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MULTIPLIERS FOR FOURIER – ULTRASPHERICAL EXPANSIONS IN WEIGHTED LOCALLY CONVEX SPACES

BY

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Abstract. Using a convolution formula of GASPER (1971), we prove a new result on multipliers for Fourier – ultraspherical series in a suitably chosen locally convex Hausdorff topological vector space.

1. Introduction. Let $X = (X, \{p_i\}_{i\in I})$ be a locally convex Hausdorff topological vector space whose topology is generated by a system of seminorms $\{p_i\}_{i\in I}$, I being an index set. Let X^* be the space of all continuous linear operators of X onto itself and let $\{P_k\}_{k\in P} \subset X^*$ be a total sequence of mutually orthogonal projections on X (cf. [1]). The sequence $\{P_k\}_{k\in P}$ is said to be fundamental provided the set of all such projections is dense in X.

The formal expansion of a function f in X in the form of Fourier series is given by

$$f \sim \sum_{k=0}^{\infty} P_k f \tag{1.1}$$

Since the sequence $\{P_k\}_{k\in P}$ is total, the series (1.1) is unique.

2. Multipliers for X. Let X be the set of all sequences $\eta = \{\eta_k\}_{k\in P}$ of scalars. A sequence $\eta \in S$ is called a multiplier for X with respect to $\{P_k\}_{k\in P}$ if $\forall f \in X$ there exists an element $f^\eta \in X$ such that

$$P_k f^\eta = \eta_k P_k f.$$

Since $\{P_k\}_{k\in P}$ is total, f^η is uniquely determined by f.

JUNGGEBURTH [4] has studied in detail the multiplier criteria for various types of Fourier–orthogonal expansions in locally convex topological vector spaces using sequence to sequence transformation methods. He has shown that if X is a locally convex space and the (C,δ), $\delta \geq 0$, transformation of the Fourier series of every $f \in X$ is uniformly bounded, then $bv_{\delta+1}$ is continuously embedded in the space of multipliers $M_c(X, \{P_k\}_{k\in P})$, where

$$bv_{\delta+1} = \left\{ \eta \in S : \|\eta\|_{bv_{\delta+1}} = \sum_{k=0}^{\infty} A_k^\delta \left|\Delta^{\delta+1}\eta_k\right| + \lim_{k\to\infty} |\eta_k| < \infty \right\}, \tag{2.1}$$

M_c is the class of all continuous multipliers on X with respect to $\{P_k\}$,

$$A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$$

and the fractional difference Δ^r is defined by

$$\Delta^r \eta_k = \sum_{\nu=0}^{\infty} A_\nu^{-r-1} \eta_{k+\nu}. \tag{2.2}$$

3. In the present paper we study the problem of approximation of functions by Fourier–ultraspherical series and use our results to characterize the space of multipliers on weighted locally convex spaces.

We suppose that X_r^p is the Banach space of all measurable functions on $[0,\pi]$ such that

$$X_r^p = \left\{ f \in L^1[0,\pi] : a_r^p(f) = \left(\int_0^\pi |f(\theta)|^p [U_r^p(\theta)]^{-1} d\theta \right)^{1/p} < \infty \right\}, \tag{3.1}$$

$1 < p < \infty$, where

$$\begin{aligned} U_r^p(\theta) &= 2^{-r/p-1/q} \left[\frac{\left\{ \Gamma\left(\frac{rq}{p} + 1 \right) \right\}^2}{2\Gamma\left\{ 2\left(\frac{rp}{p+1} \right) \right\}} \right]^{-1/q} (\sin\theta)^{r/p} \\ &= C(\sin\theta)^{r/p},\ r \in J = [1,p],\ 1/p + 1/q = 1 \end{aligned}$$

and C is a positive constant, but not necessarily the same at each occurence.

It is clear that in the sense of continuous embeddings we have

$$L^{\infty}[0,\pi]\subset X_r^p\subset L^1[0,\pi].$$

Let $P_k^{\lambda}(c_{\infty}\theta)$ denotes the kth ultraspherical polynomial of order λ. We write

$$R_k(\theta)=R_k^{\lambda}(\cos\theta)=P_k^{\lambda}(\cos\theta)/P_k^{\lambda}(1).$$

The convolution formula for any two functions $f,g\in L^1[0,\pi]$ is defined by

$$(f*g)(\theta)=\int_0^{\pi}f(\psi)(T_{\psi}g(\theta))d\mu(\psi), \tag{3.2}$$

where the generalized translation operator T_{ψ} is defined by

$$T_{\psi}g(\theta)=\int_0^{\pi}g(\phi)k(\theta,\phi,\psi)d\mu(\phi), \tag{3.3}$$

$K(\theta,\phi,\psi)$ is a non–negative symmetric function such that

$$R_k(\theta)R(\phi)=\int_0^{\pi}R_k(\psi)K(\theta,\phi,\psi)d\mu(\psi),\int_0^{\pi}K(\theta,\phi,\psi)d\mu(\psi)=1 \tag{3.4}$$

and

$$d\mu(\psi)=2^{2\lambda}(\sin\theta/2)^{2\lambda}(\cos\theta/2)^{2\lambda}.$$

GASPER [2] has shown that $L^1[0,2\pi]$ is a commutative semi–simple Banach algebra with respect to the convolution defined by (3.2) as multiplication.

We now define the mutually orthogonal projections $\{B_k\}_{k\in P}$ by

$$B_kf(\theta)=\left(\int_0^{\pi}f(\theta)R_k(\theta)d\mu(\theta)\right)h_kR_k(\theta), \tag{3.5}$$

where

$$\begin{aligned}h_k=h_k^{\lambda}&=\left(\int_0^{\pi}[R_k(\theta)]^2d\mu(\theta)\right)^{-1}=\\&=\frac{2^{1-2\lambda}(\Gamma(2\lambda))^2(k+\lambda)\Gamma(k+2\lambda)}{(\Gamma(\lambda))^2\Gamma(k+1)}.\end{aligned} \tag{3.6}$$

On the lines of TREBELS [5], it can be easily seen that the sequence $\{B_k\}_{k\in P}$ is total and fundamental in X_ρ^p. Following JUNGGEBURTH [4] we define the locally convex space

$$X = \left(\bigcup_{r\in J} X_r^p\right),\ 1 < p < \infty \text{ and } J = [1, p].$$

The formal expansion of a function $f \in X$ in the form of Fourier–ultraspherical series is given by

$$f \sim \sum_{k=0}^{\infty} B_k f. \tag{3.7}$$

We denote by $M_c \equiv M_c(X, \{B_k\}_{k\in P})$ the set of all continuous multipliers on X with respect to $\{B_k\}_{k\in P}$. We shall prove the following:

Theorem. *If* $X = \bigcup_{r\in J} X_r^p,\ 1 < p < \infty,\ J = [1, p]$ *is a locally convex space whose topology, defined by the system of semi-norms* (3.1)*, then the space* $bv_{\lambda+1}$ *is continuously embedded in* $M_c(X, \{B_k\}_{k\in P})$.

4. Proof of the Theorem. At first we shall demonstrate that the sequence of Cesàro means $\{(c, \lambda)_n f(\theta)\}$ is uniformly bounded for every $f \in X$.

Using the formulas (3.3) and (3.4), it can be easily seen that

$$(c, \lambda)_n f(\theta) - f(\theta) = \int_0^\pi (T_\psi f(\theta) - f(\theta)) K_n^\lambda(\psi) d\mu(\psi)$$

where

$$\begin{aligned} K_n^\lambda(\psi) &= \frac{1}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^\lambda \cdot h_\nu^\lambda \cdot R_\nu^\lambda(\cos\psi) \\ &= \frac{c}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^\lambda \cdot (\nu + \lambda) P_\nu^{(\lambda)}(\cos\psi) \\ &= \frac{c}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \cdot \sum_{k=0}^{\nu} (k + \lambda) P_k^\lambda(\cos\psi). \end{aligned}$$

Thus it follows that

$$
\begin{aligned}
&\int_0^\pi \{[(c,\lambda)_n f(\theta) - f(\theta)]d\theta \\
&= c\int_0^\pi \int_0^\pi [T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1} K_n^\lambda(\psi) U_r^p(\psi) d\mu(\psi) d\theta\} \leq \\
&\leq c\int_0^\pi |\int_0^\pi \{[T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1}\} \cdot \\
&\qquad \cdot \{K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\} d\psi| d\theta.
\end{aligned} \tag{4.1}
$$

We now consider the inside integral in (4.1). We write

$$
\begin{aligned}
I = C\int_0^\pi &\{[T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1} \cdot \\
&\cdot \{K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\} d\psi = \\
= \int_0^{c/n} + \int_{2/n}^{\varepsilon} + \int_{\varepsilon}^{\pi - c/n} &+ \int_{\pi - c/n}^{\pi} = I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.2}
$$

say, where ε is a sufficiently small psoitive number.

In order to evaluate the above integrals, we shall use the following order estimates for the ultraspherical polynomials:

$$
P_n^\lambda(\cos\psi) = \begin{cases} \psi^{-\lambda} O(n^{\lambda-1}), & c/n \leq \psi \leq \pi/2, \\ O(n^{2\lambda-1}), & 0 \leq \psi \leq c/n, \end{cases} \tag{4.3}
$$

where c is a constant;

$$
P_n^{(\lambda)}(\cos\psi) = \begin{cases} n^{\lambda-1} k(\psi)\cos(N\psi + \gamma) + (n\sin\psi)^{-1} O(1) \\ \qquad\qquad \text{for } c/n \leq \psi \leq \pi - c/n; \\ n^{\lambda-1} k(\psi)\cos(N\psi + \gamma) + O(n^{-3/2}), \end{cases} \tag{4.4}
$$

where

$$
k(\psi) = \pi^{-1/2}(\sin\psi/2)^{-\lambda}(\cos\psi/2)^{-\lambda},
$$

$$
N = n + \lambda \text{ and } \gamma = -\lambda\pi/2.
$$

We discuss I_1 first.

$$I_1 = c\int_0^{c/n} \{[T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1}\} \cdot$$
$$\cdot \{K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\} d\psi.$$

Applying Hölder's inequality, we obtain

$$I_1 \le c \sup_{0\le\psi\le c/n} \|T_\psi f(\theta) - f(\theta)\|_X \cdot$$
$$\cdot \left\{\int_0^{c/n} \left|K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda} U_r^p(\psi)\right|^q d\psi\right\}^{1/q}.$$

Since $\|T_\psi f(\theta) - f(\theta)\|_X \to 0$ as $\psi \to 0$, using the order estimates in (4.3), we get

$$\begin{aligned} I_1 &= o(1)O(n^{2\lambda+1}) \left(\int_0^{c/n} (\psi^{(2\lambda+r/p)})^q d\psi\right)^{1/q} \\ &= o(n^{2\lambda+1})[(c/n)^{(2\lambda+r/p)q+1}]^{1/q} \\ &= o(n^{2\lambda+1}) \left(\frac{c}{n}\right)^{2\lambda+r/p+1/q} = o(1) \text{ for } r \in [1,p]. \end{aligned} \tag{4.5}$$

Next, we consider I_2.

$$I_2 = c\int_{c/n}^{\varepsilon} \{[T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1}\} \cdot$$
$$\cdot \{K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\} d\psi.$$

Applying Hölder's inequality again, we obtain

$$I_2 \le c \sup_{c/n\le\psi\le\varepsilon/2} \|T_\psi f(\theta) - f(\theta)\|_X \cdot$$
$$\cdot \left\{\int_{c/n}^{\varepsilon} \left|K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\right|^q d\psi\right\}^{1/q}.$$

Using the first part of (4.4), we evaluate the integral in the small bracket. We have

$$W = \left\{\int_{c/n}^{\varepsilon} |K_n^\lambda(\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}|^q d\psi\right\}^{1/q} =$$
$$= \left\{\int_{c/n}^{\varepsilon} |\frac{A}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \sum_{k=0}^{\nu} (k+\lambda) P_k^\lambda(\cos\psi) U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}|^q d\psi\right\}^{1/q}$$

$$
\begin{aligned}
&= \left\{ \int_{c/n}^{\varepsilon} | \frac{1}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \frac{1}{2} \left[\frac{d}{dx} \{ P_\nu^\lambda(x) + P_{\nu+1}^\lambda(x) \} \right]_{x=\cos\psi} \cdot \right. \\
&\qquad \left. \cdot U_r^p(\psi) (\sin\psi/2)^{2\lambda} (\cos\psi/2)^{2\lambda} |^q d\psi \right\}^{1/q} = \\
&= \left\{ \int_{c/n}^{\varepsilon} | \left[\frac{1}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \frac{1}{2} \left[\frac{d}{dx} \{ P_\nu^\lambda(x) + P_{\nu+1}^\lambda(x) \} \right]_{x=\cos\psi} \sin\psi \right\} \cdot \right. \\
&\left. \cdot (\sin\psi)^{-1} U_r^p(\psi) (\sin\psi/2)^{2\lambda} (\cos\psi/2)^{2\lambda} |^q d\psi \right\}^{1/2}.
\end{aligned} \tag{4.6}
$$

Following GUPTA [3], the order estimate for the expression inside the curley bracket may be written in the form

$$
\begin{aligned}
&\frac{1}{A_n^\lambda} \sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \frac{1}{2} \left[\frac{d}{dx} \{ P_\nu^\lambda(x) + P_{\nu+1}^\lambda(x) \} \right]_{x=\cos\psi} \sin\psi = \\
&= O(\psi^{-2\lambda}) + O(n^{\lambda-1} \psi^{-1} (\sin\psi)^{-\lambda}) + \\
&+ O(n^{-1} (\sin\psi)^{-\lambda-1}) + O(n^{\lambda-1} \psi^{-\lambda} (\sin\psi)^{-1}) \\
&= O(\psi^{-2\lambda} + O(n^{\lambda-1} \psi^{-\lambda-1}).
\end{aligned} \tag{4.7}
$$

Combining (4.6) and (4.7), we get

$$
\begin{aligned}
W &= O\left[\int_{c/n}^{\varepsilon} \left| \psi^{-2\lambda} (\sin\psi)^{-1} (\sin\psi/2)^{2\lambda} (\cos\psi/2)^{2\lambda} U_r^p(\psi) \right|^q d\psi \right]^{1/q} \\
&+ O(n^{\lambda-1}) \left[\int_{c/n}^{\varepsilon} \left| \psi^{-\lambda-1} (\sin\psi)^{-1} (\sin\psi/2)^{2\lambda} (\cos\psi/2)^{2\lambda} U_r^p(\psi) \right|^q d\psi \right]^{1/q} \\
&= O\left[\int_{c/n}^{\varepsilon} \left| \psi^{r/p-1} \right|^q d\psi \right]^{1/q} + O(n^{\lambda-1} \left[\int_{c/n}^{\varepsilon} \left| \psi^{\lambda+r/2-2} \right|^q d\psi \right]^{1/q} \\
&= O(1) + O(n^{\lambda-1}) \left(\frac{c}{n} \right)^{\lambda+r/2-2+1/q} = O(1) \text{ for } r \in [1, p].
\end{aligned} \tag{4.8}
$$

Since

$$
\| T_\psi f(\theta) - f(\theta) \|_X \longrightarrow 0 \text{ as } \psi \to 0,
$$

there exists ε such that

$$
\sup_{c/n \le \psi \le \varepsilon} \| T_\psi f(\theta) - f(\theta) \|_X < c\varepsilon. \tag{4.9}
$$

Combining (4.8) and (4.9) and making $\varepsilon \to 0$, we obtain

$$I_2 = O(1). \tag{4.10}$$

Now we discuss I_4.

$$I_4 = c\int_{\pi-c/n}^{\pi} \{[T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1}\}\cdot \\ \cdot\{K_n^\lambda(\psi)U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\}d\psi.$$

Using Hölder's inequality, we get

$$I_4 \le c \sup_{\pi-c/n\le\psi\le\pi} \|T_\psi f(\theta) - f(\theta)\|_X \left[\int_{\pi-c/n}^{\pi} \left| \left\{ \frac{1}{A_n^\lambda}\sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} \right. \right.\right. \\ \cdot\frac{1}{2}\left[\frac{d}{dx}\{P_\nu^\lambda(x) + P_{\nu+1}^\lambda(x)\}\right]_{x=\cos\psi} \cdot \\ \left.\left.\left. \cdot U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\right|^q \right\} d\psi\right]^{1/q}.$$

Using the order estimate

$$\left[\frac{d}{dx}\{P_\nu^\lambda(x) + P_{\nu+1}^\lambda(x)\}\right]_{x=\cos\psi} = O(\nu^{2\lambda}) \text{ for } \pi - \frac{c}{n} \le \psi \le \pi,$$

we have

$$I_4 = O(1)\left[\int_{\pi-c/n}^{\pi} \left| \frac{1}{A_n^\lambda}\sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1}\nu^{2\lambda}U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}\right|^q d\psi\right]^{1/q} \tag{4.11}$$

$$= O(n^{2\lambda})\left[\int_0^{c/n} \left|U_r^p(\pi - z)(\sin z/2)^{2\lambda}\right|^q dz\right]^{1/q} \quad \text{for } \psi = \pi - z.$$

$$= O(n^{2\lambda})\left[\int_0^{c/n} z^{2\lambda q} dz\right]^{1/q} = O(n^{2\lambda})\left(\frac{c}{n}\right)^{2\lambda+1+1/q} = O(1).$$

Finally, we discuss I_3.

$$I_3 = c\int_{\varepsilon}^{\pi-c/n} \omega_f(\psi)K_n^\lambda(\psi)U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}d\psi,$$

where

$$\omega_f(\psi) = [T_\psi f(\theta) - f(\theta)](U_r^p(\psi))^{-1}.$$

Thus we have

$$\begin{aligned} I_3 &= c\int_\varepsilon^{\pi-c/n} \omega_f(\psi)\frac{1}{A_n^\lambda}\sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1}\frac{1}{2}\left[\frac{d}{dx}\{P_\nu^\lambda(x)+P_{\nu+1}^\lambda(x)\}\right]_{x=\cos\psi} \\ &\qquad \cdot \sin\psi(\sin\psi)^{-1}U_r^p(\psi)(\sin\psi/2)^{2\lambda}(\cos\psi/2)^{2\lambda}d\psi \\ &= c\int_\varepsilon^{\pi-c/n} \omega_f(\psi)U_r^p(\psi)(A_n^\lambda)^{-1}E_n^\lambda(\psi)(\sin\psi/2)^{2\lambda-1}d\psi \\ &\quad + \int_\varepsilon^{\pi-c/n} |\omega_f(\psi)|U_r^p(\psi)O[n^{\lambda-1}\psi^{-\lambda}(\sin\psi)^{-1}+ \\ &\qquad +n^{-1}(\sin\psi)^{-\lambda-1}(\sin\psi)^{2\lambda-1}]d\psi = I_{3,1}+I_{3,2}, \end{aligned} \tag{4.12}$$

say, where

$$\begin{aligned} E_n^\lambda(\psi) = R\Bigg\{ &-\frac{2}{\pi}\lambda\sin\lambda\pi(\cos\psi/2)^{1-\lambda}(\sin\psi/2)^{1-2\lambda}i\pi\left(\frac{1}{2}-\lambda\right) \\ &\cdot\int_{-\infty}^{\psi}(\psi-t)^{-\lambda-1}\sum_{\nu=0}^{n}A_{n-\nu}^{\lambda-1}\left[e^{i(\nu+\lambda+1/2)\psi}-e^{i(n+\lambda+1/2)t}\right]dt\Bigg\}\cdot \\ &\cdot\ell(\psi)e^{i(n+\lambda+1/2)\psi}, \end{aligned}$$

where

$$\begin{cases} \ell(\psi) = O(n^\lambda\psi^{-\lambda}), \\ \ell(\psi+\mu n)-\ell(\psi) = O(n^{2\lambda-1}\psi^{-1}\log n). \end{cases} \tag{4.13}$$

Hence we have

(4.14)

$$
\begin{aligned}
&I_{3,2} = O(n^{\lambda-1})\int_{\varepsilon}^{\pi-c/n} |\omega_f(\psi)|\psi^{-\lambda}\big|(\sin\psi)^{2\lambda-2}\big||U_r^p(\psi)|d\psi\\
&+O(n^{-1})\int_{\varepsilon}^{\pi-c/n} |\omega_f(\psi)||U_r^p(\psi)||\sin\psi|^{\lambda-2}d\psi\\
&= O(n^{\lambda-1})\int_{\varepsilon}^{\pi-c/n} |\omega_f(\psi)||U_r^p(\psi)|(\cos\psi/2)^{2\lambda-2+r/p}d\psi\\
&+O(n^{-1})\int_{\varepsilon}^{\pi-c/n} |\omega_f(\psi)||U_r^p(\psi)|(\cos\psi/2)^{2\lambda-2+r/p}d\psi\\
&\le O(n^{\lambda-1})\sup_{\varepsilon\le\psi\le\pi-c/n}\|T_\psi f(\theta)-f(\theta)\|_X\left[\int_{c/n}^{\pi-\varepsilon}\{(\sin z/2)^{r/p+2\lambda-2}\}^q dz\right]^{1/q}\\
&+O(n^{-1})\sup_{\varepsilon\le\psi\le\pi-c/n}\|T_\psi f(\theta)-f(\theta)\|_X\left[\int_{c/n}^{\pi-\varepsilon}\{(\sin z/2)^{r/p+\lambda-2}\}^q dz\right]^{1/q}\\
&= O(n^{\lambda-1})\left[\int_{c/n}^{\pi-\varepsilon} z^{(2\lambda-2+r/p)q}dz\right]^{1/q} + O(n^{-1})\left[\int_{c/n}^{\pi-\varepsilon} z^{(\lambda-2+r/p)q}dz\right]^{1/q}\\
&= O(n^{\lambda-1})\left(\frac{c}{n}\right)^{2\lambda-2+1/q+r/p} + O(n^{-1})\left(\frac{c}{n}\right)^{\lambda-2+1/q+r/p} =\\
&= O(1) \text{ for } r\in[1,p].
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
I_{3,1} &= c(A_n^\lambda)^{-1}\int_{\varepsilon}^{\pi-c/n}\omega_f(\psi)\ell(\psi)(\sin\psi)^{2\lambda+r/p-1}e^{i(n+\lambda+\frac{1}{2})\psi}d\psi\\
&= c(A_n^\lambda)^{-1}\int_{\varepsilon}^{\pi-c/n}\omega_f(\psi)\ell(\psi)m(\psi)e^{i(n+\lambda+1/2)\psi}d\psi
\end{aligned}
\tag{4.15}
$$

where

$$m(\psi) = (\sin\psi)^{2\lambda+r/p-1}.$$

The integral in (4.15) may be rewritten in the form

$$\frac{1}{2}\left\{\int_{\varepsilon}^{\pi-c/n}\omega_f(\psi)\ell(\psi)m(\psi)e^{i(n+\lambda+1/2)\psi}d\psi-\right.$$
$$\left.-\int_{\varepsilon-\mu_n}^{\pi-c/n-\mu_n}\omega_f(\psi+\mu_n)\ell(\psi+\mu_n)m(\psi+\mu_n)e^{i(n+\lambda+1/2)\psi}d\psi\right\}\leq$$
$$\leq\frac{1}{2}(L_1+L_2+L_3+L_4+L_5),$$

say, where $\mu_n=\pi/(n+\lambda+1/2)$,

$$L_1=\int_{\varepsilon-\mu}^{\varepsilon}|\omega_f(\psi+\mu_n)\ell(\psi+\mu_n)m(\psi+\mu_n)|d\psi,$$
$$L_2=\int_{\pi-c/n-\mu_n}^{\pi-c/n}|\omega_f(\psi)\ell(\psi)m(\psi)|d\psi,$$
$$L_3=\int_{\varepsilon}^{\pi-c/n-\mu_n}|\omega_f(\psi+\mu_n)-\omega_f(\psi)||\ell(\psi+\mu_n)||m(\psi+\mu_n)|d\psi,$$
$$L_4=\int_{\varepsilon}^{\pi-c/n-\mu_n}|\ell(\psi+\mu_n)-\ell(\psi)|\omega_f(\psi)|m(\psi+\mu_n)|d\psi$$

and

$$L_5=\int_{\varepsilon}^{\pi-c/n-\mu_n}|m(\psi+\mu_n)-m(\psi)|\omega_f(\psi)|\ell(\psi)|d\psi.$$

Now we see that

$$L_1=O\left[n^{\lambda}\int_{\varepsilon-\mu_n}^{q}|\omega_f(\psi+\mu_n)|\psi^{-\lambda}\psi^{2\lambda+r/p-1}d\psi\right]=$$
$$=O(n^{\lambda})\sup_{\varepsilon-\mu_n\leq\psi\leq\varepsilon}\|T_\psi f(\theta)-f(\theta)\|_X=$$
$$=O(n^{\lambda})\text{ as }n\to\infty\text{ and }\varepsilon\to 0,$$
$$L_2=O(n^{\lambda})\int_{\varepsilon}^{\pi-c/n-\mu_n}\omega_f(\psi)\cdot\psi^{-\lambda}(\sin\psi)^{2\lambda+r/p-1}d\psi,$$
$$L_3=O(n^{\lambda})\int_{\varepsilon}^{\pi-c/n-\mu_n}|\omega_f(\psi+\mu_n)-\omega_f(\psi)|(\psi+\mu_n)^{-\lambda}$$
$$[\sin(\psi+\mu_n)]^{2\lambda+r/p-1}\,d\psi=$$
$$=O(n^{\lambda})\sup_{\varepsilon\leq\psi\leq\pi-c/n-\mu_n}\|\omega_f(\psi+\mu_n)-\omega_f(\psi)\|_X=$$
$$=O(n^{\lambda})\text{ as }n\to\infty,$$

because translation is continuous in the X-norm;

$$L_4 \le c \cdot n^{2\lambda-1} \log n \int_{\varepsilon}^{\pi-c/n-\mu_n} \psi^{-1}\omega_f(\psi)\Big|[\sin(\psi+\mu_n)]^{2\lambda+r/p-1}\Big|d\psi \le$$
$$\le c \cdot n^{2\lambda-1} \log n \sup_{\varepsilon\le\psi\le\pi-c/n-\mu_n} \|T_\psi f(\theta) - f(\theta)\|_X =$$
$$= O(n^{2\lambda-1}\log n)$$

and

$$L_5 = c\mu_n \int_{\varepsilon}^{\pi-c/n-\mu_n} |\sin\psi|^{2\lambda+r/p-1}\omega_f(\psi) n^\lambda \psi^{-\lambda} d\psi =$$
$$= c \cdot n^{\lambda-1}\left(\frac{c}{n}\right)^{2\lambda+\frac{r}{p}-2+\frac{1}{q}} = O(n^\lambda) \text{ as } n \to \infty.$$

Substituting the above order estimates for L_1, L_2, L_3, L_4, L_5 in (4.15), we obtain

$$I_{3,1} = O(1). \tag{4.16}$$

Thus, on account of the relations (4.1), (4.2), (4.5), (4.10), (4.11), (4.14) and (4.16), we find that

$$\int_0^\pi [(c,\lambda)_n f(\theta) - f(\theta)]\,(U_r^p(\theta))^{-1} U_r^p(\theta) d\theta = c\int_0^\pi |I| d\theta \tag{4.17}$$
$$\text{where } I = O(1) \text{ as } n \to \infty.$$

Applying Hölder's inequality to (4.17), we get

$$\left[\int_0^\pi \left|[(c,\lambda)_n f(\theta) - f(\theta)](U_r^p(\theta))^{-1}\right|^p d\theta\right]^{1/p} \left(\int_0^\pi |U_r^p(\theta)|^q d\theta\right)^{1/q} = O(1),$$

which implies that

$$\|(c,\lambda)_n f(\theta) - f(\theta)\|_X = O(1) \text{ as } n \to \infty. \tag{4.18}$$

Combining (4.17) and (4.18), we have

$$\int_0^\pi |(c,\lambda)_n f(\theta)|^p (\sin\theta)^r d\theta \le$$
$$\le c\int_0^\pi |f(\theta)|(\sin\theta)^r d\theta,\ r \in [1,p],$$

which implies that (c,λ)–means of the Fourier–ultraspherical series (3.7) are uniformly bounded in the locally convex space X. Hence, on the lines of BUTZER, NESSEL and TREBELS [1], we have

$$f^{\eta} = \sum_{k=0}^{\infty} A_k^{\lambda} \Delta^{\lambda+1} \eta_k (c,\lambda)_k f + \eta_{\infty} f, \text{ where } \eta_{\infty} = \lim_{k\to\infty} \eta_k.$$

Thus, for each $\gamma \in [1,p]$, there exists $t \in J$ such that

$$\begin{aligned} a_r^p(f^{\eta}) \quad &\leq c(r,t;\lambda) a_t^p(f) \sum_{k=0}^{\infty} A_k^{\lambda} \left|\Delta^{\lambda+1}\eta_k\right| + \eta_{\infty} a_t^p(f) \leq \\ &\leq c(r,t;\lambda) \|\eta\|_{bv_{\lambda+1}} a_t^p(f), \end{aligned}$$

which implies that f^{η} exists in the space X.

Finally, in order to complete the proof of our theorem, it remains to show that

$$B_k f^{\eta} = \eta_k B_k f.$$

On the lines of TREBELS [5, p.22], it follows that

$$B_n(c,\lambda)_k f = \begin{cases} 0 & \text{for } k < n, \\ (A_{k-n}^{\lambda}/A_k^{\lambda}) B_n f & \text{for } k \geq n, \end{cases}$$

Hence, using the result

$$\eta_n - \eta_{\infty} = \sum_{k=0}^{\infty} A_k^{\lambda} \Delta^{\lambda+1} \eta_{k+n},$$

where η_{∞} is the limiting value of the sequence $\{\eta_k\}$ as $k \to \infty$ (cf. [4], pp.136–137), we find that

$$B_n f^n = B_n f \left(\sum_{k=0}^{\infty} A_k^{\lambda} \cdot \frac{A_{k-n}^{\lambda}}{A_k^{\lambda}} \cdot \Delta^{\lambda_+ 1} \eta_k + \eta_{\infty} \right) = \eta_n B_n f.$$

This completes the proof of our theorem.

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