ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.1.

ON SPATIAL GROWTH OR DECAY OF SOLUTIONS TO A NON–SIMPLE HEAT CONDUCTION PROBLEM IN A SEMI–INFINITE STRIP

BY

S. CHIRIŢĂ and C. D'APICE

Abstract. The present paper establishes growth and decay spatial properties for the solutions of a fourth–order initial boundary value problem describing the flow of heat in a non–simple heat conductor along a semi–infinite strip in R^2. The method of time–weighted line and area integral measures is used. When the time–weighted line integral measure is used, then an alternative of Phragmén–Lindelöf type is established. It is shown that the decay rate of the end effects is controlled by the same factor as in the steady–state case (governed by the biharmonic equation), that is $\exp(-\frac{\sqrt{2}\pi}{h}x_1)$, where h is the width of the strip and x_1 is the distance to the end of the strip. When an appropriate combination of the time–weighted line and area integrals is used as a measure, then a decay estimate of Saint–Venant type is established and it is shown that the end effects decay more rapidly as do their counterparts in the steady–state case.

AMS Subject Classification (2000): 35B40, 35B45, 74A30, 74F05.

Key words and phrases: time–weighted measures, Phragmén–Lindelöf type results, Saint–Venant decay estimate, heat conduction, non–simple heat conductor.

1. Introduction. Considerable work has been carried out in investigating the spatial behaviour of solutions of the biharmonic equation in a semi–infinite strip in R^2 (see, for example, KNOWLES [1,2], FLAVIN [3], OLEINIK and YOSIFIAN [4,5], HORGAN [6], FLAVIN and KNOPS [7] and PAYNE and SCHAEFER [8]) and in a semi–infinite cylinder in R^3 (see, for example, PAYNE and SCHAEFER [8] and LIN [9]). Concerning the time–dependent problems associated with the biharmonic operator we mention the papers by LIN [10], KNOPS and LUPOLI [11] and CHIRIŢĂ and CIARLETTA [12] in connection with the spatial behaviour of solutions for a fourth–order transformed problem associated with the slow flow of an incompressible viscous fluid along a semi–infinite strip in R^2 subject to zero velocity on the lateral sides, a prescribed time–dependent end velocity and to zero initial conditions.

In this paper we investigate the end effects for a time–dependent fourth–order differential equation modelling the flow of heat in a non–simple heat conductor (see, for example, CIARLETTA and IEŞAN [13], for a non–simple elastic material) along a semi–infinite strip in R^2, subjected to a prescribed time–dependent temperature on the end and being thermal insulated on the lateral sides and zero initial state. In this aim we use the method of time–weighted line and area integral measures (see, CHIRIŢĂ and CIARLETTA [14] for details about applications of such a method in mechanics of solids). Thus, we are able to treat both growth and decay spatial properties of the solutions for the fourth–order initial boundary value problem in question. A second–order differential inequality is established for each of the three time–weighted integral measures used in the paper, which show that the end effects for the transient case decay or grow spatially at least or more rapidly as do their counterparts in the steady–state case (governed by the biharmonic equation).

For the first two time–weighted line integral measures we establish an alternative of Phragmén–Lindelöf type and the predicted decay rate of the end effects is the same like one for the steady–state case (that is, $\exp\left(-\frac{\sqrt{2}\pi}{h}x_1\right)$, where h is the width of the strip and x_1 is the distance to the end of the strip). For the third time–weighted integral measure we establish a decay estimate of Saint–Venant type predicting a decay rate controlled by the factor $\exp\left(-r_1 x_1\right)$, where r_1 is a specified function on the parameter λ characterising the measure in question. It is worth to mention it that r_1 can be made sufficiently large by choosing an appropriate large value for the parameter λ. We also indicate how one can determine pointwise decay estimates.

2. The fourth–order initial boundary value problem. We consider a semi–infinite strip Σ in the plane x_1Ox_2 defined by

$$\Sigma = \{(x_1, x_2) \in R^2 \mid \; 0 < x_2 < h \;,\; 0 < x_1\} \;,\; h \geq 0. \tag{2.1}$$

We assume $u(x_1, x_2, t)$ to be a classical solution of the initial boundary value problem $\mathcal{P}$ defined by

$$u_{,t} + u_{,\alpha\alpha\beta\beta} = 0 \quad in \quad \Sigma \times [0, T], \tag{2.2}$$

$$u(x_1, 0, t) = u_{,2}(x_1, 0, t) = 0 \ , \quad x_1 > 0 \ , \quad t \in [0, T], \tag{2.3}$$

$$u(x_1, h, t) = u_{,2}(x_1, h, t) = 0 \ , \quad x_1 > 0 \ , \quad t \in [0, T], \tag{2.4}$$

(2.5)
$$u(0, x_2, t) = g_1(x_2, t) \ \ , \quad u_{,1}(0, x_2, t) = g_2(x_2, t) \ , \quad 0 < x_2 < h \ , \quad t \in [0, T],$$

$$u(x_1, x_2, 0) = 0 \ \ , \quad (x_1, x_2) \in \overline{\Sigma}, \tag{2.6}$$

where we make no assumption on u as $x_1 \to \infty$. Here the comma notation $u_{,\alpha}$ has been used to denote partial differentiation with respect to x_α and $u_{,t}$ has been used to denote partial differentiation with respect to t. The differentiable functions g_1 and g_2 are prescribed and assumed to satisfy appropriate compatibility conditions at $(0, 0)$ and $(0, h)$, specifically,

$$g_1(0, t) = g_1(h, t) = g_{1,2}(0, t) = g_{1,2}(h, t) = 0, \tag{2.7}$$

$$g_2(0, t) = g_2(h, t) = g_{2,2}(0, t) = g_{2,2}(h, t) = 0 \quad \text{for all} \quad t \in [0, T],$$

and, moreover, we have

$$g_1(x_2, 0) = g_2(x_2, 0) = 0 \ , \quad 0 \le x_2 \le h. \tag{2.8}$$

It is worth to note that the above fourth–order initial boundary value problem represents a simplified approach for the transient heat conduction in an isotropic non–simple material in which the gradients of the third–order for temperature are taken into consideration.

We further introduce the notations

$$L_z = \{(x_1, x_2) \in R^2 \mid \ x_1 = z \geq 0 \ , \quad 0 \leq x_2 \leq h\}, \tag{2.9}$$

$$D_{x_1^* x_1} = \{(y_1, y_2) \in R^2 \mid \quad 0 \leq x_1^* < y_1 < x_1 \ , \quad 0 < y_2 < h\}, \tag{2.10}$$

$$D_{x_1} = \{(y_1, y_2) \in R^2 \mid \quad 0 \leq x_1 < y_1 \ , \quad 0 < y_2 < h\}. \tag{2.11}$$

3. Some fundamental time–weighted identities. In this section we establish some fundamental time–weighted identities associated with the solution u of the fourth–order initial boundary value problem $\mathcal{P}$ defined in the above section.

Lemma 1. *For any solution u of the initial boundary value problem $\mathcal{P}$ defined by the relations (2.2)–(2.6), we have*

$$\begin{aligned} \int_{D_{x_1^* x_1}} e^{-\lambda t} \frac{1}{2} u^2 da + \int_0^t \int_{D_{x_1^* x_1}} e^{-\lambda s} (\frac{\lambda}{2} u^2 + u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) dads = \\ = \int_0^t \int_{L_{x_1}} e^{-\lambda s} (-uu_{,11} + u_{,1}^2 + u_{,2}^2)_{,1} dx_2 ds - \\ - \int_0^t \int_{L_{x_1^*}} e^{-\lambda s} (-uu_{,11} + u_{,1}^2 + u_{,2}^2)_{,1} dx_2 ds, \\ 0 \leq x_1^* \leq x_1, \quad t \in [0, T], \end{aligned} \tag{3.1.}$$

where λ is a prescribed positive parameter.

Proof. In view of the equation (2.2), we have

(3.2)
$$uu_{,s} + (uu_{,111})_{,1} - u_{,1}u_{,111} + 2(uu_{,112})_{,2} - 2u_{,2}u_{,112} + (uu_{,222})_{,2} - u_{,2}u_{,222} = 0,$$

so that, we get

$$(3.3)\quad (\frac{1}{2}u^2)_{,s}+(uu_{,111})_{,1}-(u_{,1}u_{,11})_{,1}+u_{,11}^2+2(uu_{,112})_{,2}-2(u_{,2}u_{,12})_{,1}+2u_{,12}^2$$

$$+(uu_{,222})_{,2}-(u_{,2}u_{,22})_{,2}+u_{,22}^2=0.$$

If we multiply (3.3) by $e^{-\lambda s}$, we can write

$$(3.4)\quad [e^{-\lambda s}\frac{1}{2}u^2]_{,s}+e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)=[e^{-\lambda s}(-uu_{,111}+u_{,1}u_{,11}$$

$$+2u_{,2}u_{,12})]_{,1}+[e^{-\lambda s}(-2uu_{,112}-uu_{,222}+u_{,2}u_{,22})]_{,2}.$$

By an integration over $D_{x_1^*x_1}\times[0,t]$ and by using the relations (2.3), (2.4) and (2.6), from (3.4) we deduce the relation (3.1) and the proof is complete.

Lemma 2. *For any solution of the problem $\mathcal{P}$, we have*

$$(3.5)\quad \int_{D_{x_1^*x_1}} e^{-\lambda t}\frac{1}{2}u_{,1}^2da+\int_0^t\int_{D_{x_1^*x_1}} e^{-\lambda s}(\frac{\lambda}{2}u_{,1}^2+u_{,111}^2+2u_{,112}^2+u_{,122}^2)dads$$

$$=\int_0^t\int_{L_{x_1}} e^{-\lambda s}(u_{,s}u_{,1}+u_{,11}u_{,111}+u_{,22}u_{,122})dx_2ds$$

$$-\int_0^t\int_{L_{x_1^*}} e^{-\lambda s}(u_{,s}u_{,1}+u_{,11}u_{,111}+u_{,22}u_{,122})dx_2ds\ ,\quad 0\le x_1^*\le x_1,\ \ t\in[0,T].$$

Proof. By means of the relation (2.2) we have

$$(3.6)\qquad \begin{aligned} u_{,s}u_{,11}+(u_{,11}u_{,111})_{,1}-u_{,111}^2+2(u_{,11}u_{,112})_{,2}-\\ -2u_{,112}^2+(u_{,11}u_{,222})_{,2}-u_{,112}u_{,222}=0,\end{aligned}$$

so that, we get

$$
\begin{aligned}
&[e^{-\lambda s}\tfrac{1}{2}u_{,1}^2]_{,s} + e^{-\lambda s}(\tfrac{\lambda}{2}u_{,1}^2 + u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2) = \\
&= [e^{-\lambda s}(u_{,1}u_{,s} + u_{,11}u_{,111} + u_{,22}u_{,122})]_{,1} + \\
&+ [e^{-\lambda s}(2u_{,11}u_{,112} + u_{,11}u_{,222} - u_{,22}u_{,112})]_{,2}.
\end{aligned} \tag{3.7}
$$

If we integrate (3.7) over $D_{x_1^* x_1} \times [0, t]$ and then we use the relations (2.3), (2.4) and (2.6) we obtain the relation (3.5) and the proof is complete.

Lemma 3. *For any solution of the problem $\mathcal{P}$, we have*

$$
\begin{aligned}
&\int_{D_{x_1^* x_1}} e^{-\lambda t}\frac{1}{2}(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)da + \\
&+ \int_0^t \int_{D_{x_1^* x_1}} e^{-\lambda s}[u_{,s}^2 + \frac{\lambda}{2}(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)]dads = \\
&= \int_0^t \int_{L_{x_1}} e^{-\lambda s}[(u_{,s}u_{,11})_{,1} - 2u_{,s}(u_{,111} + u_{,122})]dx_2 ds - \\
&- \int_0^t \int_{L_{x_1^*}} e^{-\lambda s}[(u_{,s}u_{,11})_{,1} - 2u_{,s}(u_{,111} + u_{,122})]dx_2 ds, \\
&0 \le x_1^* \le x_1\ , \quad t \in [0, T].
\end{aligned} \tag{3.8}
$$

Proof. On the basis of the relation (2.2), we get

$$
\begin{aligned}
-u_{,s}^2 &= (u_{,s}u_{,111})_{,1} - u_{,s1}u_{,111} + 2(u_{,s}u_{,112})_{,2} - \\
&-2u_{,s2}u_{,112} + (u_{,s}u_{,222})_{,2} - u_{,s2}u_{,222},
\end{aligned} \tag{3.9}
$$

so that, we further have

$$
\begin{aligned}
&u_{,s}^2 + \frac{1}{2}(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)_{,s} = [(u_{,s}u_{,11})_{,1} - \\
&-2u_{,s}(u_{,111} + u_{,122})]_{,1} + (-u_{,s}u_{,222} + u_{,s2}u_{,22} + 2u_{,s1}u_{,12})_{,2}.
\end{aligned} \tag{3.10}
$$

If we multiply (3.10) by $e^{-\lambda s}$, we can write

$$
\begin{aligned}
&[e^{-\lambda s}\tfrac{1}{2}(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)]_{,s} + \\
&+ e^{-\lambda s}[u_{,s}^2 + \tfrac{\lambda}{2}(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)] = \\
&= \{e^{-\lambda s}[(u_{,s}u_{,11})_{,1} - 2u_{,s}(u_{,111} + u_{,122})]\}_{,1} + \\
&+ \{e^{-\lambda s}(-u_{,s}u_{,222} + u_{,s2}u_{,22} + 2u_{,s1}u_{,12})\}_{,2}.
\end{aligned} \tag{3.11}
$$

By an integration over $D_{x_1^*x_1} \times [0,t]$ and by using the relations (2.3), (2.4) and (2.6), from (3.11) we deduce the relation (3.8) and the proof is complete.

Lemma 4. *For any solution of the problem $\mathcal{P}$, we have*

$$\int_{L_{x_1}} u_{,s}u_{,1}dx_2 = \int_{L_{x_1}} (-u_{,1}u_{,111} + \frac{1}{2}u_{,11}^2 + u_{,12}^2 - \frac{1}{2}u_{,22}^2)_{,1}dx_2, \quad x_1 \geq 0, \quad t \in [0,T]. \tag{3.12}$$

Proof. By taking into account of the relation (2.2), we get

$$\begin{aligned} &u_{,s}u_{,1} + (u_{,1}u_{,111})_{,1} - u_{,11}u_{,111} + 2(u_{,1}u_{,112})_{,2} - \\ &-2u_{,12}u_{,112} + (u_{,1}u_{,222})_{,2} - u_{,12}u_{,222} = 0, \end{aligned} \tag{3.13}$$

so that we deduce

$$\begin{aligned} &u_{,s}u_{,1} + (u_{,1}u_{,111})_{,1} - u_{,11}u_{,111} + 2(u_{,1}u_{,112})_{,2} - 2u_{,12}u_{,112} + \\ &+(u_{,1}u_{,222})_{,2} - (u_{,12}u_{,22})_{,2} + u_{,22}u_{,122} = 0. \end{aligned} \tag{3.14}$$

Thus, by using the relations (2.3) and (2.4), from (3.14) we deduce (3.12) and the proof is complete.

4. The first time–weighted line integral measure. Before deriving our growth–decay estimates, we establish a second–order differential inequality for the time–weighted line integral measure which is fundamental in our study on the spatial behaviour of the solution u of the problem in question. In this aim, for a solution u of the problem $\mathcal{P}$ we associate the following time–weighted line integral

$$I(x_1,t) = \int_0^t \int_{L_{x_1}} e^{-\lambda s}(-uu_{,11} + u_{,1}^2 + u_{,2}^2)dx_2ds, \quad x_1 \in [0,\infty), t \in [0,T]. \tag{4.1}$$

Then, from the Lemma 1, we deduce that

$$\begin{aligned} \frac{\partial^2 I}{\partial x_1^2}(x_1,t) = \int_{L_{x_1}} e^{-\lambda t}\frac{1}{2}u^2 dx_2 + \\ + \int_0^t \int_{L_{x_1}} e^{-\lambda s}(\frac{\lambda}{2}u^2 + u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) dx_2 ds. \end{aligned} \tag{4.2}$$

Toward to obtain an appropriate estimate for $I(x_1,t)$, we now recall the following three Wirtinger type inequalities:

$$\int_{L_z} u^2 dx_2 \leq \frac{h^2}{\pi^2}\int_{L_z} u_{,2}^2 dx_2 \quad , \quad \int_{L_z} u_{,1}^2 dx_2 \leq \frac{h^2}{\pi^2}\int_{L_z} u_{,12}^2 dx_2, \tag{4.3}$$

$$\int_{L_z} u_{,2}^2 dx_2 \leq \frac{h^2}{4\pi^2}\int_{L_z} u_{,22}^2 dx_2, \tag{4.4}$$

$$\int_{L_z} u^2 dx_2 \leq (\frac{2}{3})^4 \frac{h^4}{\pi^4} \int_{L_z} u_{,22}^2 dx_2. \tag{4.5}$$

Thus, by (4.1), the above Wirtinger inequalities and the Schwarz inequality, we have

$$\begin{aligned} |I(x_1,t)| \leq \xi_1 (\int_0^t \int_{L_{x_1}} e^{-\lambda s} u^2 dx_2 ds \int_0^t \int_{L_{x_1}} e^{-\lambda s} u_{,11}^2 dx_2 ds)^{1/2} + \\ + \frac{4h^2}{9\pi^2}(1-\xi_1)(\int_0^t \int_{L_{x_1}} e^{-\lambda s} u_{,11}^2 dx_2 ds \int_0^t \int_{L_{x_1}} e^{-\lambda s} u_{,22}^2 dx_2 ds)^{1/2} + \\ + \frac{h^2}{4\pi^2} \int_0^t \int_{L_{x_1}} e^{-\lambda s}(4u_{,12}^2 + u_{,22}^2) dx_2 ds, \end{aligned} \tag{4.6}$$

where $\xi_1 \in [0,1]$ is a positive real parameter that will be chosen later. If we now apply the arithmetic–geometric mean inequality in (4.6), then we get

$$\begin{aligned} |I(x_1,t)| \leq \frac{h^2}{2\pi^2}\int_0^t \int_{Lx_1} e^{-\lambda s}\{(\frac{2\pi^2 \xi_1 \varepsilon_1}{h^2 \lambda})\frac{\lambda}{2}u^2 + \\ + [\frac{\pi^2 \xi_1}{\varepsilon_1 h^2} + \frac{4}{9}(1-\xi_1)\varepsilon_2] u_{,11}^2 + 2u_{,12}^2 + \\ + [\frac{4}{9\varepsilon_2}(1-\xi_1) + \frac{1}{2}] u_{,22}^2\} dx_2 ds, \ \forall \varepsilon_1, \varepsilon_2 > 0. \end{aligned} \tag{4.7}$$

Now, we choose the arbitrary parameters ξ_1, ε_1, ε_2 and λ so that

$$\frac{2\pi^2\xi_1\varepsilon_1}{h^2\lambda} \leq 1 \quad , \quad \frac{\pi^2\xi_1}{\varepsilon_1 h^2} + \frac{4}{9}(1-\xi_1)\varepsilon_2 \leq 1 \quad , \quad \frac{4}{9\varepsilon_2}(1-\xi_1) + \frac{1}{2} \leq 1, \tag{4.8}$$

so that we obtain the basic second–order differential inequality

$$|I(x_1,t)| \leq \frac{h^2}{2\pi^2}\frac{\partial^2 I}{\partial x_1^2}(x_1,t) \quad , \quad x_1 \geq 0 \ , \quad t \in [0,T]. \tag{4.9}$$

As an example, we can satisfy the relation (4.8) if we set $\xi_1 = 0$ and $\xi_2 \in [\frac{8}{9}, \frac{9}{4}]$.

Consequently, we have established the following two differential inequalities

$$\frac{\partial^2 I}{\partial x_1^2}(x_1,t) + \frac{2\pi^2}{h^2} I(x_1,t) \geq 0, \tag{4.10}$$

$$\frac{\partial^2 I}{\partial x_1^2}(x_1,t) - \frac{2\pi^2}{h^2} I(x_1,t) \geq 0, \tag{4.11}$$

which will be utilized in what follows in the derivation of the alternatives.

Let us consider t be fixed in $[0,T]$. Then we have the following two possibilities: (i) there exists a value $x_1 = z_t \geq 0$ for which $\partial I/\partial x_1(z_t,t) > 0$; (ii) $\partial I/\partial x_1(x_1,t) \leq 0$ for all $x_1 \geq 0$.

We consider first the case (i). Since $\partial^2 I/\partial x_1^2(x_1,t) \geq 0$ for all $x_1 \geq 0$, we have $\partial I/\partial x_1(x_1,t) > 0$ for all $x_1 \geq z_t$. Moreover, since

$$I(x_1,t) \geq I(z_t,t) + \frac{\partial I}{\partial x_1}(z_t,t)(x_1 - z_t) \ , \quad x_1 \geq z_t, \tag{4.12}$$

it follows that $I(x_1,t)$ must eventually become positive. Let z_t^* be a value of x_1 for which both $\partial I/\partial x_1(z_t^*,t) > 0$ and $I(z_t^*,t) > 0$ hold true. Then, from (4.11), we get

$$\frac{\partial}{\partial x_1}\{\exp(-\frac{\sqrt{2}\pi}{h}x_1)[\frac{\partial I}{\partial x_1}(x_1,t) + \frac{\sqrt{2}\pi}{h}I(x_1,t)]\} \geq 0, \tag{4.13}$$

or

$$\frac{\partial}{\partial x_1}\{\exp(\frac{\sqrt{2}\pi}{h}x_1)[\frac{\partial I}{\partial x_1}(x_1,t)-\frac{\sqrt{2}\pi}{h}I(x_1,t)]\}\geq 0. \tag{4.14}$$

After an integration we deduce that, for $x_1 \geq z_t^*$, we have

$$\begin{gathered}\frac{\partial I}{\partial x_1}(x_1,t)+\frac{\sqrt{2}\pi}{h}I(x_1,t)\geq\\ \geq[\frac{\partial I}{\partial x_1}(z_t^*,t)+\frac{\sqrt{2}\pi}{h}I(z_t^*,t)]\exp[\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)],\end{gathered} \tag{4.15}$$

and

$$\begin{gathered}\frac{\partial I}{\partial x_1}(x_1,t)-\frac{\sqrt{2}\pi}{h}I(x_1,t)\geq\\ \geq[\frac{\partial I}{\partial x_1}(z_t^*,t)-\frac{\sqrt{2}\pi}{h}I(z_t^*,t)]\exp[-\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)].\end{gathered} \tag{4.16}$$

Hence, from (4.15) and (4.16), we deduce

$$\begin{gathered}\frac{\partial I}{\partial x_1}(x_1,t)\geq\frac{\partial I}{\partial x_1}(z_t^*,t)\cosh[\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)]+\\ +\frac{\sqrt{2}\pi}{h}I(z_t^*,t)\sinh[\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)].\end{gathered} \tag{4.17}$$

Thus, if we combine the relations (3.1) (with $x_1^* = z_t^*$), (4.1) and (4.16), then we deduce

$$\int_{D_{z_t^* x_1}} e^{-\lambda t}\frac{1}{2}u^2 da+\int_0^t\int_{D_{z_t^* x_1}} e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)dads \tag{4.18}$$

$$\geq\frac{\partial I}{\partial x_1}(z_t^*,t)\{\cosh[\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)]-1\}+\frac{\sqrt{2}\pi}{h}\,I(z_t^*,t)\sinh[\frac{\sqrt{2}\pi}{h}(x_1-z_t^*)],$$

and consequently

$$\lim_{x_1\to\infty}\{e^{-\frac{\sqrt{2}\pi}{h}x_1}[\int_{D_{z_t^*x_1}} e^{-\lambda t}\frac{1}{2}u^2 da+ \tag{4.19}$$
$$+\int_0^t\int_{D_{z_t^*x_1}} e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)dads]\}\geq C_1(t),$$

where

$$C_1(t)=\frac{1}{2}\exp{(-\frac{\sqrt{2}\pi}{h}z_t^*)}[\frac{\partial I}{\partial x_1}(z_t^*,t)+\frac{\sqrt{2}\pi}{h}I(z_t^*,t)]. \tag{4.20}$$

Let us further consider the case (ii). That is we assume that $\partial I/\partial x_1(x_1,t)\leq 0$ for all $x_1\geq 0$. Then it results that $I(x_1,t)\geq 0$ for all $x_1\geq 0$. In fact, in order to see this we suppose that there is a value $x_1=z_0>0$ such that $I(z_0,t)<0$. Then, by our assumption $\partial I/\partial x_1(x_1,t)\leq 0$ we have $I(x_1,t)\leq I(z_0,t)$ for all $x_1\geq z_0$. But by (4.10), we have

$$\frac{\partial I}{\partial x_1}(x_1,t)-\frac{\partial I}{\partial x_1}(z_0,t)\geq-\frac{2\pi^2}{h^2}\int_{z_0}^{x_1} I(\xi,t)d\xi\geq-\frac{2\pi^2}{h^2}I(z_0,t)(x_1-z_0), \tag{4.21}$$

and hence $\partial I/\partial x_1(x_1,t)$ cannot remain nonpositive for all x_1. By this contradiction, we conclude that $I(x_1,t)\geq 0$ for all $x_1\geq 0$.

Further, we integrate (4.14) from 0 to x_1 in order to obtain

$$-\frac{\partial I}{\partial x_1}(x_1,t)+\frac{\sqrt{2}\pi}{h}\,I(x_1,t)\leq\exp{(-\frac{\sqrt{2}\pi}{h}x_1)}[-\frac{\partial I}{\partial x_1}(0,t)+\frac{\sqrt{2}\pi}{h}\,I(0,t)]. \tag{4.22}$$

Since $I(x_1,t)$ and $-\frac{\partial I}{\partial x_1}(x_1,t)$ decay exponentially as $x_1\to\infty$, from (3.1) we deduce

$$-\frac{\partial I}{\partial x_1}(x_1,t)=\int_{D_{x_1}} e^{-\lambda t}\frac{1}{2}u^2 da+ \tag{4.23}$$
$$+\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)dads,$$

$$I(x_1,t)=\int_{x_1}^{\infty}\int_{D_\xi}e^{-\lambda t}\frac{1}{2}u^2 da_\xi d\xi+ \tag{4.24}$$
$$+\int_0^t\int_{x_1}^{\infty}\int_{D_\xi}e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)da_\xi d\xi ds.$$

Consequently, in this case we have

$$\int_{D_{x_1}}e^{-\lambda t}\frac{1}{2}u^2 da+\int_0^t\int_{D_{x_1}}e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)dads \tag{4.25}$$
$$\leq \exp(-\frac{\sqrt{2}\pi}{h}x_1)[-\frac{\partial I}{\partial x_1}(0,t)+\frac{\sqrt{2}\pi}{h}I(0,t)].$$

Moreover, by means of the relations (4.22), (4.23) and (4.24), we deduce that

$$\int_{x_1}^{\infty}\int_{D_\xi}e^{-\lambda t}\frac{1}{2}u^2 da_\xi d\xi+ \tag{4.26}$$
$$+\int_0^t\int_{x_1}^{\infty}\int_{D_\xi}e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)da_\xi d\xi ds\leq$$
$$\leq\frac{h}{\sqrt{2}\pi}\exp(-\frac{\sqrt{2}\pi}{h}x_1)[-\frac{\partial I}{\partial x_1}(0,t)+\frac{\sqrt{2}\pi}{h}I(0,t)].$$

Thus, we have established the following Phragmén-Lindelöf type alternative.

Theorem 1. *If u is a solution of the problem $\mathcal{P}$, then for each fixed $t\in[0,T]$, either*

$$\lim_{x_1\to\infty}\{\exp(-\frac{\sqrt{2}\pi}{h}x_1)\mathcal{E}(x_1,t)\}\geq C_1(t), \tag{4.27}$$

or

$$\mathcal{E}(x_1,t)\leq C_2(t)\exp(-\frac{\sqrt{2}\pi}{h}x_1), \tag{4.28}$$

where

$$\mathcal{E}(x_1,t)=\int_{D_{x_1}} e^{-\lambda t}\frac{1}{2}u^2da+ \\ +\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\frac{\lambda}{2}u^2+u_{,11}^2+2u_{,12}^2+u_{,22}^2)dads, \tag{4.29}$$

$$C_2(t)=-\frac{\partial I}{\partial x_1}(0,t)+\frac{\sqrt{2}\pi}{h}I(0,t). \tag{4.30}$$

A useful consequence of the above theorem is the following pointwise decay estimate result.

Corollary 1. *Let u be a solution with the finite time–weighted energy $\mathcal{E}(x_1,t)$. Then, for each fixed $t\in[0,T]$, the following pointwise decay estimate holds true*

$$\frac{2\sqrt{2}\pi^2}{h^2}\int_0^t e^{-\lambda s}u^2(x_1,x_2,s)ds\leq \\ \leq C_2(t)\exp{(-\frac{\sqrt{2}\pi}{h}x_1)},\quad x_1\in[0,\infty)\ ,\quad x_2\in[0,h]. \tag{4.31}$$

Proof. On the basis of the relations (2.3), (2.4) and (4.3), we have

$$u^2(x_1,x_2,s)=2\int_0^{x_2}u(x_1,\xi,s)u_{,2}(x_1,\xi,s)d\xi= \\ =-2\textstyle\int_{x_2}^h u(x_1,\xi,s)u_{,2}(x_1,\xi,s)d\xi= \\ =(\displaystyle\int_0^{x_2}-\int_{x_2}^h)[u(x_1,\xi,s)u_{,2}(x_1,\xi,s)]d\xi\leq \\ \leq\int_0^h|u(x_1,\xi,s)u_{,2}(x_1,\xi,s)|d\xi\leq\frac{h}{\pi}\int_0^h u_{,2}^2(x_1,\xi,s)d\xi. \tag{4.32}$$

Since $\mathcal{E}(x_1,t)$ is bounded, we further get, by (4.4), that

$$
\begin{aligned}
\frac{h}{\pi}\int_0^h u_{,2}^2 dx_2 &= -\frac{h}{\pi}\int_{D_{x_1}} 2u_{,2}u_{,12}da \leq \\
&\leq \frac{2h}{\pi}(\int_{D_{x_1}} u_{,2}^2 da \int_{D_{x_1}} u_{,12}^2 da)^{1/2} \leq \\
&\leq \frac{h^2}{\pi^2\sqrt{2}}(\int_{D_{x_1}} u_{,22}^2 da \int_{D_{x_1}} 2u_{,12}^2 da)^{1/2} \leq \\
&\leq \frac{h^2}{2\sqrt{2}\pi^2}\int_{D_{x_1}} (u_{,22}^2 + 2u_{,12}^2) da.
\end{aligned} \tag{4.33}
$$

Thus, by combining the results described by the relations (4.28), (4.29), (4.32) and (4.33) we get the relation (4.31) and the proof is complete.

5. The second time–weighted line integral measure. In this section we associate with a solution u of the problem $\mathcal{P}$ defined by the relations (2.2)–(2.6), the following time–weighted line integral function

$$
\begin{gathered}
J(x_1,t) = \int_0^t \int_{L_{x_1}} e^{-\lambda s}(-u_{,1}u_{,111} + u_{,11}^2 + u_{,12}^2) dx_2 ds, \\
x_1 \geq 0\ ,\quad t \in [0,T],
\end{gathered} \tag{5.1}
$$

where λ is a positive parameter to be chosen later.

If we substitute the relation (3.12) into (3.5), then we can see that

$$
\begin{aligned}
\frac{\partial^2 J}{\partial x_1^2}(x_1,t) &= \int_{L_{x_1}} e^{-\lambda t}\frac{1}{2}u_{,1}^2 dx_2 + \\
&+ \int_0^t \int_{L_{x_1}} e^{-\lambda s}(\frac{\lambda}{2}u_{,1}^2 + u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2) dx_2 ds.
\end{aligned} \tag{5.2}
$$

On the other hand, by using the relation (5.1), the Wirtinger inequalities (4.3)–(4.5) and the Schwarz inequality, we have

$$
\begin{aligned}
|J(x_1,t)| &\leq \zeta_1(\int_0^t \int_{L_{x_1}} e^{-\lambda s}u_{,1}^2 dx_2 ds \int_0^t \int_{L_{x_1}} e^{-\lambda s}u_{,111}^2 dx_2 ds)^{1/2} \\
&+ \frac{4h^2}{9\pi^2}(1-\zeta_1)(\int_0^t \int_{L_{x_1}} e^{-\lambda s}u_{,122}^2 dx_2 ds \int_0^t \int_{L_{x_1}} e^{-\lambda s}u_{,111}^2 dx_2 ds)^{1/2}
\end{aligned} \tag{5.3}
$$

$$+\frac{h^2}{4\pi^2}\int_0^t\int_{L_{x_1}} e^{-\lambda s}(4u_{,112}^2+u_{,122}^2)dx_2ds\ ,\qquad \zeta_1\in[0,1],$$

so that, by using the arithmetic–geometric mean inequality, we get

$$\begin{aligned}|J(x_1,t)|\le \frac{h^2}{2\pi^2}\int_0^t\int_{L_{x_1}} e^{-\lambda s}\{(\frac{2\zeta_1\pi^2\varepsilon_1}{h^2\lambda})\frac{\lambda}{2}u_{,1}^2+\\ +[\frac{\zeta_1\pi^2}{h^2\varepsilon_1}+\frac{4}{9\varepsilon_2}(1-\zeta_1)]u_{,111}^2+2u_{,112}^2+\\ +[\frac{1}{2}+\frac{4}{9}(1-\zeta_1)\varepsilon_2]u_{,122}^2\}dx_2ds\quad ,\quad \forall\varepsilon_1,\varepsilon_2>0.\end{aligned}\tag{5.4}$$

Further, we set

$$\frac{2\zeta_1\pi^2\varepsilon_1}{h^2\lambda}\le 1\quad ,\quad \frac{\zeta_1\pi^2}{h^2\varepsilon_1}+\frac{4}{9\varepsilon_2}(1-\zeta_1)\le 1\quad ,\quad \frac{4}{9}(1-\zeta_1)\varepsilon_2\le\frac{1}{2},\tag{5.5}$$

so that the relation (5.4) gives

$$|J(x_1,t)|\le\frac{h^2}{2\pi^2}\int_0^t\int_{L_{x_1}} e^{-\lambda s}(\frac{\lambda}{2}u_{,1}^2+u_{,111}^2+2u_{,112}^2+u_{,122}^2)dx_2ds.\tag{5.6}$$

In fact, we can satisfy (5.5) if we set $\zeta_1=0$ and $\varepsilon_2\in[\frac{4}{9},\frac{9}{8}]$.

On the basis of the relations (5.2) and (5.6), we deduce that

$$|J(x_1,t)|\le\frac{h^2}{2\pi^2}\frac{\partial^2 J}{\partial x_1^2}(x_1,t),\quad x_1\ge 0\ ,\ \ t\in[0,T].\tag{5.7}$$

The consequences of the second–order differential inequality (5.7) upon the spatial behaviour of the solution u can be established by means of a similar analysis with that in the above section. Thus, we can state the following Phragmén–Lindelöf type alternative.

Theorem 2. *If u is a solution of the problem $\mathcal{P}$, then for each fixed $t\in[0,T]$, either*

$$\lim_{x_1\to\infty}\{\exp(-\frac{\sqrt{2}\pi}{h}x_1)\mathcal{F}(x_1,t)\}\ge\tilde{C}_1(t),\tag{5.8}$$

or

$$\mathcal{F}(x_1,t) \leq \tilde{C}_2(t)\exp\left(-\frac{\sqrt{2}\pi}{h}x_1\right), \tag{5.9}$$

where

$$\begin{aligned}\mathcal{F}(x_1,t) = \int_{D_{x_1}} e^{-\lambda t}\frac{1}{2}u_{,1}^2 da + \\ + \int_0^t \int_{D_{x_1}} e^{-\lambda s}\left(\frac{\lambda}{2}u_{,1}^2 + u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2\right) da ds,\end{aligned} \tag{5.10}$$

$$\tilde{C}_1(t) = \frac{1}{2}\exp\left(-\frac{\sqrt{2}\pi}{h}z_t^*\right)\left[\frac{\partial J}{\partial x_1}(z_t^*,t) + \frac{\sqrt{2}\pi}{h}J(z_t^*,t)\right] \ , \quad z_t^* \in [0,\infty), \tag{5.11}$$

$$\tilde{C}_2(t) = -\frac{\partial J}{\partial x_1}(0,t) + \frac{\sqrt{2}\pi}{h}J(0,t).$$

By using a similar procedure with that in the above section we can obtain the following consequence of the theorem 2.

Corollary 2. *Let u be a solution of the initial boundary value problem $\mathcal{P}$ for which $\mathcal{F}(x_1,t)$ is bounded. Then, for each fixed $t \in [0,T]$, the following pointwise decay estimate holds true*

$$\begin{aligned}\frac{2\sqrt{2}\pi^2}{h^2}\int_0^t e^{-\lambda s}u_{,1}^2(x_1,x_2,s)ds \leq \tilde{C}_2(t)\exp\left(-\frac{\sqrt{2}\pi}{h}x_1\right), \\ x_1 \in [0,\infty), x_2 \in [0,h].\end{aligned} \tag{5.12}$$

6. The third time–weighted line–area integral measure.

Throughout this section we assume that the solution u of the problem $\mathcal{P}$ satisfies the following conditions at infinity

$$\begin{aligned}u\,,\, u_{,s}\,,\, u_{,1}\,,\, u_{,2}\,,\, u_{,11}\,,\, u_{,12}\,,\, u_{,22}\,,\, u_{,111}\,,\, u_{,112}\,,\, u_{,122} = \\ = o(x_1^{-1/2}) \quad \text{uniformly in} \quad x_2\ ,\ s \quad \text{as} \quad x_1 \to \infty.\end{aligned} \tag{6.1}$$

By combining the results described by the Lemma 2 and the Lemma 3, we introduce the time-weighted line integral function

$$
\begin{aligned}
K(x_1,t) = \int_0^t \int_{L_{x_1}} e^{-\lambda s}\{k_1[(u_{,s}u_{,11})_{,1} - 2u_{,s}(u_{,111}+u_{,122})]+ \\
+k_2[u_{,s}u_{,1} + \frac{1}{2}(u_{,11}^2 + u_{,22}^2)_{,1}]\}dx_2 ds, x_1 \in [0,\infty), t \in [0,T],
\end{aligned} \tag{6.2}
$$

where k_1 and k_2 are arbitrary positive parameters to be chosen later. Clearly, the relations (3.5), (3.8) and (6.2) imply that

$$
\begin{aligned}
K(x_1,t) = K(x_1^*,t) + \int_{D_{x_1^* x_1}} e^{-\lambda t}\frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2]da \\
+ \int_0^t \int_{D_{x_1^* x_1}} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] + k_1 u_{,s}^2 \\
+k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\}dads \ , \quad 0 \le x_1^* < x_1 \ , \quad t \in [0,T].
\end{aligned} \tag{6.3}
$$

Further, if we make x_1 to tend to infinity in (6.3), then by means of the relation (6.1), we have $K(\infty,t)=0$ and hence, we obtain

$$
\begin{aligned}
K(x_1,t) = -\int_{D_{x_1}} e^{-\lambda t}\frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2]da \\
-\int_0^t \int_{D_{x_1}} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] \\
+k_1 u_{,s}^2 + k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\}dads \le 0 \ , \quad x_1 > 0, \quad t \in [0,T].
\end{aligned} \tag{6.4}
$$

We now introduce the time–weighted integral measure

$$
S(x_1,t) = -\int_{x_1}^{\infty} K(\xi,t)d\xi \ , \quad x_1 \in [0,\infty) \ , \quad t \in [0,T], \tag{6.5}
$$

which, by means of the relations (6.1) and (6.2), can be rewritten as a combination of time–weighted line and area integrals

$$
\begin{aligned}
S(x_1,t) = &\int_0^t \int_{L_{x_1}} e^{-\lambda s}[k_1 u_{,s}u_{,11} + \frac{k_2}{2}(u_{,11}^2 + u_{,22}^2)]dx_2 ds - \\
&- \int_0^t \int_{D_{x_1}} e^{-\lambda s}[-2k_1 u_{,s}(u_{,111} + u_{,122}) + k_2 u_{,s}u_{,1}]da ds.
\end{aligned} \tag{6.6}
$$

We note that the relations (6.4) and (6.5) give

$$
S(x_1,t) \geq 0 \quad \text{for all} \quad x_1 \in [0,\infty)\ ,\ \ t \in [0,T], \tag{6.7}
$$

and

$$
\begin{aligned}
S(x_1,t) = &\int_{x_1}^{\infty} \int_{D_\xi} e^{-\lambda t}\frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2]da d\xi \\
&+ \int_0^t \int_{x_1}^{\infty} \int_{D_\xi} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] \\
&+ k_1 u_{,s}^2 + k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\}da_\xi d\xi ds\ , \quad x_1 \in [0,\infty)\ ,\ t \in [0,T].
\end{aligned} \tag{6.8}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{\partial S}{\partial x_1}(x_1,t) = K(x_1,t) = \\
&= -\int_{D_{x_1}} e^{-\lambda t}\frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2]da - \\
&- \int_0^t \int_{D_{x_1}} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] + \\
&+ k_1 u_{,s}^2 + k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\}da ds,
\end{aligned} \tag{6.9}
$$

and

(6.10)
$$\begin{aligned}\frac{\partial^2 S}{\partial x_1^2}(x_1,t) = \int_{L_{x_1}} e^{-\lambda t}\frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2]dx_2 + \\ + \int_0^t \int_{L_{x_1}} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] + \\ + k_1 u_{,s}^2 + k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\}dx_2 ds.\end{aligned}$$

We further proceed to obtain an appropriate estimate for $S(x_1,t)$ in terms of $-\partial S/\partial x_1(x_1,t)$ and $\partial^2 S/\partial x_1^2(x_1,t)$. With this in mind, we write $S(x_1,t)$ as

(6.11)
$$S(x_1,t) = S_1(x_1,t) + S_2(x_1,t),$$

where

(6.12)
$$S_1(x_1,t) = \int_0^t \int_{L_{x_1}} e^{-\lambda s}[k_1 u_{,s} u_{,11} + \frac{k_2}{2}(u_{,11}^2 + u_{,22}^2)]dx_2 ds,$$

(6.13)
$$S_2(x_1,t) = \int_0^t \int_{D_{x_1}} e^{-\lambda s}[2k_1 u_{,s}(u_{,111} + u_{,122}) - k_2 u_{,s} u_{,1}]da ds.$$

By applying the Schwarz's inequality, the relations (4.3)–(4.5) and the arithmetic–geometric mean inequality to (6.12), we get

(6.14)
$$\begin{aligned}|S_1(x_1,t)| \le \eta_1 \int_0^t \int_{L_{x_1}} e^{-\lambda s}[\frac{k_2}{2}(u_{,11}^2 + u_{,22}^2) + \frac{k_1}{2}(\delta_1 u_{,s}^2 + \frac{1}{\delta_1}u_{,11}^2)]dx_2 ds + \\ +(1-\eta_1)\int_0^t \int_{L_{x_1}} e^{-\lambda s}[\frac{k_1}{2}\delta_2 u_{,s}^2 + \frac{k_2}{2}u_{,22}^2 + \\ +(\frac{k_2}{2} + \frac{k_1}{2\delta_2})\frac{h^2}{\pi^2}u_{,112}^2]dx_2 ds,\ \eta_1 \in [0,1], \forall \delta_1, \delta_2 > 0,\end{aligned}$$

that is

(6.15)
$$\begin{aligned}|S_1(x_1,t)| \le \\ \le \alpha \int_0^t \int_{L_{x_1}} e^{-\lambda s}(k_1 u_{,s}^2 + \frac{\lambda k_1}{2}u_{,11}^2 + \frac{\lambda k_1}{2}u_{,22}^2 + 2k_2 u_{,112}^2)dx_2 ds\ ,\end{aligned}$$

where

$$\alpha = \max_{1\le i\le 4} c_i \ , \tag{6.16}$$

and

$$c_1 = \frac{1}{2}[\eta_1\delta_1 + (1-\eta_1)\delta_2] \ , \quad c_2 = \frac{\eta_1}{\lambda}(\frac{k_2}{k_1}+\frac{1}{\delta_1}) \ , \quad c_3 = \frac{k_2}{\lambda k_1} \ , \tag{6.17}$$

$$c_4 = \frac{1-\eta_1}{4}(1+\frac{k_1}{k_2\delta_2})\frac{h^2}{\pi^2}.$$

Thus, by means of the relations (6.10) and (6.15), we obtain the estimate

$$|S_1(x_1,t)| \le \alpha \ \frac{\partial^2 S}{\partial x_1^2}(x_1,t) \ , \quad x_1 \ge 0 \ , \quad t\in[0,T]. \tag{6.18}$$

By using a similar procedure for (6.13), we deduce

$$|S_2(x_1,t)| \le \int_0^t\int_{D_{x_1}} e^{-\lambda s}\{k_1(\delta_3 u_{,s}^2+\frac{1}{\delta_3}u_{,111}^2+\delta_4 u_{,s}^2+\frac{1}{\delta_4}u_{,122}^2)\}dads \tag{6.19}$$

$$+\eta_2\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\frac{k_2\delta_5}{2}u_{,s}^2+\frac{k_2}{2\delta_5}u_{,1}^2)dads+$$
$$+\eta_3\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\frac{k_2\delta_6}{2}u_{,s}^2+\frac{k_2}{2\delta_6}\frac{h^2}{\pi^2}u_{,12}^2)dads+$$
$$+\eta_4\int_0^t\int_{D_{x_1}} e^{-\lambda s}[\frac{k_2\delta_7}{2}u_{,s}^2+\frac{k_2}{2\delta_7}(\frac{2h}{3\pi})^4u_{,122}^2]dads, \forall\delta_3,\delta_4,\delta_5,\delta_6,\delta_7>0,$$

where η_2, η_3, $\eta_4 \in [0,1]$ and $\eta_2+\eta_3+\eta_4=1$. Thus, we have

$$|S_2(x_1,t)| \le$$
$$\le \beta\int_0^t\int_{D_{x_1}} e^{-\lambda s}[k_1u_{,s}^2+\frac{\lambda k_2}{2}u_{,1}^2+\lambda k_1 u_{,12}^2+k_2(u_{,111}^2+u_{,122}^2)]dads, \tag{6.20}$$

where

$$\beta = \max_{1\leq i\leq 5} d_i, \tag{6.21}$$

and

$$d_1 = \delta_3 + \delta_4 + \frac{1}{2}(\eta_2\delta_5 + \eta_3\delta_6 + \eta_4\delta_7)\frac{k_2}{k_1} \quad , \quad d_2 = \frac{\eta_2}{\lambda\delta_5} \quad , \quad d_3 = \frac{1}{\delta_3}\frac{k_1}{k_2} \ , \tag{6.22}$$

$$d_4 = \frac{h^2}{\pi^2}\frac{\eta_3}{2\lambda\delta_6}\frac{k_2}{k_1} \quad , \qquad d_5 = \frac{1}{\delta_4}\frac{k_1}{k_2} + \frac{\eta_4}{2\delta_7}(\frac{2h}{3\pi})^4.$$

Further, the relations (6.9) and (6.20) give

$$|S_2(x_1,t)| \leq -\beta\frac{\partial S}{\partial x_1}(x_1,t) \quad , \quad x_1 \geq 0, \quad t \in [0,T]. \tag{6.23}$$

Consequently, the relations (6.11), (6.18) and (6.23) lead to the following second–order differential inequality

$$S(x_1,t) \leq \alpha\,\frac{\partial^2 S}{\partial x_1^2}(x_1,t) - \beta\,\frac{\partial S}{\partial x_1}(x_1,t), \tag{6.24}$$

which can be rewritten as

$$\frac{\partial^2 S}{\partial x_1^2}(x_1,t) - a\,\frac{\partial S}{\partial x_1}(x_1,t) - b\,S(x_1,t) \geq 0, \tag{6.25}$$

where

$$a = \frac{\beta}{\alpha} \quad , \quad b = \frac{1}{\alpha}. \tag{6.26}$$

Furthermore, following [15], we can write (6.25) in the form

$$(\frac{\partial}{\partial x_1} + r_1)(\frac{\partial S}{\partial x_1} - r_2 S) \geq 0, \tag{6.27}$$

where

$$r_1 = \frac{1}{2}(-a + \sqrt{a^2 + 4b}) \quad , \quad r_2 = \frac{1}{2}(a + \sqrt{a^2 + 4b}). \tag{6.28}$$

Then, from (6.27), we deduce

$$\frac{\partial}{\partial x_1}[\exp(r_1 x_1)(\frac{\partial S}{\partial x_1} - r_2 S)] \geq 0, \tag{6.29}$$

so that, by integration from 0 to x_1, we get

$$\begin{gathered} -\frac{\partial S}{\partial x_1}(x_1, t) + r_2 S(x_1, t) \leq \\ \leq e^{-r_1 x_1}[-\frac{\partial S}{\partial x_1}(0, t) + r_2 S(0, t)] \ , \quad t \in [0, T]. \end{gathered} \tag{6.30}$$

Theorem 3. *Let u be a solution of the problem $\mathcal{P}$ satisfying the conditions (6.1) at infinity. Then, for each fixed $t \in [0, T]$, the spatial behaviour of the solution is described by the following Saint–Venant type decay estimate*

$$\mathcal{G}(x_1, t) \leq C_1^*(t) \exp(-r_1 x_1) \ , \quad x_1 \geq 0, \tag{6.31}$$

where

$$\begin{gathered} \mathcal{G}(x_1, t) = \int_{D_{x_1}} e^{-\lambda t} \frac{1}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] da + \\ + \int_0^t \int_{D_{x_1}} e^{-\lambda s}\{\frac{\lambda}{2}[k_1(u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2) + k_2 u_{,1}^2] + \\ + k_1 u_{,s}^2 + k_2(u_{,111}^2 + 2u_{,112}^2 + u_{,122}^2)\} da ds \geq 0, \end{gathered} \tag{6.32}$$

$$C_1^*(t) = -\frac{\partial S}{\partial x_1}(0, t) + r_2 S(0, t). \tag{6.33}$$

Corollary 3. *Let u be a solution of the initial boundary value problem $\mathcal{P}$, satisfying the conditions (6.1). Then, for each fixed $t \in [0,T]$, the following pointwise decay estimate holds true*

$$\begin{gathered}\frac{3\sqrt{2}\pi^2 k_1}{h^2}\, e^{-\lambda t}\, u^2(x_1,x_2,t)+\\ +\int_0^t e^{-\lambda s}\{\frac{\pi^2\lambda k_1}{h^2}\, u^2(x_1,x_2,s)+\frac{2\pi^2 k_2}{h^2}\, u_{,1}^2(x_1,x_2,s)+\\ +\frac{\pi\sqrt{2\lambda k_1 k_2}}{h}\, u_{,2}^2(x_1,x_2,s)\}ds\leq\frac{3}{2}\, C_1^*(t)\exp\left(-r_1x_1\right),\\ x_1\geq 0\ ,\quad x_2\in[0,h].\end{gathered}\tag{6.34}$$

Proof. By using the relations (4.32) and (4.33), we deduce that

$$\frac{3\sqrt{2}\pi^2 k_1}{2h^2}\, e^{-\lambda t}u^2(x_1,x_2,t)\leq\frac{3}{2}\int_{D_{x_1}} e^{-\lambda t}\frac{k_1}{2}(2u_{,12}^2+u_{,22}^2)da.\tag{6.35}$$

Further, we use a procedure similar with that in deduction to the relations (4.32) and (4.33), in order to obtain

$$\begin{gathered}\frac{\pi^2\lambda k_1}{h^2}\int_0^t e^{-\lambda s}\, u^2(x_1,x_2,s)ds\leq\\ \leq\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\varepsilon_1\frac{\lambda k_1}{2}u_{,12}^2+\frac{\lambda k_1}{2\varepsilon_1}u_{,22}^2)dads\ ,\quad\forall\varepsilon_1>0,\end{gathered}\tag{6.36}$$

$$\begin{gathered}\frac{2\pi^2 k_2}{h^2}\int_0^t e^{-\lambda s}\, u_{,1}^2(x_1,x_2,s)ds\leq\\ \leq\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\varepsilon_2 k_2 u_{,112}^2+\frac{k_2}{\varepsilon_2}u_{,122}^2)dads\ ,\quad\forall\varepsilon_2>0,\end{gathered}\tag{6.37}$$

$$\begin{gathered}\frac{\pi\sqrt{2\lambda k_1 k_2}}{h}\int_0^t e^{-\lambda s}\, u_{,2}^2(x_1,x_2,s)ds\leq\\ \leq\int_0^t\int_{D_{x_1}} e^{-\lambda s}(\varepsilon_3\frac{\lambda k_1}{2}u_{,22}^2+\frac{1}{\varepsilon_3}k_2u_{,122}^2)dads\ ,\ \forall\varepsilon_3>0.\end{gathered}\tag{6.38}$$

If we combine the estimates (6.35) to (6.38) and we take into account the relations (6.31) to (6.33) and we set $\varepsilon_1 = \varepsilon_2 = 3$, $\varepsilon_3 = 1$, then we get the estimate (6.34) and the proof is complete.

Finally we proceed to prove that the decay rate of the end effects can be controlled by the parameter λ characterising the measure $S(x_1, t)$. In fact, we prove that for appropriate large values for λ the decay rate predicted in the present section is larger than that obtained in the above sections. In this aim we recall that from the relations (6.26) and (6.28) we have

$$r_1 = \frac{2}{\beta + \sqrt{\beta^2 + 4\alpha}}. \tag{6.39}$$

If we set $\eta_1 = 1$ and $\delta_1 = (2/\lambda)^{1/2}$ in (6.17), then we get

$$c_1 = \frac{1}{\sqrt{2\lambda}}\ , \quad c_2 = \frac{1}{\sqrt{2\lambda}} + \frac{1}{\lambda}\frac{k_2}{k_1}\ , \quad c_3 = \frac{1}{\lambda}\frac{k_2}{k_1}\ , \quad c_4 = 0, \tag{6.40}$$

so that

$$\alpha = \frac{1}{\sqrt{2\lambda}} + \frac{1}{\lambda}\frac{k_2}{k_1}. \tag{6.41}$$

On the other hand, if we set

$$\eta_2 = \eta_4 = 0\ , \quad \eta_3 = 1\ , \quad \delta_3 = \delta_4 = (\frac{k_1}{k_2})^{1/2}\ , \quad \delta_6 = 2\,(\frac{k_1}{k_2})^{3/2}, \tag{6.42}$$

in (6.22), then we obtain

$$d_1 = 3\sqrt{\frac{k_1}{k_2}}\ , \quad d_2 = 0\ , \quad d_3 = d_5 = \sqrt{\frac{k_1}{k_2}}\ , \quad d_4 = \frac{h^2}{4\pi^2\lambda}(\frac{k_2}{k_1})^{5/2}. \tag{6.43}$$

Further, if we set

$$\frac{k_1}{k_2} = (\frac{h^2}{4\pi^2\lambda})^{1/3}, \tag{6.44}$$

then we get

$$\alpha = \frac{1}{\sqrt{2\lambda}} + (\frac{4\pi^2}{h^2})^{1/3}\frac{1}{\lambda^{2/3}} \quad , \quad \beta = 3(\frac{h^2}{4\pi^2\lambda})^{1/6}. \tag{6.45}$$

Now it can be easily seen from the relations (6.39) and (6.45) that the decay rate in (6.31) is controlled by the parameter λ. Thus, it is seen that for appropriate large values for λ the decay rate in (6.31) is more rapidly than that described by the relations (4.28) and (5.9).

REFERENCES

1. Knowles, J.K. – *On Saint–Venant's principle in the two–dimensional linear theory of elasticity,* Arch. Rational Mech. Anal., 21 (1966), 1–22.
2. Knowles, J.K. – *An energy estimate for the biharmonic equation and its application to Saint–Venant's principle in plane elastostatics,* Indian J. Pure Appl. Math., 14(1983), 791–805.
3. Flavin, J.N. – *On Knowles' version of Saint-Venant's principle in two–dimensional elastostatics,* Arch. Rational Mech. Anal., 53 (1974), 366–375.
4. Oleinik, O.A. and Yosifian, G.A. – *On Saint–Venant's principle in plane elasticity theory,* Dokl. Akad. Nauk SSSR, 239 (1978), 530–533; English transl. in Soviet Math. Dokl., 19 (1978), 364–368.
5. Oleinik, O.A. and Yosifian, G.A. – *The Saint–Venant principle in the two–dimensional theory of elasticity and boundary problems for a biharmonic equation in unbounded domains,* Sibirsk. Math. Zh., 19 (1978), 1154–1165; English transl. in Siberian Math. J., 19 (1978), 813–822.
6. Horgan, C.O. – *Decay estimates for the biharmonic equation with applications to Saint–Venant principles in plane elasticity and Stokes flow,* Quart. Appl. Math., 47 (1989), 147–157.
7. Flavin, J.N. and Knops, R.J. – *Some convexity considerations for a two–dimensional traction problem,* J. Appl. Math. and Phys. (ZAMP), 39 (1988), 166-176.
8. Payne, L.E. and Schaefer, P.W. – *Some Phragmén–Lindelöf type results for the biharmonic equation,* J. Appl. Math. and Phys. (ZAMP), 45 (1994), 414–432.
9. Lin, C. – *Energy estimates for the biharmonic equation in three dimensions,* Quart. Appl. Math., 52 (1994), 649–663.

10. Lin, C. – *Spatial decay estimates and energy bounds for the Stokes flow equation,* Stability and Appl. Anal. of Continuous Media, 2 (1992), 249–264.

11. Knops, R.J. and Lupoli, C. – *End effects for plane Stokes flow along a semi–infinite strip,* J. Appl. Math. and Phys. (ZAMP), 48 (1997), 905–920.

12. Chiriţă, S. and Ciarletta, M. – *Spatial behaviour of solutions in the plane Stokes flow,* to appear.

13. Ciarletta, M. and Ieşan, D. – *Non–classical elastic solids,* Pitman Research Notes Mathematical Series, Longman Scientific & Technical, Essex, 1993.

14. Chiriţă, S. and Ciarletta, M. – *Time–weighted surface power function method for the study of spatial behaviour in dynamics of continua,* Eur. J. Mech. A/Solids, 18 (1999), 915–933.

15. Ames, K.A., Payne, L.E. and Schaefer, P. W. – *Spatial decay estimates in time–dependent Stokes flow,* SIAM J. Math. Anal., 24 (1993), 1395–1413.

Received: 3.IX.2001

Faculty of Mathematics
University "Al. I. Cuza" Iaşi
Iaşi – 6600
ROMANIA

Department of Information Engineering and Applied Mathematics
University of Salerno
via Ponte don Melillo
84084 Fisciano (SA)
ITALY