we consider the functions $g_{ij}(x, y) = g_{ij}(x)$, $\forall (x, y) \in \pi^{-1}(U)$ and put

$$\|y\| = \sqrt{g_{ij}(x)y^i y^j}. \tag{1.1}$$

Then, $\|y\|$ is globally defined on TM, differentiable on $\widetilde{TM}$ and continuous on the null section. Moreover, the pair $F^n = (M, \|y\|)$ is a particular Finsler space, having $\|y\|$ as fundamental function and $g_{ij}(x)$ as fundamental tensor field. Applying the theory of Finsler spaces, [5], [6], [7], we notice that:

1^o. The Cartan tensor field vanishes:

$$C^i_{jk} = 0 \tag{1.2}$$

2^o. The canonical spray S of the space F^n has the coefficients

$$G^i = \frac{1}{2}\gamma^i_{00} = \frac{1}{2}\gamma^i_{jk}(x)y^j y^k, \tag{1.3}$$

where $\gamma^i_{jk}(x)$ are the Christoffel symbols of the metric g.

3^o. The canonical nonlinear connection N has the coefficients

$$N^i_j = \gamma^i_{j0} = \gamma^i_{jk}(x)y^k \tag{1.4}$$

4^o. N determines an horizontal distribution on $\widetilde{TM}$, which is supplementary to the vertical distribution V. Such that, we have:

$$T_u\widetilde{TM} = N_u \oplus V_u, \ \forall u \in \widetilde{TM}. \tag{1.5}$$

5^o. The adapted basis to N and V is given by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, (i=1,...,n) and its dual basis is $(dx^i, \delta y^i)$, $(i = 1..., n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \ \delta y^i = dy^i + N^i_j dx^j. \tag{1.6}$$

Therefore the Sasaki lift of g to TM is defined by

$$\overset{0}{G}(x,y) = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j, \forall (x,y) \in \widetilde{TM}. \tag{1.7}$$

The following properties follow:

7^o. $\overset{0}{G}$ is globally defined on $\widetilde{TM}$.

8^o. $\overset{0}{G}$ is a Riemannian metric on $\widetilde{TM}$.

9^o. $\overset{0}{G}$ is not homogeneous on the fibres of TM.

Namely, for the homothety $h_t : (x,y) \to (x,ty), \forall t \in R^+$ we get

$$(\overset{0}{G} \circ h_t)(x,y) = g_{ij}(x)dx^i \otimes dx^j + t^2 g_{ij}(x)\delta y^i \otimes \delta y^j \neq \overset{0}{G}(x,y).$$

Let us consider the $\mathcal{F}(\widetilde{TM})$-linear mapping $\overset{0}{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, given in the adapted basis by

$$\overset{0}{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \overset{0}{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i},\ i = \overline{1,n}. \tag{1.8}$$

It follows that:

10^o. $\overset{0}{F}$ is globally defined on $\widetilde{TM}$ and it is a tensor field of type (1.1).

11^o. $\overset{0}{F}$ is an almost complex structure.

12^o. $\overset{0}{F}$ depends only on g.

13^o. $\overset{0}{F}$ is a complex structure on $\widetilde{TM}$ if and only if the Riemann space M^n is locally flat.

14^o. The pair $(\overset{0}{G}, \overset{0}{F})$ is an almost Hermitian structure on $\widetilde{TM}$.

15^o. The associated almost symplectic structure $\overset{0}{\theta}$ to the Hermitian structure $(\overset{0}{G}, \overset{0}{F})$ is integrable (i.e. the exterior differential of $\overset{0}{\theta}$ vanishes).

Consequently, we get:

Theorem 1.1. *The space (TM, G, F) is almost Kählerian space depending only on the Riemannian metric g.*

The previous space, called "the geometrical model on TM of the Riemannian space (M, g)", is important in the study of the geometry of the initial Riemannian space (M, g) [5], [6].

2. The homogeneous lift of the Riemannian metric g. We can eliminate the inconvenient of the Sasaki lift $\overset{0}{G}$ given by the property 9^o introducing a new kind of lift to TM of the Riemannian metric g.

Definition 2.1. We call the homogeneous lift of the Riemannian metric g the following tensor field on $\widetilde{TM}$

$$G(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{a^2}{||y||^2} g_{ij}(x)\delta y^i \otimes \delta y^j, \quad \forall (x,y) \in \widetilde{TM}. \tag{2.1}$$

where $a > 0$ is a constant imposed by applications (in order to preserve the physical dimensions of the components of G), [1,2,3], and $||y||^2$ is the square of the norm of Liouville vector field:

$$||y||^2 = g_{ij}(x)y^i y^j. \tag{2.2}$$

We get, without difficulties:

Theorem 2.1. *The following properties hold:*

1^o. *The pair $(\widetilde{TM}, G)$ is a Riemannian space, depending only on the metric g.*

2^o. *G is 0-homogeneous on the fibres of the tangent bundle TM.*

3^o. *The distributions N and V are orthogonal with respect to G.*

We can write G in the form

$$G = G^H + G^V, \quad G^H = g_{ij}dx^i \otimes dx^j, \quad G^V = h_{ij}\delta y^i \otimes \delta y^j \tag{2.3}$$

where

$$h_{ij} = \frac{a^2}{||y||^2} g_{ij}. \tag{2.4}$$

In order to study the geometry of the Riemannian space $(\widetilde{TM}, G)$ we can apply the theory of the (h, v)-Riemannian metric on TM given in the books [4,6,7]. Looking at the relations (1.7) and (2.1) we can assert:

Proposition 2.1. *The lifts* $\overset{0}{G}$ *and* G *coincide on the hypersphere* $g_{ij}(x_o)y^iy^j = a^2$, *for every point* $x_o \in M$.

A linear connection D on TM is called a metrical N-connection with respect to G if $DG = 0$ and D preserves by parallelism the horizontal distribution N, [6], [7].

We can prove easily the existence of the metrical N-connections in the adapted basis. To this aim we represent a linear connection D in the adapted basis in the following form

$$
\begin{aligned}
&D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = \overset{H}{L}{}^i_{jk}\frac{\delta}{\delta x^i} + \widetilde{L^i_{jk}}\frac{\partial}{\partial y^i}, \quad D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j} = \widetilde{\widetilde{L^i_{jk}}}\frac{\delta}{\delta x^i} + \overset{V}{L}{}^i_{jk}\frac{\partial}{\partial y^i},\\
&D_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j} = \overset{H}{C}{}^i_{jk}\frac{\delta}{\delta x^i} + \widetilde{C^i_{jk}}\frac{\partial}{\partial y^i}, \qquad D_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j} = \widetilde{\widetilde{C^i_{jk}}}\frac{\delta}{\delta x^i} + \overset{V}{C}{}^i_{jk}\frac{\partial}{\partial y^i}.
\end{aligned} \tag{2.5}
$$

The systems of functions $(\overset{H}{L}{}^i_{jk}, \widetilde{L^i_{jk}}, \widetilde{\widetilde{L^i_{jk}}}, \overset{V}{L}{}^i_{jk}, \overset{H}{C}{}^i_{jk}, \widetilde{C^i_{jk}}, \widetilde{\widetilde{C^i_{jk}}}, \overset{V}{C}{}^i_{jk})$ are the coefficients of D.

It is not difficult to prove:

Theorem 2.2. *There exist the metrical* N*-connections* D *on* $\widetilde{TM}$, *with respect to the homogeneous lift* G, *which depend on the metric tensor* g, *only.*

One of them has the following coefficients:

$$
\begin{cases}
\widetilde{L^i_{jk}} = \widetilde{\widetilde{L^i_{jk}}} = \widetilde{C^i_{jk}} = \widetilde{\widetilde{C^i_{jk}}} = \overset{H}{C}{}^i_{jk} = 0,\\
\overset{H}{L}{}^i_{jk}(x,y) = \overset{V}{L}{}^i_{jk}(x,y) = \gamma^i_{jk}(x),\\
\overset{V}{C}{}^i_{jk} = \dfrac{1}{||y||^2}(\delta^i_j y_k + \delta^i_k y_j - g_{jk}y^i),
\end{cases} \tag{2.6}
$$

where $y_k = g_{kj}(x)y^j$.

The simplicity of this metrical N-connection and the fact that it is determined only on the Riemannian metric g, allows to call it the canonical metrical N-connection of the space $(\widetilde{TM}, G)$.

The structure equations of the canonical metrical N-connection are very simple, too, [6], [7]. Its structure forms are as follows:

$$\omega_j^i = \gamma_{jk}^i(x)dx^k, \qquad \widetilde{\omega}_j^i = \gamma_{jk}^i(x)dx^k + \overset{V}{C}{}_{jk}^i\delta y^k. \tag{2.7}$$

Theorem 2.3. *The structure equations of the canonical metrical N-connection D of the Riemann space $(\widetilde{TM}, G)$, G being the homogeneous lift of the metric g, are given by:*

$$\begin{cases} d(dx^i) - dx^k \wedge \omega^i{}_k = -\Omega^i, \\ d(\delta y^i) - \delta y^k \wedge \widetilde{\omega^i{}_k} = -\widetilde{\Omega^i}, \end{cases} \tag{2.8}$$

$$\begin{cases} d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = -\Omega^i{}_j, \\ d\widetilde{\omega^i{}_j} - \widetilde{\omega^k{}_j} \wedge \widetilde{\omega^i{}_k} = -\widetilde{\Omega^i{}_j}, \end{cases} \tag{2.8$'$}$$

where the 2-forms of torsion Ω^i and $\widetilde{\Omega^i}$ are

$$\Omega^i = 0, \quad \widetilde{\Omega^i} = \frac{1}{2}R^i{}_{jk}dx^j \wedge dx^k \tag{2.9}$$

and the 2-forms of curvature $\Omega^i{}_j$ and $\widetilde{\Omega^i{}_j}$ are

$$\begin{cases} \Omega^i{}_j = \dfrac{1}{2}R_j{}^i{}_{kh}dx^k \wedge dx^h, \\ \widetilde{\Omega^i{}_j} = \dfrac{1}{2}\widetilde{R_j{}^i{}_{kh}}dx^k \wedge dx^h + \dfrac{1}{2}S_j{}^i{}_{kh}\delta y^k \wedge \delta y^h \end{cases} \tag{2.9$'$}$$

where $R_j{}^i{}_{kh}(x)$ is the curvature tensor of the metric g, $S_j{}^i{}_{kh}$ is the $(vv)v$ curvature tensor of D, i.e.

$$\begin{cases} R_j{}^i{}_{kh} = \dfrac{\partial\gamma_{jk}^i}{\partial x^h} - \dfrac{\partial\gamma_{jh}^i}{\partial x^k} + \gamma_{jk}^s\gamma_{sh}^i - \gamma_{jh}^s\gamma_{sk}^i, \\ S_j{}^i{}_{kh} = \dfrac{\partial\overset{V}{C}{}_{jk}^i}{\partial y^h} - \dfrac{\partial\overset{V}{C}{}_{jh}^i}{\partial y^k} + \overset{V}{C}{}_{jk}^s\overset{V}{C}{}_{sh}^i - \overset{V}{C}{}_{jh}^s\overset{V}{C}{}_{sk}^i, \\ R^i{}_{kh}(x,y) = y^jR_j{}^i{}_{kh}(x); \qquad \widetilde{R_j{}^i{}_{kh}} = R_j{}^i{}_{kh} + \overset{V}{C}{}_{js}^iR_{kh}^s. \end{cases} \tag{2.9$''$}$$

Finally, by a straighforward calculus we can determine the Levi–Civita connection of the Riemannian metric G given by the homogeneous lift of g.

Theorem 2.4. *In the adapted basis the coefficients* $(\overset{H}{L}{}^i_{jk}, ..., \overset{V}{C}{}^i_{jk})$ *from* (2.5) *of the Levi–Civita connection of the Riemannian metric given by the homogeneous lift* G *from* (2.1) *are as follows:*

$$
\begin{aligned}
&\widetilde{C^i_{jk}} = \widetilde{\widetilde{C^i_{jk}}} = 0\\
&\overset{H}{L}{}^i_{jk} = \overset{V}{L}{}^i_{jk} = \gamma^i_{jk}, \quad \widetilde{L^i_{jk}} = -\frac{1}{2}R^i_{jk}, \quad \widetilde{\widetilde{L^i_{jk}}} = \overset{H}{C}{}^i_{kj} = \frac{1}{2}g^{is}h_{mj}R^m_{sk},\\
&\overset{V}{C}{}^i_{jk} = -\frac{1}{||y||^2}(\delta^i_j y_k + \delta^i_k y_j - g_{jk}y^i).
\end{aligned}
\tag{2.10}
$$

Of course, the structure equations of the Levi–Civita connection can be written without difficulties.

3. The almost Hermitian structure (G, F)**.** The almost complex structure $\overset{0}{F}$ defined in (1.8) has not the property of homogeneity. The $\mathcal{F}(\widetilde{TM})$-linear mapping $\overset{0}{F} : \chi(\widetilde{TM}) \longrightarrow \chi(\widetilde{TM})$, applies the 1-homogeneous vector fields $\dfrac{\delta}{\delta x^i}$ into a 0-homogeneous vector fields $\dfrac{\partial}{\partial y^i}$, $(i = 1, .., n)$. Therefore, we consider the $\mathcal{F}(\widetilde{TM})$-linear mapping $F : \chi(\widetilde{TM}) \longrightarrow \chi(\widetilde{TM})$, given on the adapted basis by

$$
F\left(\frac{\delta}{\delta x^i}\right) = -\frac{||y||}{a}\frac{\partial}{\partial y^i}, \quad F\left(\frac{\partial}{\partial y^i}\right) = \frac{a}{||y||}\frac{\delta}{\delta x^i}, \quad (i = 1, .., n).
\tag{3.1}
$$

It is not difficult to prove:

Theorem 3.1. *F has the following properties:*

1^o *F is a tensor field of type* $(1, 1)$ *on* $\widetilde{TM}$;

2^o *F is an almost complex structure on* $\widetilde{TM}$:

$$
F \circ F = -I
\tag{3.2}
$$

3^o *F depends only on the metric g;*

4^o *F is homogeneous on the fibres of TM.*

Theorem 3.2. *The almost complex structure F is a complex structure on $\widetilde{TM}$ if, and only if the Riemann space (M, g) is of constant curvature $\left(= \frac{1}{a^2}\right)$.*

Proof. In the adapted basis, the Nijenhuis tensor field $\mathcal{N}_F$ vanishes if, and only if we have:

$$R^i{}_{jk} = \frac{1}{a^2}(y_j\delta^i_k - y_k\delta^i_j).$$

Taking into account $(2.9)''$ and the fact that $y_i = g_{ij}(x)y^j$ the previous equality is equivalent to

$$R_h{}^i{}_{jk} = \frac{1}{a^2}(g_{hj}\delta^i_k - g_{hk}\delta^i_j). \qquad \text{q.e.d.}$$

The pair of geometrical structures G and F has some special properties.

Theorem 3.3. *We have:*

1^o *(G, F) is an almost Hermitian structure on $\widetilde{TM}$;*

2^o *(G, F) depends only on the metric g of the Riemann space (M, g);*

3^o *The associated almost simplectic structure θ is given in adapted basis by*

$$\theta = \frac{a}{\|y\|} g_{ij}(x)\delta y^i \wedge dx^j \tag{3.3}$$

4^o *θ and $\overset{\circ}{\theta}$ from (1.9) are related by*

$$\theta = \frac{a}{||y||}\overset{\circ}{\theta} \tag{3.3$'$}$$

5^o *The pair (G, F) is a conformal almost Kählerian structure.*

Proof.

1^o Follows from the equation $G(FX, FY) = G(X, Y)$ on $\widetilde{TM}$.

2^o G and F depending only on g, the Hermitian structure (G, F) has the same property.

3^o In the adapted basis $\theta(X, Y) = G(FX, Y)$ gives us (3.3).

4^o Is obvious.

5^o It is a consequence of the property 4^o. q.e.d.

The exterior differential of θ satisfies the equation:

$$d\theta = -\frac{a}{||y||} d||y|| \wedge \theta. \tag{3.4}$$

It follows, without difficulties that $d\theta \neq 0$ on $\widetilde{TM}$. That means: the space $(\widetilde{TM}, G, F)$ can not be an almost Kählerian one.

The conformal almost Kählerian space $K^{2n} = (\widetilde{TM}, G, F)$ is called the homogeneous geometrical model of the Riemannian space (M, g) with respect to the homogeneous lift G. It can be used in a gauge theory of the Riemannian metric defined by the homogeneous lift G, [6,7].

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Universitatea "AL.I.Cuza" Iaşi
6600, Iaşi
ROMANIA