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GENERIC MEAN VALUE PROPERTIES FOR HALF–CONTINUOUS FUNCTIONS

BY

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Dedicated to Professor Viorel Barbu, on the occasion of his 60th birthday

Abstract. Some generic type mean value properties are given, for a class of half–continuous real/vector functions.

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1. Introduction. Let f, g be a couple of functions from R to itself. Take the points $a, b \in R$, $a < b$, in accordance with

(1H1) f and g are continuous over $[a, b]$;

and assume that, for a certain denumerable part A of $[a, b]$,

(1H2) the right derivatives $d^{(+)}f(t), d^{(+)}g(t)$ exist, for all $t \in]a, b[\setminus A$.

Consider the generic logical propositions

$(P(t, \triangleleft))$ $\quad (g(b) - g(a))d^{(+)}f(t) \triangleleft (f(b) - f(a))d^{(+)}g(t)$.

Here, $(\triangleleft)$ is one of the relations $\{\leq, >, \geq, <, =, \neq\}$. By a *generic* mean value property for (f, g) over $[a, b]$ we mean an evaluation of the formula $(P(t, \triangleleft))$ when t describes certain (generic) subsets of $]a, b[$. The standard cases are

(1D1) $\Big\|$ [the formula] $(P(t, \triangleleft))$ holds for countably (respectively uncountably) many t in $]a, b[$

(1D2) $\Big\|$ $(P(t, \triangleleft))$ holds for t in $]a, b[$ describing a zero Lebesgue measure (respectively, a nonzero Lebesgue exterior measure) part of it.

The logical equivalent of the first half of (1D1) will be referred to as

(1D3) [the formula] $(P(t, \triangleright))$ holds, for nearly all t in $]a, b[$;
while, the one of the first half of (1D2),as

(1D4) $(P(t, \triangleright))$ holds, for almost all t in $]a, b[$.

(Here, by definition, $(\triangleright)$ is the *negation* of $(\triangleleft)$). The following result involving our data is basic to developments below.

Theorem 1. *Suppose that*

(1H3) $g(a) < g(b)$.
Then, one of the alternatives below are true: either

(i) $(P(t, >))$ *holds for uncountably many* $t \in]a, b[\setminus A$*;*
or else

(ii) $(g(b) - g(a))(f(t) - f(a)) = (f(b) - f(a))(g(t) - g(a)), \forall t \in [a, b]$
(and then, $(P(t, =))$ *holds, for all* $t \in]a, b[\setminus A$*).*

A left counterpart of this statement, based on (1H2) replaced by
(1H4) the left derivatives $d^{(-)}f(t)$, $d^{(-)}g(t)$ exist, for $t \in]a, b[\setminus A$
is directly obtainable from the above one, by simply reversing the usual order of the real axis. Note that both (1H2) and (1H4) hold under
(1H5) $f'(t)$ and $g'(t)$ exist, for all $t \in]a, b[\setminus A$.

The stated result above seems to have being obtained for the first time by Flett [4,ch.1,Sect.6]. It may be viewed as a completion of the one due to Aziz and Diaz [2]; see also McLeod [6]. This, in the particular case of

(1H6) $g = i$ (=the identity function from R to itself)
is essentially related to the possibility of evaluating the size of the intermediary points appearing in mean value theorems (under an inequality form). So, further extensions of Theorem 1 – which, technically speaking, are naturally connected with (1H1) being removed – are not without interest. It is our main aim in this exposition to indicate such a device, in a half–continuity seting; details wil be given in Section 3. The basic tool to be used is a differential comparison principle, similar with the one in Turinici [9]; we refer to Section 2 for details. Finally, Section 4 is devoted to some vectorial aspects of the problem.

2. A differential comparison principle. Let $f : R \to R$ be a function. Denote, for $r, s \in R$, $r < s$

(2D1) $R_f(r, s) = f(s) - f(r)$, $Q_f(r, s) = \frac{1}{s-r} R_f(r, s)$

These will be referred to as the *incrementary* difference/quotient of f on the interval $[r, s]$. Put also, for simplicity,

$$R[+\infty] = R \cup \{+\infty\},\ R[-\infty] = R \cup \{-\infty\},\ R[\mp\infty] = R \cup \{-\infty, +\infty\}.$$

Fix some $t \in R$. Call $x \in R[\mp\infty]$, a right/left *derivative value* for f at t, when

$$\text{(2D2)} \quad \left\| \begin{array}{l} Q_f(t, s_n) \to x, \text{ for some } (s_n) \text{ with } s_n \downarrow t \\ (Q_f(r_n, t) \to x, \text{ for some } (r_n) \text{ with } r_n \uparrow t). \end{array} \right.$$

The set of all these will be denoted $\mathcal{D}^{(+)}f(t)$ (resp., $\mathcal{D}^{(-)}f(t)$). Put also

$$\text{(2D3)} \quad \left\| \begin{array}{ll} \Delta_{(+)}f(t) = \liminf_{s \to t+} Q_f(t, s), & \Delta^{(+)}f(t) = \limsup_{s \to t+} Q_f(t, s), \\ \Delta_{(-)}f(t) = \liminf_{r \to t-} Q_f(r, t), & \Delta^{(+)}f(t) = \limsup_{r \to t-} Q_f(r, t). \end{array} \right.$$

(the inferior/superior *Dini* right/left *derivative* of f at t). Note that

$$\text{(2.1)} \quad \left\| \begin{array}{l} \Delta_{(+)}f(t) = \inf \mathcal{D}^{(+)}f(t) \le \sup \mathcal{D}^{(+)}f(t) = \Delta^{(+)}f(t) \\ \text{(and similarly for the left part);} \end{array} \right.$$

when the extreme terms of this relation are equal, their common value is just

$$\left\| \begin{array}{l} d^{(+)}f(t) \text{=the right derivative of } f \text{ at } t \\ (d^{(-)}f(t) \text{=the left derivative of } f \text{ at } t). \end{array} \right.$$

Further, call $u \in R[\mp\infty]$, a right/left *oscillation value* for f at t, provided

$$\text{(2D4)} \quad \left\| \begin{array}{l} R_f(t, s_n) \to u, \text{ for some } (s_n) \text{ with } s_n \downarrow t \\ (R_f(r_n, t) \to u, \text{ for some } (r_n) \text{ with } r_n \uparrow t). \end{array} \right.$$

The set of all these will be denoted $\mathcal{E}^{(+)}F(t)$ (resp., $\mathcal{E}^{(-)}F(t)$). Put also

$$\text{(2D5)} \quad \left\| \begin{array}{ll} \Omega_{(+)}f(t) = \liminf_{s \to t+} R_f(t, s), & \Omega^{(+)}f(t) = \limsup_{s \to t+} R_f(t, s), \\ \Omega_{(-)}f(t) = \liminf_{r \to t-} R_f(r, t), & \Omega^{(-)}f(t) = \limsup_{r \to t-} R_f(r, t). \end{array} \right.$$

(the inferior/superior *Dini* right/left *oscillation* of f at t). Note that

$$\text{(2.2)} \quad \left\| \begin{array}{l} \Omega_{(+)}f(t) = \inf \mathcal{E}^{(+)}f(t) \le \sup \mathcal{E}^{(+)}f(t) = \Omega^{(+)}f(t) \\ \text{(and similarly for the left part);} \end{array} \right.$$

when the extreme terms of this relation are equal, their common value is just

$$\left\| \begin{array}{l} f(t+0) - f(t) \text{=the right oscillation of } f \text{ at } t \\ (f(t) - f(t-0) \text{=the left oscillation of } f \text{ at } t). \end{array} \right.$$

Let in the following f, g be a couple of functions from R to itself and $a, b \in R$, $a < b$, two points. Assume that

(2H1) $f - g$ is continuous from the left on $]a, b]$

(2H2) $\left\Vert \begin{array}{l} \text{for each } t \text{ in } [a,b[\text{, there exist compatible couples} \\ (u,v) \text{ in } \mathcal{E}^{(+)}f(t) \times \mathcal{E}^{(+)}g(t) \text{ with } u \neq +\infty,\ v \neq -\infty. \end{array} \right.$

(Here, by convention, $(u,v) \in \mathcal{E}^{(+)}f(t) \times \mathcal{E}^{(+)}g(t)$ is *compatible* when both u and v are introduced with the same sequence (s_n) with $s_n \downarrow t$). Let also A be some denumerable part of $]a, b[$ with

(2H3) $\left\Vert \begin{array}{l} \text{for each } t \text{ in } [a,b[\setminus A\text{, there exist compatible couples} \\ (x,y) \text{ in } \mathcal{D}^{(+)}f(t) \times \mathcal{D}^{(+)}g(t) \text{ with } x \neq +\infty,\ y \neq -\infty. \end{array} \right.$

Finally, let B stand for a zero Lebesgue measure part of $]a, b[$ which includes A. The following differential comparison principle will be useful for us.

Proposition 1. *Suppose that*

(2H4) $\left\Vert \begin{array}{l} \textit{for each } t \textit{ in }]a,b[\setminus B\textit{, at least one compatible couple} \\ (x,y) \textit{ taken as in } \text{(2H3)} \textit{ fulfils } x \leq y. \end{array} \right.$

(2H5) $\left\Vert \begin{array}{l} \textit{for each } t \textit{ in } [a,b[\textit{, at least one compatible couple} \\ (u,v) \textit{ taken as in } \text{(2H2)} \textit{ fulfils } u \leq v. \end{array} \right.$

Then, we necessarily have

(2.3) $f(b) - f(a) \leq g(b) - g(a)$.

Moreover, this property is valid over all subintervals of $[a, b]$*; i.e.,*

(2.4) $f - g$ *is decreasing over* $[a, b]$.

Before effectively developing the argument, we need some auxiliary facts, to be found, e.g., in Aumann [1,ch.7,Sect.2] and Flett [4,ch.1,Sect.10].

Lemma. *The following are valid*

(i) *for each denumerable part* A *of* $]a, b[$*, there exists an increasing function* $h = h_A$ *from* R *to itself, with* $h(t + 0) - h(t) > 0,\ \forall t \in A$*;*

(ii) *for each zero Lebesgue measure part* B *of* $]a, b[$*, there exists an increasing continuous function* $k = k_B$ *from* R *to itself, with* $k'(t) = \infty$*, for all* $t \in B$*.*

Proof of Prop.1. Let $\varepsilon > 0$ be arbitrary fixed and put

$$g_\varepsilon(t) = g(t) + \varepsilon(t + h(t) + k(t)), \quad t \in R.$$

Here, $h = h_A$ and $k = k_B$ are the functions from R to itself given by the Lemma. Define a new ordering $\preceq$ over $[a, b]$ by the convention

(2D6) $t \preceq s$ iff $t \leq s$ and $f(s) - f(t) \leq g_\varepsilon(s) - g_\varepsilon(t)$.

It is not hard to see, via (2H1), that

(2.5) $\left\|\begin{array}{l}\text{each ascending (modulo } \preceq) \text{ sequence in } [a,b] \\ \text{is a Cauchy one, bounded from above (modulo } \preceq).\end{array}\right.$

So, by the maximality principle in TURINICI [8], there exists, for the starting point c in $]a,b[$, some maximal (modulo $\preceq$) point r in $[a,b]$,with $c \preceq r$. We intend to show that $r = b$. Assume not (hence $r \in]a,b[$). For each t in $]r,b[$, a relation like $r \preceq t$ is impossible. So, necessarily,

(2.6) $f(t)-f(r) > g(t)-g(r)+\varepsilon[t-r+h(t)-h(r)+k(t)-k(r)],\ \forall t \in]r,b[.$

Now, three cases are open before us.

(i) r is outside B. Then, by (2.6),

$Q_f(r,t) > Q_g(r,t) + \varepsilon$, for all $t \in]r,b[$.

In particular, this must be valid for that common sequence (s_n) appearing in (2H4). So, taking the limit as $n \to \infty$ it follows that the derivative values x, y given by this condition are both finite and $x \geq y + \varepsilon > y$, contradiction (to (2H4)).

(ii) r is in $B \setminus A$. By (2.6) again,

$Q_f(r,t) > Q_g(t) + \varepsilon Q_k(t)$, for all $t \in]r,b[$.

This, in particular, must be valid for the common sequence (s_n) given by (2H3) (and yielding the couple of derivative values x, y with the precised properties). But then, passing to limit as $n \to \infty$ one gets (by the Lemma) $x \geq y + \varepsilon\infty = \infty$, contradiction (to (2H3)).

(iii) r belongs to A. We get, by (2.6),

$f(t) - f(r) > g(t) - g(r) + \varepsilon(h(t) - h(r))$, for all $t \in]r,b[$.

In particular, this must be true for the common sequence (s_n) given by (2H5). So, taking the limit as $n \to \infty$, it follows that the oscillation values u, v are both finite and $u \geq v + \varepsilon(h(r+0) - h(r)) > v$, contradiction (to (2H5)).

Having explored all possibilities, our claim follows; i.e.,

$f(b) - f(c) \geq g(b) - g(c) + \varepsilon(b - c + h(b) - h(c) + k(b) - k(c)),\ \forall \varepsilon > 0.$

So, if we let ε (> 0) tends to zero,

(2.7) $f(b) - f(c) \geq g(b) - g(c), \quad$ for any c in $]a,b[$.

Let (c_n) be the common sequence involved in the definition of a compatible couple of oscillation values (u,v), given by (2H5). Writing (2.7) for this sequence, and then taking the limit as $n \to \infty$ gives

(2.8) $f(b) - f(a) - u \leq g(b) - g(a) - v.$

This, by (2H2, shows that u, v are both finite; and, combining with $u \leq v$, conclusion (2.3) follows. For the last part, it will suffice noting that the general assumptions (2H1)–(2H3) as well as the specific hypotheses

(2H4)+(2H5) are hereditary with respect to the subintervals of $[a, b]$. Hence the result. ■

Now, as a completion of this, the following statement is available.

Proposition 2. *Let the above data be such that*

(2H6) *at least one of the inequalities in* (2H4) *and/or* (2H5) *is strict.*

Then, (2.3) *may be written in the stronger form*

(2.9) $f(b) - f(a) < g(b) - g(a)$.

Proof. Suppose this would be not true; i.e. (combining with (2.3))

(2H7) $f(b) - f(a) = g(b) - g(a)$.

This, via (2.4), gives a contradiction to (2H6). Hence the conclusion. ■

Some remarks are in order. The general condition (2H1) has the only purpose of deducing the property (2.5) (which, as above said, allows the application of the maximality principle in TURINICI [op.cit.]). In this direction we note that, a more general counterpart of it is

(2H8) $\Omega^{(-)}(f - g)(t) \leq 0, \quad$ for all $t \in]a, b]$.

On the other hand, the specific assumption (2H5) was imposed so as to handle the alternative **(ii)** above, as well as the last part of the argument. In this case, it is not difficult to verify that a more general way of getting the same conclusion is represented by

(2H9) $\Omega_{(+)}(f - g)(t) \leq 0, \quad$ for all $t \in [a, b[$.

Note that these are exactly the basic assumptions used in AUMANN [1,ch.7,Sect.2] to deduce a similar statement; see also Gal [5] and MIRICA [7]. But, for the developments below, this is not essential. A left counterpart of these developments is directly obtainable by reversing the natural order of the reals and passing to the functions

(2D7) $F(t) = f(-t), \quad G(t) = g(-t), \quad t \in R$.

Since the whole procedure is clear, we do not give details.

3. Main results. The information in the statements above consists, essentially, of two implications:

(2H4)+(2H5) $\Longrightarrow$ (2.3); (2H4)+(2H5)+(2H6) $\Longrightarrow$ (2.9).

These may now be exploited from a logical perspective. (For example, the former says that, whenever (2.3) is not true, then conditions (2H4), (2H5) cannot hold simultaneously; a similar conclusion is also valid for the latter). It is our aim in the following to make precise this fact. Let f, g be a couple of functions from R to itself, and fix $a, b \in R$, $a < b$. Assume (2H1) holds, as well as

(3H1) $\left\| \begin{array}{l} \text{for each } t \text{ in } [a, b[,\ \mathcal{E}^{(+)} f(t) \cap R[-\infty] \\ \text{is nonempty and } g(t+0) - g(t) \text{ exists in } R[+\infty]. \end{array} \right.$

Let also A be some denumerable part of $]a, b[$ with

(3H2) $\left\| \begin{array}{l} \text{for each } t \text{ in }]a, b[\setminus A,\ \mathcal{D}^{(+)} f(t) \cap R[-\infty] \neq \infty \\ \text{and } d^{(+)} g(t) \text{ exists in } R[+\infty]. \end{array} \right.$

Further, let $t \vdash D^{(+)} f(t)$ stand for a *selection* of the multivalued map $t \vdash \mathcal{D}^{(+)} f(t) \cap R[-\infty]$, in the sense

(3D1) $\left\| \begin{array}{l} D^{(+)} f(t) \in \mathcal{D}^{(+)} f(t) \cap R[-\infty], \text{ for } t \in]a, b[\setminus A \\ D^{(+)} f(t) = \text{an element of } R[\mp\infty], \text{ otherwise.} \end{array} \right.$

We also consider an extension $t \vdash d^{(+)} g(t)$ of the right derivative map attached to f, in the sense

(3D2) $\left\| \begin{array}{l} d^{(+)} g(t) = \text{the right derivative, for } t \in]a, b[\setminus A \\ d^{(+)} g(t) = \text{an element of } R[\mp\infty], \text{ otherwise.} \end{array} \right.$

Finally, we let $t \vdash E^{(+)} f(t)$ stand for a selection of the multivalued map $t \vdash \mathcal{E}^{(+)} f(t) \cap R[-\infty]$; and $t \vdash (g(t+0) - g(t))$, an extension of the right oscillation map attached to g. (The precise meaning of these are similar to the ones described by (3D1) and (3D2) respectively; we do not give details). These, in turn, make possible the consideration of the logical propositions

$(M(t, \lhd))$ $D^{(+)} f(t) \lhd d^{(+)} g(t)$

$(N(t, \lhd))$ $E^{(+)} f(t) \lhd g(t+0) - g(t)$.

(Here, as already precised, $\lhd$ is one of the relations $\{\leq, >, \geq, <, =, \neq, \}$). The main result of the present exposition is

Theorem 2. *Suppose that*

(3H3) $f(b) - f(a) \geq g(b) - g(a)$.

Then, one of the three alternatives below is true

(i) *relation* $(M(t, >))$ *holds over a subset of points* t *in* $]a, b[\setminus A$ *with nonzero Lebesgue exterior measure*

(ii) *relation* $(N(t, >))$ *holds for at least one* t *in* $[a, b[$*;*

or else

(iii) $f - g$ *is constant over* $[a, b]$ *(hence* $(M(t, =))$ *holds over all* t *in* $]a, b[\setminus A$*, and* $(N(t, =))$ *holds over all* t *in* $[a, b[$*).*

Proof. Suppose that both **(i)** and **(ii)** were false. The negation of **(i)** is just the specific assumption (2H5), if we take (3H1) into account. And the negation of **(i)** tells us that the subset

$C = \{t \in]a,b[\setminus A;\ \text{the formula } (M(t,>)) \text{ is true }\}$

has zero Lebesgue measure; so necessarily, the specific assumption (2H4) is also true (with $B = A \cup C$), if we take (3H2) into account. Summing up, Proposition 1 is aplicable to our data. As a consequence, (2.4) is necessarily true; and this, combined with (3H3), shows that the last alternative **(iii)** must be valid. Hence the result. ∎

A basic situation described by these developments is the one below. Let f, g be a couple of functions from R to itself and fix $a, b \in R$, $a < b$. Assume the basic condition (2H1) is in use, as well as (3H1)+(3H2). Finally, let the selection maps $t \vdash D^{(+)}f(t)$, $t \vdash E^{(+)}f(t)$ and the extension maps $t \vdash d^{(+)}f(t)$, $t \vdash (g(t+0) - g(t))$ be introduced as before.

Theorem 3. *Under the precised hypotheses, one of the three alternatives below is true*

(i) $(g(b) - g(a))d^{(+)}f(t) > (f(b) - f(a))d^{(+)}g(t)$, *over a subset of* t *in* $]a,b[\setminus A$ *with nonzero Lebesgue exterior measure*

(ii) $(g(b) - g(a))E^{(+)}f(t) > (f(b) - f(a))(g(t+0) - g(t))$, *for at least one* t *in* $[a,b[$ *;*

or else

(iii) $(g(b) - g(a))(f(t) - f(a)) = (f(b) - f(a))(g(t) - g(a))$, $\forall t \in [a,b]$ *(hence,* **(i)** *and* **(ii)** *hold with equality for all admissible* t*).*

Proof. Define a new couple F, G of functions from R to itself as

(3D3) $F(t) = (g(b) - g(a))f(t),\ \ G(t) = (f(b) - f(a))g(t),\ t \in R.$

The general conditions (2H1) and (3H1)+(3H2) are holding for this couple, as well as the specific assumption (3H3). In other words,Theorem 2 applies to these data. And, from the conclusion of that result, we are done. ∎

Now, an interesting particular case of the described situation is that of the function g being taken as in (1H6). Precisely, let $f : R \to R$ be a function and fix $a, b \in R$,$a < b$. Assume

(3H4) f is continuous from the left on $]a,b]$

(3H5) $E^{(+)}f(t) \cap R[-\infty]$ is nonempty, for each t in $[a,b[$.

Also, let A be some denumerable part of $]a,b[$ with

(3H6) $\mathcal{D}^{(+)}f(t) \cap R[-\infty]$ is nonempty, for each t in $]a,b[\setminus A$.

Finally, take the selections $t \vdash D^{(+)}f(t)$ and $t \vdash E^{(+)}f(t)$ of the multivalued maps $t \vdash \mathcal{D}^{(+)}f(t) \cap R[-\infty]$ and $t \vdash \mathcal{E}^{(+)}f(t) \cap R[-\infty]$ respectively.

Theorem 4. *Under these assumptions, one of the three alternatives below is true*

(i) $D^{(+)}f(t) > Q_f(a,b)$*, over a subset of* t *in* $]a,b[\setminus A$ *with nonzero Lebesgue exterior measure*

(ii) $E^{(+)}f(t) > 0$*, for at least one* t *in* $[a,b[$*;*

or else

(iii) f *is linear over* $[a,b]$ *(hence, the relations in* **(i)** *and* **(ii)** *hold with equality for all admissible* t*).*

Finally, as a by–product of these, we have the following synthetic fact:

Theorem 5. *Let the functions* (f,g) *be as in Theorem 3 with, in addition,*

(3H7) $(g(b)-g(a))E^{(+)}f(t) \le (f(b)-f(a))(g(t+0)-g(t)), \quad \forall t \in [a,b[.$

Then, necessarily,

(3.1) $\|$ $(g(b)-g(a))D^{(+)}f(t) \ge (f(b)-f(a))d^{(+)}g(t)$*, over a subset of* t *in* $]a,b[\setminus A$ *with nonzero Lebesgue exterior measure.*

Hence, in particular, if (f,g) *are taken as in Theorem 4, with*

(3H8) $E^{(+)}f(t) \le 0, \quad$ *for all* $t \in [a,b[$

then we must have

(3.2) $\|$ $D^{(+)}f(t) \ge Q_f(a,b)$*, over a subset of* t *in* $]a,b[\setminus A$ *with nonzero Lebesgue exterior measure.*

For example, the extra–assumption (3H7) is fulfilled under (1H1). In this case, the obtained result is reductible to the one described by Theorem 1 (due, as above said, to Flett [4,ch.1,Sect.6]). Likewise, the extra–assumption (3H8) is necessarily holding under

(3H9) f is continuous over $[a,b]$.

And then, the derived fact may be viewed as a completion of the one due to Aziz and Diaz [2]. Finally, a left counterpart of these statements is immediately obtainable by the remarks made at the end of the preceding section; we do not give further details.

4. Some vectorial aspects. Let X be a (real) toplogical vector space,and $F : R \to X$, a function. Also, take some t in R. Call the vector $x \in X$, a *right derivative value* for F at t, provided

(4D1) $Q_F(t, s_n) \to x$, for some (s_n) with $s_n \downarrow t$.
The set of all such elements will be denoted $\mathcal{D}^{(+)}F(t)$. Note that
(4.1) $\mathcal{D}^{(+)}F(t) = \{d^{(+)}F(t)\}$, when the right derivative exists.
Likewise, call the vextor $u \in X$, a *right oscillation value* for F at t, when
(4D2) $R_F(t, s_n) \to u$, for some (s_n) with $s_n \downarrow t$.
The set of all such elements will be denoted $\mathcal{E}^{(+)}F(t)$. Note that
(4.2) $\mathcal{E}^{(+)}F(t) = \{F(t+0) - F(t)\}$, when the right oscillation exists.
It is to be added here that the left counterpart of these notions are to be introduced in a dual (ordering) manner.

Further, let $p : X \to R$ be a functional. We call it *sublinear* when
(4D3) $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = \lambda p(x)$, $\lambda \in R_+$, $x, y \in X$;
i.e., p is *subadditive* and *positively homogeneous*. Note that, the continuity over X of p is equivalent with its continuity at the origin (of X); see, for instance, Bourbaki [3,ch.2,Sect.2].

We are now passing to the effective part of our developments. Let $F : R \to X$ and $g : R \to R$ be two functions. Fix $a, b \in R$, $a < b$, and assume
(4H1) F, g are continuous from the left on $]a, b]$
(4H2) $\Big\|$ g is strictly increasing on $[a, b]$ (hence, for each t in $[a, b[$, the right oscillation $g(t+0) - g(t)$ exists (in R)); as well as
(4H3) for each t in $[a, b[$, $\mathcal{E}^{(+)}F(t)$ is nonvoid (as a subset of X).
In addition, let the denumerable part A of $]a, b[$ be such that
(4H4) $\Big\|$ for each $t \in]a, b[\setminus A$, $\mathcal{D}^{(+)}F(t)$ is nonvoid (as a part of X) and $d^{(+)}g(t)$ exists (in $R[+\infty]$).
Let $t \vdash D^{(+)}F(t)$ stand for a selection of the multivalued map $t \vdash \mathcal{D}^{(+)}F(t)$. Likewise, take a selection $t \vdash E^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{E}^{(+)}F(t)$. For example, noting that (by (4H4))
(4.3) $0 \in \mathcal{E}^{(+)}F(t)$, for all $t \in]a, b[\setminus A$,
a concrete selection of this type (denoted $e_A^{(+)}F$) may be defined as
(4D5) $e_A^{(+)}F(t) = 0$, if $t \in]a, b[\setminus A$; $e_A^{(+)}F(t) \in \mathcal{E}^{(+)}F(t)$, otherwise.
Note that, by (4H2), the quantities
(4D6) $Q_{F,g}(r, s) = R_F(r, s)/R_g(r, s)$, $a \leq r < s \leq b$
(referred to as the *incrementary quotients* of (F, g) on the interval $[r, s]$) are well defined (as elements of X). More information about these is to be derived from

Theorem 6. *Suppose that*

(4H5) *p is sublinear and continuous.*

Then, one of the three alternatives below is true

(i) $p(E^{(+)}F(t)) > p(Q_{F,g}(a,b))(g(t+0)-g(t))$*, for at least one t in $[a,b[$*

(ii) $p(D^{(+)}F(t)) \geq p(Q_{F,g}(a,b))d^{(+)}g(t)$*, over a subset of $t \in]a,b[\setminus A$ with nonzero Lebesgue exterior measure;*

or else

(iii) $p(Q_{F,g}(r,s)) = p(Q_{F,g}(a,b))$*, for all (r,s) with $a \leq r < s \leq b$ (hence* **(i)** *and* **(ii)** *hold with equality for all possible t).*

Proof. Suppose that the alternative **(i)** were false:

(4H6) $p(E^{(+)}F(t)) \leq p(Q_{F,g}(a,b))(g(t+0)-g(t)), \quad \forall t \in [a,b[;$

as well as **(ii)**. We claim that in this case, the existence of a relation like

(4H7) $p(Q_{F,g}(a,b)) < p(Q_{F,g}(r,s))$, for some (r,s) with $a \leq r < s \leq b$

is contradictory. Indeed, let x^* be some linear continuous functional over X with

(4H8) $x^*(Q_{F,g}(r,s)) = p(Q_{F,g}(r,s)); \quad x^*(x) \leq p(x), \ x \in X.$

(The existence of such an object follows at once from the Hahn–Banach theorem; see for instance, Bourbaki [3,ch.2,Sect.3]). Define

(4D6) $f(t) = x^*(F(t)), \quad t \in R.$

This function satisfies (3H4) (on $[r,s]$), as it can be directly seen. Moreover, by

(4.4) $x^*(\mathcal{E}^{(+)}F(t)) \subseteq \mathcal{E}^{(+)}f(t) \cap R, \quad t \in [a,b[,$

(4.5) $x^*(\mathcal{D}^{(+)}F(t)) \subseteq \mathcal{D}^{(+)}f(t) \cap R, \quad t \in]a,b[\setminus A,$

it also fulfils conditions (3H5)+(3H6)(on $[r,s]$). This, combined with (4H6), shows that Theorem 5 (the first half) is applicable to these data on the subinterval $[r,s]$. So, by the conclusion of that result

(4.6) $\Big\|$ $x^*(D^{(+)}F(t)) \geq x^*(Q_{F,g}(r,s))d^{(+)}g(t)$, over a subset of t in $]r,s[\setminus A$ with nonzero Lebesgue exterior measure.

But then, the choice (4H8) of x^* and the specific assumption (4H7) yield **(ii)**, contradiction. In other words, we must have an evaluation like

(4.7) $p(Q_{F,g}(r,s)) \leq p(Q_{F,g}(a,b))$, when $a \leq r < s \leq b$.

If one of these relations would be strict then in view of

$$Q_{F,g}(a,b) \leq \alpha Q_{F,g}(a,r) + \beta Q_{F,g}(r,s) + \gamma Q_{F,g}(s,b),$$

where

$$\alpha = \frac{g(r)-g(a)}{g(b)-g(a)},\quad \beta = \frac{g(s)-g(r)}{g(b)-g(a)},\quad \gamma = \frac{g(b)-g(s)}{g(b)-g(a)},$$
one has, by the sub–linearity condition (4H5),

$p(Q_{F,g}(a,b)) < p(Q_{F,g}(a,b))$, contradiction.

Hence, we must have equality in (4.7); and this proves **(iii)**. As a consequence,

$p(Q_F(r,s)) = p(Q_{F,g}(a,b))Q_g(r,s)$, for all (r,s) with $a \le r < s \le b$;
and from this, the last part follows by a limiting process. ■

Remark. Suppose that, in addition,

(4H9) g is continuous over $[a,b]$.
Then, the alternative **(i)** of Theorem 6 may be written as

(i*) $p(E^{(+)}F(t)) > 0$, for at least one $t \in [a,b[$;
and the quoted result includes the one in Flett [4,ch.1,Sect.6],to which it reduces when (4H1) is substituted by (1H1) (with $f = F$).

Now, the written extra–condition (4H9) is fulfilled whenever the function g is taken as in (1H6). An interesting application of this fact is to be given under

(4H10) X is a (real) locally convex space.
Precisely, let $F : R \to X$ be a function and fix $a,b \in R$, $a < b$. Assume that (4H1) is true (relative to F) as well as (4H3). Likewise, let the denumerable part A of $]a,b[$ be such that (4H4) (relative to F) is valid. Take the selections $t \vdash D^{(+)}F(t)$ and $t \vdash E^{(+)}F(t)$ of the multivalued maps $t \vdash \mathcal{D}^{(+)}F(t)$ and $t \vdash \mathcal{E}^{(+)}F(t)$ respectively. Finally, let B stand for a zero Lebesgue measure part of $]a,b[$ which includes A.

Theorem 7. *Let the convex part Z of X be such that*

(4H11) $D^{(+)}F(t) + \xi E^{(+)}F(s) \in Z$, *for all* $t \in]a,b[\setminus B$, $s \in [a,b[$, $\xi \ge 0$.

Then, necessarily,

(4.8) $Q_F(a,b)$ *is adherent to* Z.

Moreover, if

(4H12) *Z has a nonempty interior (in X)*

it follows that one of the alternatives below is true

(j) $Q_F(a,b)$ *is interior to* Z

(jj) $\{Q_F(r,s); a \le r < s \le b\}$ *is included in each closed hyperplane which supports* Z *at* $Q_F(a,b)$.

Proof. Let x^* be a continuous linear functional over X and λ, some real number with

(4H13) $x^*(z) \le \lambda$, for each $z \in Z$.

By a standard reasoning one derives, via (4H11) above

(4.9) $x^*(E^{(+)}F(t) \le 0$, for all $t \in [a, b[$

(4.10) $x^*(D^{(+)}F(t) \le \lambda$, for all $t \in]a, b[\setminus B$.

The former of these shows that the negation of the alternative **(i)** in Theorem 6 is realized, with $p = x^*$ (and $g = i$, as above said). Hence, the alternatives **(ii)** or **(iii)** of that result must be true; and from this (combined with (4.10))

(4.11) $x^*(Q_F(a, b)) \le \lambda$.

Summing up, (4H13) implies (4H12). And then, (4.8) follows by a well known characterization of closed convex sets (cf. Bourbaki [3,ch.2,Sect.4]). For the second part, assume the alternative **(j)** above would be false; i.e., (by (4.8))

(4H14) $Q_F(a, b)$ is a boundary point of Z.

Let $Y = \{x \in X; x^*(x) = \lambda\}$ stand for a supporting hyperplane for Z at $Q_F(a, b)$; i.e.,

(4.12) $x^*(x) \le \lambda = x^*(Q_F(a, b))$, $x \in Z$.

By a previous remark, (4.9)+(4.10) must be true in this case; so (by the argument we just developed)

(4.13) $x^*(Q_F(r, s)) \le \lambda$, when $a \le r < s \le b$.

This, along with the convexity method of Theorem 6, yields (for all such (r, s))

(4.14) $x^*(Q_F(r, s)) = \lambda$ (hence $Q_F(r, s) \in Y$)

and **(jj)** follows. ∎

In particular, (4H11) is fulfilled for

(4D7) $Z = \mathrm{co}\{D^{(+)}F(t); t \in]a, b[\setminus B\} + \mathrm{cone}\{E^{(+)}F(t); t \in [a, b[\}$.

(Here, co and cone stand for the *convex* and respectively, *conical* cover). Therefore, (4.8) must hold (with Z above). In addition, suppose that (4H12) is valid (for this Z). If the alternative **(jj)** would be valid then, by a limit process,

$Z \subseteq Y$, for all these (support hyperplanes) Y;

which, in view of $\mathrm{int}(Y) = \emptyset$, yields a contradiction to the assumption we just made. So, necessarily, the alternative **(j)** must be valid for our data.

This is nothing but the main result in TURINICI [9]. On the other hand, when the structural condition (4H1) is substituted by (1H1) (with $f = F$) the statement above is just the one due to FLETT [op.cit.]; see also MCLEOD [6]. Finally, a left counterpart of these is to be obtained by the construction indicated at the end of Section 3; we do not give further details.

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