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THE HYPERSPACIAL TOPOLOGIES AND ALMOST LINEAR SPACES

BY

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1. Introduction. Let X be a linear space over the field of scalars $\mathbb{K}$ and $\mathcal{P}(X)$ the family of nonempty subsets of X. The study of the algebraic operations $(A, B) \longmapsto A+B$ and $(\lambda, A) \longmapsto \lambda A$, with $\lambda \in \mathbb{K}$, on $\mathcal{P}(X)$ show that the most axioms from the definition of the linear space are verified (in fact, without the existence of the symmetrical element and the distributivity with respect to the sum of scalars).

This suggest us to introduce a new notion, namely "almost linear space" (a.l.s.);if we endow the space X with a topology,we can topologize the family $\mathcal{P}(X)$ with certain topologies, as Hausdorff, lower Vietoris or proximal topologies.We wanted the operations of almost linear space to be compatible with the topology on $\mathcal{P}(X)$.This is the source of the notion "almost linear topological space" (a.l.t.s.).

The aim of the present work is to study different properties of the a.l.t.s. (introduce by the author in [1]), following a parallel between these spaces and linear topological spaces.

The notions and the notations in the work will be recall in the Section 2. We also give the abstract notions of a.l.s., a.l.t.s. and some examples.

The Section 3 is dedicated to the translation of a topology (or the translation topology) on a a.l.t.s., emphasing necessary and sufficient conditions for the translation topology to be almost linear (Theorem 3.2, Proposition 3.2).

2. Terminology and notations. We denote by X a linear topological space;let be

$$\mathcal{P}b(X) = \{A \in \mathcal{P}(X);\ A \text{ is bouded}\};$$
$$\mathcal{D}(X) = \{A \in \mathcal{P}(X);\ A \text{ is open}\};$$
$$\mathcal{K}(X) = \{A \in \mathcal{P}(X);\ A \text{ is compact}\};$$
$$\mathcal{K}^w(X) = \{A \in \mathcal{P}(X);\ A \text{ is } w\text{–compact}\},$$

where by w we denote the weak topology on X.

Now let X be a normed space.We write

$S_\varepsilon(A) = \{x \in X;\ \exists\, a \in A \text{ such that } d(x,a) < \varepsilon\},\ A \subset X,\ \varepsilon > 0,$
for the *ε–enlargement of the set A*.

The Husdorff topology τ_H ([3], [4] or [7]) on $\mathcal{P}b(X)$ is the topology induced by the Pompeiu-Hausdorff semi-metric

$$H(A,B) = \max\{e(A,B).e(B,A)\},$$

where $e(A,B) = \sup\{d(a,B);\ a \in A\}$ is *the Hausdorff excess* of A over B.

Equivalently, τ_H on a subfamily $\mathcal{A}$ is the supremum of two topologies, namely the lower Hausdorff topology and upper Hausdorff topology ([9]): $\tau_H = \tau_H^- \vee \tau_H^+$, where

τ_H^- is generated by all the sets of form
$U_-(A_0,\varepsilon) = \{A \in \mathcal{A};\ A_0 \subset S_\varepsilon(A)\}$
with A_0 from $\mathcal{A}$ and ε positively number;

τ_H^+ is generated by all the sets of form
$U_+(A_0,\varepsilon) = \{A \in \mathcal{A};\ A \subset S_\varepsilon(A_0)\}$
with A_0 from $\mathcal{A}$ and ε positively number.

The lower Vietoris tipology τ_V^- ([9]) on $\mathcal{P}(X)$ is generated by all the sets of form
$V^- = \{A \in \mathcal{A};\ A \cap V \neq \emptyset\}$, with V open in X.

The proximal topology τ_P ([12], [13]) on $\mathcal{P}b(X)$ is $\tau_P = \tau_V^- \vee \tau_H^+$.

Analyzing the behavior of the τ_V^- and τ_H topologies concerning different set-operations, we retain especially the properties with algebraic character. Actually,regarding their allowance to the sum and multiplication with scalars, we might accept as a model the $\tau_H^-, \tau_H^+, \tau_H, \tau_V^-$ and, of course, $\tau_P = \tau_V^- \vee \tau_H^+$.

As for the definition of an algebraic structure on classes of sets there are some results which imply the family of bounded closed convex subsets of X, with the Hausdorff topology, taken as a abstract convex cone of certain linear topological space. This construction is due to MINKOVSKI, RADSTRÖM and HÖRMANDER (see [11] or [5] for example).

In the following we would like to introduce otherwise (in a different way) an algebraic structure on the family $\mathcal{P}(X)$ and on its subfamilies. We consider the following abstract definition:

Definition 2.1. Let L be a nonempty set, $\mathbb{\Gamma}$ a scalar field, "$+$": $L\times L\longrightarrow L$ and "$\cdot$"$\mathbb{\Gamma}\times L\longrightarrow L$ two operations satisfying the axioms:

S1. $(x+y)+z=x+(y+z),\ \forall x,y,z\in L$;

S2. there exists an unique element $0\in L$ such that $x+0=0+x=x,\ \forall x\in L$;

S3. $x+y=y+x,\ \forall x,y\in L$;

S4. $\lambda(\mu x)=(\lambda\mu)x,\ \forall\lambda,\mu\in\mathbb{\Gamma},\ \forall x\in L$;

S5. $1\cdot x=x,\ \forall x\in L$;

S6. $\lambda(x+y)=\lambda x+\lambda y,\ \forall\lambda\in\mathbb{\Gamma},\ \forall x,y\in L$.

S7. $0\cdot x=0,\ \forall x\in L$.

In this situation we say that $(L,+,\cdot)$ is un *almost linear space*.

As a immediate consequence of the axioms from this definition, it results:

Proposition 2.1. *If $(L,+,\cdot)$ is an almost linear space, then $\lambda\cdot 0=0$, for all $\lambda\in\mathbb{\Gamma}$.*

Proof: If $\lambda=0$, the property results from the axiom S7. In converse case, let $x\in X$ be arbitrarely; from the axioms S2 and S4 we obtain that $\lambda(0+\lambda^{-1}x)=x$, and from S6 we find $\lambda(0+\lambda^{-1}x)=\lambda\cdot 0+x$. From the unicity of zero element with respect to sum it follows that $\lambda\cdot 0=0$. ∎

We recall that if on the set L only the sum is defined and this verifies the axioms S1–S3, the structure $(L,+)$ is called *abelian semigroup*.

Let Γ be a scalar field.Notice that every abelian semigroup can be considered as a trivial almost linear space over $\mathbb{I}\Gamma$ if we endow it with the following multiplication with scalars:

$$\cdot : \mathbb{I}\Gamma \times L \longrightarrow L,\ \lambda x = \begin{cases} x, & \text{if } \lambda \neq 0 \\ 0, & \text{if } \lambda = 0 \end{cases} \tag{2.1}$$

as in the case of the family $\mathcal{L}(X)$ of the linear subspaces of a linear space X; yet this situation doesn't strayus to much from the notion of semigroup.

There are also untrivial examples of almost linear spaces among the classes of nonempty subsets of a linear space, such as the class of the affine sets, the classes of the convex and the balanced subsets; or among those of a topological linear space, as $\mathcal{P}b(X)$, $\mathcal{K}(X)$, or $\mathcal{K}^w(X)$.

Obviously, $\mathcal{P}(X) \supset \mathcal{P}b(X) \supset \mathcal{K}(X)$, so $\mathcal{P}b(X), \mathcal{K}(X)$, respectively $\mathcal{K}^w(X)$, ar almost linear subspaces of the a.l.s. $(\mathcal{P}(X), +, \cdot)$.

Definition. The structure $(L, +, \cdot, \tau)$ is called almost linear topological space if $(L, +, \cdot)$ is an almost linear space, and the operations "+": $L \times L \longrightarrow L$ and "$\cdot$": $\mathbb{I}\Gamma \times L \longrightarrow L$ are both continuous in the topology τ.

Coming back to the example of the abelian semigroup with the operation (2.1), we see that the multiplication with scalars is continuous to the points $(\lambda_0, 0)$ for every topology τ; nevertheless the continuity of the same operation to the points of form $(0, x_0)$, with $x_0 \neq 0$, leed us to the condition

$$\forall x_0 \in L,\ \forall V \text{ a origin neighbourhood, we have } x_0 \in V,$$

which tell us that τ can not be but the coarse topology. As the sum is continuous in this topology, it follows that the only almost linear topology is the coarse one, on the abelian semigroupes endowed with the multiplication with scalars given by (2.1).

Hence the following remark:

If we have an almost linear topology τ on a family of sets that contains the subfamily $\mathcal{L}(X)$, τ is not T_2–separated because the restriction of τ at $\mathcal{L}(X)$ is the coarse topology.

When we want to work with separated topologies, a more adequate family might be $\mathcal{P}b(X)$.

We also have other (untrivial) examples of a.l.t.s. (see [1], Theorem 4.3 and Corollary 4.2):

Let X be linear normed space. The following spaces are almost linear topological:

1) $(\mathcal{P}(X), +, \cdot, \tau_V^-)$;

2) $(\mathcal{P}b(X), +, \cdot, \tau_H^-)$;

3) $(\mathcal{P}b(X), +, \cdot, \tau_H^+)$;

4) $(\mathcal{P}b(X), +, \cdot, \tau_H)$;

5) $(\mathcal{P}b(X), +, \cdot, \tau_P)$.

In consequence, $(\mathcal{K}^w(X), +, \cdot, \tau)$ and $(\mathcal{K}(X), +, \cdot, \tau)$ are almost linear topological subspaces of $(\mathcal{P}b(X), +, \cdot, \tau)$, where τ is one of five above topologies.

3. The translation of a topology on an almost linear space. In the following, we estabish some results concerning the fundamental origin neighbourhoods system for a.l.t.s., similar to those in the case of linear topological spaces:

We first give some definitions on an a.l.t.s. $(L, +, \cdot)$, known from the case of linear spaces :

Definition 3.1. A subset $A \subset L$ is called *absorbent* if $\forall x \in L \ \exists \ \lambda > 0$ such that $\lambda x \in A$.

Definition 3.2. A subset $A \subset L$ is called *balanced* if $\forall \lambda A \subseteq A$, $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq 1$.

Definition 3.3. By *the balanced involving of a set* $A \subset L$ (denoted $\mathcal{E}(A)$) we mean the intersection of all balanced sets of L which contain A.

Remark 3.1. The following formula from the case of the linear spaces for the balanced involving of the set A is holds:

$$\mathcal{E}(A) = \bigcup_{|\lambda| \leq 1} \lambda A.$$

As the linear (respectively the linear topological) spaces are particular cases of a.l.s. (respectively a.l.t.s.) many properties of linear (topological) spaces are valid. A specially distinction from the two types of spaces consists in the fact that on a.l.t.s., the translations are not necessarily inversable (because the elements of $(L, +)$ doesn't admit, generally, a symmetric). So, if $\mathcal{V}(x_0)$ is the system of the neighbourhoods of a point x_0, it is possible that

$-x_0 + \mathcal{V}(x_0)$ be not the system of the neighbourhoods of the origin. Also, for an almost linear topology, a fundamental system of neighbourhoods for a point x_0 isn't, generally, the translation with x_0 of a fundamental system of neighbourhoods for the origin.

About the omotheties, we make the following remark:

Let $(L, +, \cdot, \tau)$ be an a.l.t.s. over the scalar field $\mathbb{K}$. If $\lambda_0 \neq 0$ then $x \to \lambda_0 x$ is bicontinuous, so V is a neighbourhood for a point x_0 iff $\lambda_0 V$ is neighbourhood for $\lambda_0 x_0$. Particularly, if V is a τ–neighbourhood of the origin in L, and $\lambda_0 \in \mathbb{K}\backslash\{0\}$, then $\lambda_0 V$ is a τ–neighbourhood of the origin, too.

Now we recall the axioms satisfied by any origin fundamental system $\mathcal{V}(0)$ of neighbourhoods :

(V1) $\forall V_1, V_2 \in \mathcal{V}(0)\ \exists\, V_3 \in \mathcal{V}(0)$ such that $V_3 \subset V_1 \cap V_2$;

(V2) $\forall V \in \mathcal{V}(0)\ \exists\, V_1 \in \mathcal{V}(0)$ such that $V_1 + V_1 \subset V$;

(V3) $\forall V \in \mathcal{V}(0),\ V$ is absorbent;

(V4) $\forall V \in \mathcal{V}(0)\ \exists\, V_1 \in \mathcal{V}(0)$ such that $\mathcal{E}(V_1) \subset V$.

We also consider the axiom:

(V0) $0 \in V$ for any $V \in \mathcal{V}(0)$.

Adopting the same proof as in the case of a linear topological space, we have the following theorem:

Theorem 3.1. *If $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin in an a.l.t.s. L, then $\mathcal{V}(0)$ satisfied the axioms* (V1)–(V4).

Let be now σ an almost linear topology on L and $\mathcal{V}(0)$ a fundamental system of neighbourhoods of the origin in this topology. We might wonder if there exists a topology τ such that, for any $x \in L$, the family $x + \mathcal{V}(0)$ could be a fundamental system of neighbourhoods for x.

If the answer is affirmatively, the system of neighbourhoods of x in the topology τ should be:

$$\mathcal{U}(x) = \{U \subset L;\ \exists\, V \in \mathcal{V}(0) \text{ such that } x + V \subset U\}. \tag{2.2}$$

As $\mathcal{V}(0)$ satisfies the axioms (V1)–(V4) (in fact, (V0), (V1), (V2)) are sufficient to be satisfied), it is easy to verify that we have generated a g topology

τ on L, in which a fundamental system of neighbourhoods of x is the desired system. Even more, $\sigma \leq \tau$.

Definition 3.4. Let σ be a topology on an a.l.t.s. and $\mathcal{V}(0)$ a fondamental system of neighbourhoods of the origin, satisfying (V0)–(V2). We define the *translation* of the topology σ ,as the topology τ obtained from σ by the construction (2.2).

Generally, these two topologies are different, as it result from:

Example 3.1. We endow $\mathcal{P}b(\mathbb{R})$ with the topologies τ_H^+ and its translation τ. We consider in $\mathcal{P}b(\mathbb{R})$ the set

$$A_0 = \{a_1, a_2\}, \text{ where } a_1 \neq a_2.$$

Let be $\varepsilon_0 > 0$ arbitrarily fixed. For every $\delta > 0$, the set $B_\delta = \{a_1 + \delta\} \in \mathcal{P}b(\mathbb{R})$ satisfies the relation $B_\delta \in B_{H^+}(A_0, \delta)$ (where $B_{H^+}(A_0, \delta)$ denotes the ball with center A_0 and radius δ in the topology τ_H^+) :

$$e(B_\delta, A_0) = \sup_{b \in B} \inf_{a \in A_0} |b - a| = \min\{\delta, |a_1 - a_2 + \delta|\} \leq \delta.$$

Let be now $U \in \mathcal{P}b(\mathbb{R})$ a nonempty set from the ball $B_{H^+}(0, \varepsilon_0)$. Obviously, $B_\delta \neq A_0 + U$, as the set $A_0 + U$ contains at least two distinct elements, while the set B_δ contains only one element. We thus found, for A_0, ε_0 and for every $\delta > 0$ a set $B_\delta \in B_{H^+}(A_0, \delta)$ such that $B_\delta \notin A_0 + B_{H^+}(0, \varepsilon_0)$. This tell us that

$$B_{H^+}(A_0, \delta) \not\subset A_0 + B_{H^+}(0, \varepsilon_0).$$

Taking into accounts the relation $\tau_H^+ \leq \tau$ it follows that τ is finner that τ_H^+.

Even more, the translation topology τ might not be almost linear even the topology σ is almost linear,as we observe by the

Example 3.2. Let be $A_n = \left[-\frac{n+1}{n}, \frac{n+1}{n}\right]$ a set in the space $\mathcal{P}b(\mathbb{R})$ endowed with the translation τ of the topology τ_H^+. If we denote $A_0 = [-1, 1]$, we may easily observe that $A_n \xrightarrow{\tau} A_0$:

$$\forall \varepsilon > 0 \; \exists \; n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1 \text{ such that } \forall n \geq n_\varepsilon \text{ we have } A_n \in A_0 + B_{H^+}(0, \varepsilon)$$

(that is, $A_n = A_0 + B_n$, with $B_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ verifying the relation

$$e(B_n, 0) = \sup_{b \in B_n} |b| = \frac{1}{n} < \varepsilon).$$

Let be now the sequence $\lambda_n = \left(\frac{n}{n+1}\right)^2$, convergent at $\lambda_0 = 1$. We want to show that $\lambda_n A_n \overset{\tau}{\not\longrightarrow} \lambda_0 A_0$. Let be $\varepsilon_0 > 0$ fixed. Then, for every $n \in \mathbb{N}^*$ we have $\lambda_n A_n \not\subset \lambda_0 A_0 + B_{H^+}(0, \varepsilon_0)$:
Otherwise, let be $B_n \in \mathcal{P}b(\mathbb{R})$ with $B_n \in B_{H^+}(0, \varepsilon_0)$ such that $\lambda_n A_n = \lambda_0 A_0 + B_n$, that is:

$$\left[-\frac{n}{n+1}, \frac{n}{n+1}\right] = [-1, 1] + B_n.$$

If B_n contained an element $b \geq 0$, then $1 + b \in \lambda_0 A_0 + B_n$, but $1 + b \notin \lambda_n A_n$. An analogous reasoning is made if B_n contained an element $b < 0$.

Remark 3.2. In fact, in the previous example it was no essential intervention of the origin neighbourhoods form in the topology τ_H^+. We may conclude that, on $\mathcal{P}(\mathbb{R})$ (and other subclasses containing intervals) no translation of any almost linear topology represents an almost linear topology, because the multiplication with scalars is not continuous. Adapting the example to the case of a linear normed space X ($A_n = \left\{\frac{n+1}{n} t e_j;\ t \in [-1, 1]\right\}$, e_j vector of a base of X, $A_0 = \{t e_j;\ t \in [-1, 1]\}$ and $\lambda_n = \left(\frac{n}{n+1}\right)^2$) the result still remains valid for any topology on $\mathcal{P}(X)$.

Nevertheless, in the case of an abstract almost linear space (L, σ), we will give a necessary and sufficient condition satisfied by the family $\mathcal{V}(0)$ such that the translation topology τ be almost linear; in this condition, which is obviously stronger then the hypothesis (V1)–(V4)), τ will be the finest almost linear topology on L , for whom $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin.

Proposition 3.1. *Let be an a.l.s. L and $\mathcal{V}(0) \subset \mathcal{P}(L)$ a nonempty family, such that there exists a topology τ with the property othat for any $x \in L$, the family $x + \mathcal{V}(0)$ is a fundamental system of neighbourhoods of x in the topology τ. Then:*

(i) *If* $\mathcal{V}(0)$ *satisfies the condition* (V2)*, the sum is continuous in any point* $(x_0, y_0) \in L \times L$;

(ii) *If* $\mathcal{V}(0)$ *satisfied the axioms* (V2) *and* (V4)*, the multiplication with scalars is continuous in any point* $(0, x_0) \in \mathbb{\Gamma} \times L$.

Particularly, the conditions i) *and* ii) *are verified by the translation* τ *of an almost linear topology* σ.

Proof i) From the axiom (V2) it follows that: $\forall V \in \mathcal{V}(0)\ \exists\, V_1 \in \mathcal{V}(0)$ such that $(x_0 + V_1) + (y_0 + V_1) \subset x_0 + y_0 + V$.

ii) Let be V an arbitrarily chosen neighbourhood of the element $0 = 0 \cdot x_0$, $V_1 \in \mathcal{V}(0)$ such that $V_1 + V_1 \subset V$ given by the axiom (V2) and $U \in \mathcal{V}(0)$ such that $\mathcal{E}(U) \subset V_1$ given by the condition (V4).

Since V_1 is absorbent, there exists a $\delta_1 > 0$ such that $\lambda x_0 \in V_1$ for any $\lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq \delta_1$.

Let be $\delta = \min\{\delta_1, 1\}$. Then, for any $\lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq \delta$ we have

$$\lambda U \subset \mathcal{E}(U) \subset V_1.$$

We found that $V \in \mathcal{V}(0)$ there exists $U \in \mathcal{V}(0)$ and $\delta > 0$ such that if $\lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq \delta$, then $\lambda(x_0 + U) \subset \lambda x_0 + V_1 \subset V_1 + V_1 \subset V$, that is exactly the continuity with the multiplication with scalars in $(0, x_0)$. ∎

Remark 3.3. In the case of the linear topological spaces X, the continuity of the multiplication with real scalars in a arbitrarely chosen point $(\lambda_0, x_0) \in \mathbb{R} \times X$ will be prove firstly for $\lambda_0 \in \mathbb{R}$ with $|\lambda_0| \leq 1$, then for the scalars decomposed in a sum of a entire and a decimal part; this method is no long available for the almost linear topological spaces because in this case the axiom of the distributivity of the multiplication towards the scalar addition is not longer valid. This inconvenient must be encomassed by reinforcing one of the axioms related to the multiplication, that is (V3) or (V4).

Definition 3.5. Let be $(L, +, \cdot)$ an a.l.s. and $k > 0$. A subset $M \subset L$ is called *k-balanced* if $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq k$ and $\forall x \in M$ we have $\lambda x \in M$ (that is $\lambda M \subset M$, $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq k$).

For $k = 1$, we obtain the definition of the balanced set.

Now we consider the following hypothesis associated with a nonempty family $\mathcal{V}(0) \subset \mathcal{P}(L)$ which satisfied the axioms (V0)–(V2):

(V3)′ $\forall V \in \mathcal{V}(0),\ \forall \lambda_0 \in \mathbb{\Gamma},\ \forall x_0 \in L\ \exists \delta > 0$ such that $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda - \lambda_0| < \delta$ we have $\lambda x_0 \in \lambda_0 x_0 + V$

and

(V4)′ $\forall V \in \mathcal{V}(0),\ \forall k > 0,\ \exists V_1 \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq k$ we have $\lambda V_1 \subset V$.

Remark 3.4. The axiom (V3)′ represents the continuity of the application $\lambda \longmapsto \lambda x_0$ in every $\lambda_0 \in \mathbb{\Gamma}$, in the topology given by construction (2.2), for any $x_0 \in L$ fixed, while the axiom (V4)′ assures us that, for every $k > 0$, any neighbourhood of the origin contains a k–balanced neighbourhood.

Theorem 3.2. *Let be $(L, +, \cdot)$ an almost linear space.*

(i) *If the translation of a topology σ on L is almost linear, then any fundamental system of neighbourhoods of the origin in the topology σ verifies the axioms* (V3)′ *and* (V4)′.

(ii) *If a nonempty family $\mathcal{V}(0) \subset \mathcal{P}(L)$ satisfies the conditions* (V1), (V2), (V3)′ *and* (V4)′, *then for every $x \in L$, the family $x + \mathcal{V}(0)$ forms a fundamental system of neighbourhoods of x in an almost linear topology.*

Particularly, the translation of a topology that verifies the axioms (V3)′ *and* (V4)′ *is an almost linear topology.*

Proof.

i) (V3)′ is obvious, according the Remark 3.4.

Let be $k > 0$ arbitrarely, $V \in \mathcal{V}(0)$ and $\mu \in \mathbb{\Gamma}$ with $|\mu| \leq k$. According with the Remark 3.2, the set $\frac{1}{k}V$ is also a neighbourhood of the origin.

As σ is an almost linear topology, it satisfies the axiom (V4) (Theorem 3.1). Particularly, for $\frac{1}{k}V$ there exists $V_1 \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda| \leq 1$ we have $\lambda V_1 \subset \frac{1}{k}V$.

By choosing $\lambda = \frac{\mu}{k}$ we obtain $\frac{\mu}{k}V_1 \subset \frac{1}{k}V$, and it follows that $\mu V_1 \subset V$.

ii) According to the Proposition 3.1, we have to verify only the continuity of the multiplication with scalars.

Let be $(\lambda_0, x_0) \in \mathbb{K} \times L$ and $V \in \mathcal{V}(0)$ arbitrarely; we consider $V_1 \in \mathcal{V}(0)$ given by the axiom (V2): $V_1 + V_1 \subset V$.

From the condition (V3)$'$ for V_1, λ_0 and x_0 we have:

$$\exists \delta > 0 \text{ such that } \forall \lambda \in \mathbb{K} \text{ with } |\lambda - \lambda_0| < \delta \text{ avem } \lambda x_0 \in \lambda_0 x_0 + V_1.$$

We choose $k = |\lambda_0| + \delta > 0$ and we write (V4)$'$ for $V_1 \in \mathcal{V}(0)$ and this k:

$$\exists U \in \mathcal{V}(0) \text{ such that } \lambda U \subset V_1,\ \forall \lambda \in \mathbb{K} \text{ with } |\lambda| \leq k,$$

also for $\lambda \in \mathbb{K}$ with $|\lambda - \lambda_0| < \delta$.

So, we obtain:

$$\lambda x_0 + \lambda U \subset \lambda_0 x_0 + V_1 + \lambda U \subset \lambda_0 x_0 + V_1 + V_1 \subset \lambda_0 x_0 + V.$$

■

Since on the families of parts of a normed linear space X, the translation of no topologies almost linear, we may impose for $\mathcal{V}(0)$ some weaker conditions that (V3)$'$ and (V4)$'$ (by maintaining (V1) and (V2)) which may lead to some weaker properties than those of the almost linear condition, for the translation topology.

We will consider the next axiom attached to a fundamental system $\mathcal{V}(0)$ of neighbourhoods of the origin in L, which represent precisely the continuity of the multiplication with scalars in the points $(\lambda_0, 0) \in \mathbb{K} \times L$.

(A4) $\forall V \in \mathcal{V}(0)$, $\forall \lambda_0 \in \mathbb{K}$ $\exists\, \delta > 0$ and $\exists\, W \in \mathcal{V}(0)$ such that for any $\lambda \in \mathbb{K}$ with $|\lambda - \lambda_0| < \delta$ we have $\lambda W \subset V$.

Remark 3.5. On the family $\mathcal{P}(X)$ (with X a normed space), the condition (A4) is verified both for the topologies τ_H^+, and τ_H^-, considering, for $V = B(0, \varepsilon)$, any $\delta > 0$ and $W = B\left(0, \dfrac{\varepsilon}{|\lambda_0| + \delta}\right)$.

In the following, we will establish the relation between the condition (A4) and some other continuity properties.

Lemma 3.1. *Let L be an a.l.s. $\mathcal{V}(0) \subset \mathcal{P}(L)$ a family of nonvoid subsets of L.*

(i) *If $\mathcal{V}(0)$ satisfies the axiom* (A4)*, then* (V0) *is also valid.*

(ii) *If* $\mathcal{V}(0)$ *satisfies the conditions* (V1) *and* (A4), *then it also satisfies the condition* (V4).

(iii) *We consider the hypothesis:*

(A4)′ $\forall V \in \mathcal{V}(0)\ \forall \lambda_0 \in \mathbb{K}\ \exists\, W \in \mathcal{V}(0)$ *such that* $\lambda_0 W \subset V$. *and*

(A4)″ $\forall V \in \mathcal{V}(0)\ \exists\, \delta_1 > 0$ *and* $\exists\, U \in \mathcal{V}(0)$ *such that* $\forall \mu \in \mathbb{K}$ *with* $|\mu - 1| < \delta_1$ *we have* $\mu U \subset V$.

Then (A4) $\Longrightarrow$ (A4)′ *and* (A4) $\Longrightarrow$ (A4)″.

If the axiom (V4) *is valid, then its converse statement is also valid:* (A4)′ *and* (A4)″ $\Longrightarrow$ (A4). *More,* (V4)′ $\Longrightarrow$ (A4)′.

Proof.

i) We cons8ider in (A4) $\lambda = \lambda_0 = 0$.

ii) Let be $D = \{\lambda \in \mathbb{K};\ |\lambda| \leq 1\}$. We rewrite the continuity of the multiplication with scalars in a element $(\lambda, 0) \in D \times L$, that is the axiom (A4) for a $\lambda \in D$ and a neighbourhood $V \in \mathcal{V}(0)$, there exists $\delta_\lambda > 0$ and $W_\lambda \in \mathcal{V}(0)$ such that $\forall \mu \in \mathbb{K}$ with $|\mu - \lambda| < \delta_\lambda$ we have $\mu W_\lambda \subset V$.

We denote by $D_\lambda = \{\mu \in \mathbb{K};\ |\mu - \lambda| < \delta_\lambda\}$ for every $\lambda \in D$. Then $D \subset \subset \bigcup_{|\lambda| \leq 1} D_\lambda$, and from the compoctity of D in $\mathbb{K}$, it follows that there exists a finite number of elements $\lambda_1, \lambda_2, ..., \lambda_k \in D$ and $D_j = D_{\lambda_j}$, $j = \overline{1,k}$ such that $D \subset \bigcup_{j=1}^{k} D_j$.

Let be $\lambda \in D$ arbitrarily fixed and the index $j_\lambda \in \overline{1,k}$ such that $\lambda \in D_{j_\lambda}$; then $\lambda W_{j_\lambda} \subset V$.

According with the axiom (V1), let be now the set $W_0 \in \mathcal{V}(0)$ with $W_0 \subset \bigcup_{j=1}^{k} W_j$. It follows that $\lambda W_0 \subset V,\ \forall \lambda \in D$, i.e. $\mathcal{E}(W_0) \subset V$.

iii) We suppose that the axiom (A4) is respected. Letting $\lambda = \lambda_0$ we have (A4)′, and for $\lambda_0 = 1$ we find (A4)″.

Conversely,we consider the hypothesis (V4), (A4)′ and (A4)″.

Let be $V \in \mathcal{V}(0)$ and $\lambda_0 \in \mathbb{K}$.

If $\lambda_0 = 0$, from (A4)′ we have $\forall \lambda \in \mathbb{K}\ \exists\, W_\lambda \in \mathcal{V}(0)$ such that $\lambda W_\lambda \subset V$. Let be $\delta > 0$ arbitrar. For $W_\delta \in \mathcal{V}(0)$, according the hypothesis (V4), there exists $W_0 \in \mathcal{V}(0)$ balanced such that $W_0 \subset W_\delta$.

If $\lambda \in \mathbb{K}$ fulfills the condition $|\lambda| \leq \delta$, as W_0 is balanced and $\left|\frac{\lambda}{\delta}\right| \leq 1$ it follows that $\lambda W_0 \subset \delta W_0 \subset \delta W_\delta$, so $\lambda W_0 \subset V$; we obtained the condition (A4) for $\lambda_0 = 0$ with $\delta > 0$ arbitrarily fixed and $W_0 \in \mathcal{V}(0)$.

Let be now $\lambda_0 \neq 0$. By writing (A4)$'$ for V and $\lambda_0 \neq 0$ we find $W \in \mathcal{V}(0)$ with $\lambda_0 W \subset V$. From (A4)$''$ written for $W \in \mathcal{V}(0)$ let $\delta_1 > 0$ and $U \in \mathcal{V}(0)$.

We choose $\delta = \delta_1 \cdot |\lambda_0| > 0$ and let be $\lambda \in \mathbb{K}$ such that $|\lambda - \lambda_0| < \delta$.

Considering $\mu = \dfrac{\lambda}{\lambda_0} \in \mathbb{K}$ we have $|\mu - 1| = \dfrac{1}{|\lambda_0|}|\lambda - \lambda_0| < \delta_1$. From (A4)$''$ we have $\mu U \subset W$, that is $\lambda U \subset \lambda_0 W$. Together with $\lambda_0 W \subset V$, we obtain (A4) for $\lambda_0 \neq 0$. ∎

By taking $\lambda_0 = 0$ in (V3)$'$, the condition is automatically fulfilled by the translation of a topology on an almost linear space. We may enforce this property with the following axiom:

(V5) $\forall x_0 \in L$, $\forall V \in \mathcal{V}(0)$ $\exists\, \delta > 0$ and $\exists\, U \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbb{K}$ with $|\lambda| < 1$ we have $\lambda x_0 + U \subset V$.

Remark 3.6. When $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin, the condition (V5) is a formulation with an algebraic aspect for the translation topology of one of the axioms satisfied by any of the neighbourhood system, that is: Any neighbourhood V of the origin is also neighbourhood for the points λx_0 (with $|\lambda| < \delta$) "sufficiently close" to the origin.

This suggest us to consider a similar condition in which to replace the origin by a point $x_0 \in L$ arbitrarily chosen (which is not obtained from (V5)!):

(A5) $\forall x_0 \in L$, $\forall V \in \mathcal{V}(0)$ $\exists\, \delta > 0$ $\exists\, U \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbb{K}$ with $|\lambda - 1| < \delta$ we have $\lambda x_0 + U \subset x_0 + V$;

respectively, by the arbitrary point $\mu_0 x_0 \in L$:

(A3) $\forall x_0 \in L$, $\forall V \in \mathcal{V}(0)$, $\forall \mu_0 \in \mathbb{K}$ $\exists\, \delta_0 > 0$ and $\exists\, W \in \mathcal{V}(0)$ such that $\forall \mu \in \mathbb{K}$ with $|\mu - \mu_0| < \delta$ we have $\mu x_0 + W \subset \mu_0 x_0 + V$.

The condition (V5), (A5) and (A3) are effectively related to the translation topology, while the hypothesis (A4) refers to the initial topology of the space.

More,they are easier to fulfill. For example, (V5), for a set $A \in \mathcal{P}b(X)$ and topology τ_H^+ on $\mathcal{P}b(X)$, it take the form:

$\forall \varepsilon > 0\ \exists \eta > 0\ \exists \delta > 0$ such that $\forall U \subset X$ with $\sup\limits_{u\in U} \|u\| < \delta$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| < \eta$ we have $\sup\limits_{\substack{a\in A\\ u\in U}} \|\lambda a + u\| < \varepsilon$.

The condition is satisfied for $\delta = \dfrac{\varepsilon}{2}$ and $\eta = \dfrac{\varepsilon}{2\|A\|}$.

In the following, we establish the relation between the conditions (V5), (A3), (A5) and those who have been analyzed in the Lemma 3.1.

Lemma 3.2. *We consider an a.l.s. L and the nonempty family $\mathcal{V}(0) \subset \mathcal{P}(L)$. Then, the following assertions hold:*

(i) (A3) $\Longrightarrow$ (V3)*;* (A3) $\Longrightarrow$ (A5)*;* (A3) $\Longrightarrow$ (V5)*;*

(ii) (V2) *and* (V3) $\Longrightarrow$ (V5)*;*

(iii) *The converse statement takes place under stronger conditions (see the Lemma 3.1, iii):*
(A4)$'$ *and* (A5) $\Longrightarrow$ (A3)*, if* $\mu_0 \neq 0$,
(V2) *and* (V3) $\Longrightarrow$ (A3)*, if* $\mu_0 = 0$.

Proof.
i) By taken $\mu_0 = 0$ in (A3) we find for $x_0 \in L$ and $V \in \mathcal{V}(0)$ a set $W \in \mathcal{V}(0)$ and $\delta_0 > 0$ such that $\forall \mu \in \mathbb{K}$ with $|\mu| < \delta$ we have $\mu x_0 + W \subset V$, so that $\mu x_0 \in V$.

ii) Let be $V \in \mathcal{V}(0)$. From the condition (V2) there exists $U \in \mathcal{V}(0)$ such that $U + U \subset V$. According to (V3), U is absorbent, so for $x_0 \in L$ there exists $\delta > 0$ such that ,for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq \delta$ we have $\lambda x_0 \in U$. Then $\lambda x_0 + U \subset U + U \subset V$. By taken $\mu_0 = 1$ in (A3), we obtain (A5) and (V5).

iii) Let be $\mu_0 = 0$ and $V \in \mathcal{V}(0)$, $x_0 \in L$. In this case, (A3) $\equiv$ (V5) and we apply ii).

We consider now $\mu_0 \neq 0$, $x_0 \in L$ and $V \in \mathcal{V}(0)$ arbitrarily fixed. From (A4)$'$, for V and μ_0 we find $W_1 \in \mathcal{V}(0)$ such that $W_1 \subset \dfrac{1}{\mu_0} V$.

Let be $\delta > 0$ and $U \in \mathcal{V}(0)$ from (A5) according to W_1.

We take $\delta_0 = |\mu_0| \cdot \delta > 0$ and $\mu \in \mathbb{K}$ with $|\mu - \mu_0| < \delta_0$.

by taking $\lambda = \dfrac{\mu}{\mu_0}$ in (A5) we have $|\lambda - 1| = \dfrac{1}{|\mu_0|}|\mu - \mu_0| < \delta$ and then $\lambda x_0 + U \subset x_0 + W_1$.

We apply for $U \in \mathcal{V}(0)$ the condition (A4)$'$ with $\lambda_0 = \dfrac{1}{\mu_0}$:

$$\exists\, W \in \mathcal{V}(0) \text{ such that } \frac{1}{\mu_0} W \subset U.$$

Then $\lambda x_0 + \dfrac{1}{\mu_0} W \subset \lambda x_0 + U \subset x_0 + W_1 \subset x_0 + \dfrac{1}{\mu_0} V$, and so $\dfrac{\mu}{\mu_0} x_0 + \dfrac{1}{\mu_0} W \subset x_0 + \dfrac{1}{\mu_0} V$, i.e. $\mu x_0 + W \subset \mu_0 x_0 + V$. ∎

Remark 3.7. From the proof of the Lemma 3.3, i) it follows that for $\mu_0 = 0$, the condition (A3) is much stronger that the absorbence axiom, presentinh itself as a "locally uniform" absorbence.

Finally, we obtain:

Proposition 3.2. *Let be $(L, +, \cdot)$ an a.l.s. $\mathcal{V}(0) \subset \mathcal{P}(L)$ a nonempty family. Then, the conditions* (V1), (V2), (V3), (A4), (A5) *are equivalent with* (V1), (V2), (A3) *and* (A4). *if $\mathcal{V}(0)$ is a fundamental system of neighbourhood of the origin for a topology in L, then both groups of axioms assure the almost liniarity for the translate topology.*

Proof.
First, the two groups of axioms are equivalent:

– from the Lemma 3.3, iii) we have (V2), (V3), (A4) and (A5) imply (A3);

– also from the Lemma 3.3, i) we know that (A3) implies (V3) and (A5).

We prove the direct theorem by using the first group of axioms. We will use the Theorem 3.1 and so the conditions (V1), (V2) and (V3) are verified.

(A4) represent the continuity of the multiplication with scalars in the points $(\lambda_0, 0) \in \mathbb{K} \times L$, and (A5), in the points $(1, x_0) \in \mathbb{K} \times L$, so both of them are satisfied.

Conversely: for the family $\mathcal{V}(0)$ which satisfies (V1)–(V3), (A4) and (A5) we consider the topology τ given by system of neighbourhoods $\mathcal{U}(x)$ defined by (2.2) for every $x \in L$.

From the Lemma 3.2,i) the five conditions of the proposition imply also (V1)–(V4), and from the Proposition 3.1 we have that the sum in any point (x_0, y_0) and the multiplication with scalars in $(0, x_0) \in \mathbb{K} \times L$ are continuous.

We have also to show the continuity of the multiplication with scalars in the points $(\lambda_0, x_0) \in \mathbb{\Gamma} \times L$ with $\lambda_0 \neq 0$. From the equivalence of the two groups of axioms, (A3) is valid for every $x_0 \in L$, a arbitrary set $V \in \mathcal{V}(0)$ and $\lambda_0 : \exists\, \delta_1 > 0$ and $\exists\, W \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda - \lambda_0| < \delta_1$ we have $\lambda x_0 + W \subset \lambda_0 x_0 + V$.

We apply the axiom (A4) for W and $\lambda_0 : \exists\, V_1 \in \mathcal{V}(0)$ and $\exists\, \delta_2 > 0$ such that $\forall \lambda \in \mathbb{\Gamma}$ with $|\lambda - \lambda_0| < \delta_2$ implies $\lambda V_1 \subset W$. Considering $\delta = \min\{\delta_1, \delta_2\}$, for $\lambda \in \mathbb{\Gamma}$ with $|\lambda - \lambda_0| < \delta$ we obtain the inclusion $\lambda(x_0 + V_1) \subset \lambda_0 x_0 + V$.

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