ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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A GOULD TYPE INTEGRAL WITH RESPECT TO A MULTIMEASURE I

BY

ANCA-MARIA PRECUPANU and ANCA CROITORU

Abstract. In this paper we consider two types of multivalued integrals for real bounded functions with respect to finite additive multimeasures φ of finite variation; we call them respectively φ-integral and strong φ-integral of Gould type.

These integrals are limits of Gould type integral sums [4] with multimeasure instead of vector measure.

We also establish some properties of φ-totally measurable functions, φ-integrable functions and strong φ-integrable functions, pointing out relationships between them.

Introduction. In this paper we define two types of integrability for bounded real valued functions $f : S \to \mathbb{R}$ defined on a measurable space $(S, \mathcal{A})$ with respect to a finite additive multimeasure $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ (defined on the σ-algebra $\mathcal{A}$ taking values in $\mathcal{P}_{kc}$, the family of all non-void compact convex subsets of a Banach space X) of finite variation ν, using integral sums of Gould type [4].

In §2 we define the φ-integrability and the φ-integral as the limit of a net of integral sums in Hausdorff distance and next, in §3, we define the strong φ-integrability and the strong φ-integral as a limit of the same net in $|\cdot|$, where by $|A|$ we mean $D(A, \{0\})$ (D is the Hausdorff distance on $\mathcal{P}_{kc}$ and 0 is the zero element of X).

We also study several properties of the two integrals and we point out some relationships between the two types of integrability and the total measurability of functions, on one hand, and the Brooks integrability, on the other hand.

§1. Terminology and Notations. Let $(S, \mathcal{A})$ be a measurable space, where S is a nonempty set S, and $\mathcal{A}$ a σ-algebra of subsets of S, and let $(X, \|\cdot\|)$ be a real Banach space. $\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ denotes the family of all nonempty compact convex subsets of X and D the usual Hausdorff metric defined on $\mathcal{P}_{kc}$. We define $|\!| A |\!| = D(A, \theta)$, $A \in \mathcal{P}_{kc}$, where $\theta = \{0\}$. If $A, B \in \mathcal{P}_{kc}$, $\lambda \in \mathbb{R}$,

$$A + B = \{x + y \mid x \in A, y \in B\},$$
$$\lambda A = \{\lambda x \mid x \in A\}.$$

Definition 1.1.

a) **A partition** of S is a finite family $P = \{E_i\}_{i=1}^n \subset \mathcal{A}$, $E_i \cap E_j = \phi$ $(i \neq j)$ such that $\bigcup_{i=1}^n E_i = S$.

b) Let $P = \{E_i\}_{i=1}^n$ and $P' = \{B_j\}_{j=1}^m$ be two partitions of S. P' is **finer than** P, denoted $P' \geq P$, if for any $j \in \{1, ..., m\}$, there is $i \in \{1, ..., n\}$ such that $B_j \subseteq E_i$.

c) The **common refinement** of two partitions $P = \{E_i\}_{i=1}^n$, $P' = \{B_j\}_{j=1}^m$ is the partition $P \wedge P' = \{E_i \cap B_j\}$.

Definition 1.2. Let $\mu : \mathcal{A} \to X$ be a finitely additive measure. We call the **semi-variation** of μ, the function $\widetilde{\mu}$ defined on $\mathcal{A}$ by

$$\widetilde{\mu}(E) = \sup\{\|\mu(A)\|; A \in \mathcal{A}, A \subseteq E\}, \ (\forall) A \in \mathcal{A}.$$

This definition differes from the definition given in [3], 10.3-IV, but $\widetilde{\mu}$ has the same properties as the function semi-variation given in [3]. Thus we have

Proposition 1.3. *The function $\widetilde{\mu}$ is monotone and subadditive and if μ is countably additive, then $\widetilde{\mu}$ is countably subadditive.*

In what follows we consider only multifunctions $\varphi : A \to \mathcal{P}_{kc}$ with $\varphi(\phi) = 0$.

Definition 1.4.

a) A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is said to be a **multimeasure** (cf. [2]) if it is finitely additive that is

$$\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2), \forall E_1, E_2 \in \mathcal{A},\ E_1 \cap E_2 = \phi.$$

b) A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is said to be **countable additive** or σ-**additive** if for every sequence $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$ with $E_n \cap E_m = \phi$, $n \neq m$,

$$\varphi\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \varphi(E_n),$$

where the sum in the right side of equality is the limit in Hausdorff distance of the sequence $\left\{\sum_{k=1}^{n} \varphi(E_k)\right\}_{n=1}^{\infty}$.

Definition 1.5. Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multifunction.

a) The **total variation** of φ is the non-negative (possibly infinite) set function $v(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$\nu(A) = v(\varphi, A) = \sup \sum_{i=1}^{n} \left| \varphi(E_i) \right|,$$

where the supremum is taken over all finite partitions $\{E_i\}_{i=1}^{n}$ of $A \in \mathcal{A}$, with $E_i \in \mathcal{A}$, $i \in \{1, ..., n\}$.

b) If $\nu(S) < \infty$ we say that the multifunction φ is of **finite variation**.

If φ is a multimeasure, then ν is finitely additive and if φ is σ-additive the same is ν. As in definition 1.2 we give for multimeasures

Definition 1.6. Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multimeasure. The real function $\widetilde{\varphi} : \mathcal{A} \to [0, +\infty]$ given by

$$\widetilde{\varphi}(A) = \sup\left\{ \left| \varphi(E) \right| ; E \in \mathcal{A}, E \subseteq A \right\}, \forall A \in \mathcal{A}$$

is said to be the **semi-variation** of φ.

Clearly, $\widetilde{\varphi}$ is monotone, subadditive and $\widetilde{\varphi}(E) \leq v(\varphi, E), \forall E \in \mathcal{A}$.

We also associate to the multifunction φ another function $\nu^* : P(S) \to [0, +\infty]$, defined by

$$\nu^*(E) = \inf\{\nu(B) ; B \in \mathcal{A}, E \subseteq B\},$$

where ν is the total variation of φ.

Definition 1.7. If φ is a multimeasure we say that a property P holds φ-**almost everywhere** (briefly φ-**a.e.**) if the property P is valid on S except a set A with $\nu^*(A) = 0$.

Definition 1.8. Let $(S, \mathcal{A})$ be a measurable space, $\mu : \mathcal{A} \to X$ a finitely additive measure and $f : S \to \mathbb{R}$ a given function.

a) f is said to be $\mu-$ **totally measurable** over $(S, \mathcal{A}, \mu)$ (briefly $\mu-$ **totally measurable**) if for each $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ of S such that $\widetilde{\mu}(E_0) < \varepsilon$ and $\sup_{s,t \in E_i} |f(s) - f(t)| = osc(f, E_i) < \varepsilon$, $\forall i \in \{1, ..., n\}$.

b) If $E \in \mathcal{A}$, f is called $\mu-$ **totally measurable over** E if the restriction $f/_E$ of f to $E$is $\mu-$ totally measurable over $(E, \mathcal{A}_E, \mu_E)$, where $\mathcal{A}_E = \{A \cap E \mid A \in \mathcal{A}\}$ and $\mu_E = \mu/_{\mathcal{A}_E}$.

For φ a multimeasure we can easily show

Lemma 1.9. *If $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is a multimeasure, $A, B \in \mathcal{A}$ with $A \cap B = \phi$ and for every $\varepsilon > 0$ we have $\left| \varphi(A \cup B) \right| \leq \varepsilon$ and $\left| \varphi(B) \right| \leq \varepsilon$, then $\left| \varphi(A) \right| \leq 2\varepsilon$.*

§2. A multivalued Gould type integral. In the sequel, we suppose that $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is a multimeasure of finite variation ν. Taking into account that if $A_1, A_2..., A_k \in \mathcal{P}_{kc}$, then the function $f(x) = \left| x_1A_1 + x_2A_2 + ... + x_kA_k \right|$, $(\forall)$ $x = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$, is continuous on $\mathbb{R}^k$, we can easily obtain the following multivalued variants of the lemmas 4.4, 4.5 and theorem 4.6 from [4]:

Proposition 2.1.

(i) If $A, B \in \mathcal{P}_{kc}$ and $\lambda \in \mathbb{R}$, then the function $g(\lambda) = \left| A + \lambda B \right|$, $(\forall)$ $\lambda \in \mathbb{R}$, is either constant or has no maximum.

(ii) Let $A_1, A_2..., A_n$ be compact, convex nonvoid subsets of X and let $x_1, x_2..., x_n$ be real numbers. Then $\left| x_1A_1 + x_2A_2 + ... + x_nA_n \right|$ as a function of $(x_1..., x_n)$ over the n-cube $||x_r|| \leq 1$ assumes its maximum at a vertex, i.e. at some point for which $||x_r|| = 1$ for $r = 1, 2...n$.

(iii) Let $\{A_i\}_{i=1}^n$ be elements of $\mathcal{P}_{kc}$ and $c_i \in \mathbb{R}$, $i \in \{1, ..., n\}$. If $\left| \sum_{i \in J} c_iA_i \right| \leq \alpha$ for any $J \subseteq \{1, ..., n\}$, then $\left| \sum_{i \in J} d_iA_i \right| \leq 2\alpha$ for any

$d_i \in \mathbb{R}$, $|d_i| \leq |c_i|$, $i \in \{1, ..., n\}$ *and any* $J \subseteq \{1, ..., n\}$.

Definition 2.2. Let $f : S \to \mathbb{R}$ be a real function.

a) f is said to be $\varphi-$ **totally measurable over** $(S, \mathcal{A}, \varphi)$ (shortly $\varphi-$ **totally measurable**) if for each $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ such that $\widetilde{\varphi}(E_0) < \varepsilon$ and $osc(f, E_i) < \varepsilon$, $\forall i \in \{1, ..., n\}$.

b) f is said to be ν-**totally measurable** if in a) $\widetilde{\varphi}$ is replaced by ν,the total variation of φ.

c) If $E \in \mathcal{A}$, f is called $\varphi-$ **totally measurable on** E if the restriction $f/_E$ is $\varphi-$ totally measurable on $(E, \mathcal{A}_E, \varphi_E)$,where $\varphi_E = \varphi/_{\mathcal{A}_E}$.

Remark 2.3. We observe that every bounded ν-totally measurable function is Brooks integrable [2] with respect to φ.

Indeed, if in the definition 2.2 - b) we take $\varepsilon = \dfrac{1}{n}$, $n \in N^*$, we obtain a partition $P_n = (E_i^n)_{i=0}^{p_n}$ of S for which $\nu(E_0^n) < \dfrac{1}{n}$ and $osc(f, E_i^n) < \dfrac{1}{n}$, for every $i \in \{1, 2, ..., p_n\}$.

Defining the sequence of simple functions $f_n = \sum_{i=0}^{p_n} f(s_i^n) 1_{E_i^n}$ (1_E is the characteristic function of E), for $s_i^n \in E_i^n$, we easily see that f_n is convergent to f in φ-measure and $\lim_{n,m\to\infty} \int_S |f_n - f_m| \, d\nu = 0$ (the integral is the classical integral of Lebesgue [3]).

Now we can point out some properties of $\varphi-$ totally measurable functions.

Theorem 2.4. *Let* $f : S \to \mathbb{R}$ *be a real function and the following statements:*

(i) f *is* $\nu-$ *totally measurable.*

(ii) f *is* $\varphi-$ *totally measurable.*

(iii) f *is* $\varphi-$ *totally measurable over* E, $\forall E \in \mathcal{A}$

(iv) Given $\varepsilon > 0$, *there is a partition* $P_0 = \{E_i\}_{i=1}^n$ *such that for any* $B_i \in \mathcal{A}, B_i \subseteq E_i$, $i \in \{1, ..., n\}$,

$$\left| \sum_{i=1}^{n} osc(f, E_i) \varphi(B_i) \right| < \varepsilon.$$

(v) Given $\varepsilon > 0$, *there is a partition* $P_0 = \{E_i\}_{i=0}^n$ *such that*

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \lambda_{ij} \varphi\left(B_j^i\right) \right| < \varepsilon$$

for any $\left\{B_j^i\right\}_{j=1}^{k_i}$ *partition of* E_i *and any* $\lambda_{ij} \in \mathbb{R}$, $|\lambda_{ij}| \leq osc(f, E_i)$, $\forall j \in \{1, ..., k_i\}$, $i \in \{1, ..., n\}$. Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof (i) $\Rightarrow$(ii). This follows directly by the inequality : $\widetilde{\varphi}(E) \leq \nu(E)$, $\forall E \in \mathcal{A}$.

(ii) $\Rightarrow$ (iii). Let $E \in \mathcal{A}$ and $\varepsilon > 0$. Then there is $P_\varepsilon = \{B_i\}_{i=0}^n$ a partition of S such that $\widetilde{\varphi}(B_0) < \varepsilon$ and $osc(f, B_i) < \varepsilon, \forall i \in \{1, ..., n\}$. Let us take $P'_\varepsilon = \{E \cap B_i\}_{i=0}^n$. Since $\widetilde{\varphi}$ is monotone, it follows that P'_ε satisfies the conditions of Definition 2.2.

(iii) $\Rightarrow$ (ii). This is evident.

(ii) $\Rightarrow$ (iv). Let $M_1 = \sup\{|f(s)|; s \in S\}$, $M_2 = \widetilde{\varphi}(S)$ and $\varepsilon > 0$. Then, for $\delta = \dfrac{\varepsilon}{2(M_1 + M_2)}$, there is $P_0 = \{E_i\}_{i=0}^n$ a partition of S such that $\widetilde{\varphi}(E_0) < \delta$ and $osc(f, E_i) < \delta$, $i \in \{1, ..., n\}$. For any $J \subseteq \{1, ..., n\}$ and $B_i \in \mathcal{A}$, $B_i \subseteq E_i$, $i \in \{0, 1, ..., n\}$ we have

$$\left| \sum_{i \in J} \delta \varphi(B_i) \right| = \delta \left| \varphi\left(\bigcup_{i \in J} B_i \right) \right| \leq \delta M_2. \tag{1}$$

Now we can apply Proposition 2.1 taking as A_i, c_i, d_i respectively $\varphi(B_i)$, $\delta, osc(f, E_i), 1 \leq i \leq n$ and $\alpha = \delta M_2$. Hence, by (1), we obtain

$$\left| \sum_{i \in J} osc(f, E_i) \varphi(B_i) \right| \leq 2\delta M_2 \tag{2}$$

and using (2) we have

$$\begin{aligned} \left| \sum_{i=0}^{n} osc(f, E_i) \varphi(B_i) \right| &\leq osc(f, E_0) \cdot \left| \varphi(B_0) \right| + \\ &+ \left| \sum_{i=1}^{n} osc(f, E_i) \varphi(B_i) \right| \leq 2\delta M_1 + 2\delta M_2 = \varepsilon. \end{aligned}$$

(iv) $\Rightarrow$ (v). Given $\varepsilon > 0$, let $P_0 = \{E_i\}_{i=1}^n$ be a partition of S such that

$$\left|\sum_{i=1}^{n} osc\left(f,E_i\right)\varphi\left(B_i\right)\right| \leq \varepsilon \tag{3}$$

for any $\{B_i\}_{i=1}^{n} \subset \mathcal{A}$, $B_i \subseteq E_i$, $i \in \{1,...,n\}$.

For each $i \in \{1,...,n\}$, $j \in \{1,...,k_i\}$, let $\left\{A_j^i\right\} \subset \mathcal{A}$ be an arbitrary partition of E_i. Let λ_{ij} be arbitrary scalars such that $|\lambda_{ij}| \leq osc\left(f,E_i\right)$.

Then by Proposition 2.1, we obtain

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\lambda_{ij}\varphi\left(A_j^i\right)\right| \leq 2\cdot\max\left(\left|\sum_{i\in J}\sum_{j\in J_i} osc\left(f,E_i\right)\varphi\left(A_j^i\right)\right|;\right. \tag{4}$$

$$\left.J\subseteq\{1,...,n\},J_i\subseteq\{1,...,k_i\}\right) = 2\left|\sum_{i\in J_0}\sum_{j\in J_{i_0}^0} osc\left(f,E_i\right)\varphi\left(A_j^i\right)\right| = 2\alpha,$$

where $\left\{J_0, J_{i_0}^0\right\}$ is a suitable choice of $\{J, J_i\}$ which maximizes the right-hand side of (4).

If we note $B_i = \bigcup_{j\in J_{i_0}^0} A_j^i \subseteq E_i$, $i \in J_0$, then $\sum_{j\in J_{i_0}^0}\varphi\left(A_j^i\right) = \varphi\left(B_i\right)$, $i \in J_0$ and if we consider $B_i = \phi$ for $i \in \{1,...,n\} \setminus J_0$, then

$$\alpha = \left|\sum_{i=0}^{n} osc\left(f,E_i\right)\varphi\left(B_i\right)\right|.$$

From (3) and (4) it follows

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\lambda_{ij}\varphi\left(A_j^i\right)\right| \leq 2\left|\sum_{i=1}^{n} osc\left(f,E_i\right)\varphi\left(B_i\right)\right| < 2\varepsilon. \blacksquare$$

Theorem 2.5. *Let $f : S \to \mathbb{R}$ be such that $f = 0$, φ-almost everywhere. Then f is ν-totally measurable.*

Proof. Let $A = \{s \in S; f(s) \neq 0\}$. By virtue of hypothesis we have $\nu^*(A) = 0$.

Since $\inf\{\nu(E); A \subseteq E, E \in \mathcal{A}\} = 0$ we obtain for every $\varepsilon > 0$ a set $E_\varepsilon \in \mathcal{A}, A \subseteq E_\varepsilon$, such that $\nu(E_\varepsilon) < \varepsilon$.

Let us now consider the partition $P_\varepsilon = \{E_\varepsilon, S \setminus E_\varepsilon\}$ of S.

Since $A \subseteq E_\varepsilon$ we have $cE_\varepsilon \subseteq cA$ and hence

$$osc(f, S\setminus E_\varepsilon) = \sup\{|f(s) - f(t)|; s, t \in S \setminus E_\varepsilon\} = 0 < \varepsilon.$$

Consequently P_ε is a partition of S such that $\nu(E_\varepsilon) < \varepsilon$ and $osc(f, S \setminus E_\varepsilon) < \varepsilon$, that is f is ν-totally measurable. ■

Remark 2.6. According to theorem 2.4 we see that f is also φ-totally measurable.

Definition 2.7. Let $f : S \to \mathbb{R}$ be a bounded function. For every partition $P = \{E_i\}_{i=1}^n$ of S and every $s_i \in E_i, i = 1, 2, ..., n$ we denote by $\sigma(P) = \sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i)$.

a) We say that f is φ-**integrable** on S if the net $\{\sigma(P)\}_{P\in(\mathcal{P},\leq)}$ is convergent in $(\mathcal{P}_{kc}, D)$ ($\mathcal{P}$ is the set of all partitions of S and $\leq$ is the order relation on $\mathcal{P}$ given in the definition 1.1) for every choice of the points $s_i \in E_i$ and its limit is called **the integral** of f on S with respect to φ, denoted by $\int_S f d\varphi$.

Hence f is φ-integrable on S if f there exists $A \in \mathcal{P}_{kc}$ with the property that for every $\varepsilon > 0$ there exists a partition P_ε of S such that for every other partition $P = \{E_i\}_{i=1}^n$ with $P \geq P_\varepsilon$ and every choice of points $s_i \in E_i, i = 1, 2, ..., n$, we have

$$D(\sigma(P), A) < \varepsilon.$$

b) If $E \in \mathcal{A}$, f is said to be φ-**integrable over** E if the restriction $f/_E$ of f to E is φ-integrable on $(E, \mathcal{A}_E, \varphi_E)$.

Theorem 2.8. *Let $f : S \to \mathbb{R}$ be a bounded function. If $f = 0, \varphi$-a.e., then f is φ-integrable and $\int_S f d\varphi = 0$.*

Proof. Suppose $f = 0, \varphi$-a.e. According to the theorem 2.5, f is ν-integrable. Because f is bounded there exists $M > 0$ such that $|f(s)| \leq$

$M, (\forall)\, s \in S$. Let $\varepsilon > 0$ arbitrary. As in the proof of the theorem 2.5 let us consider the partition $P_\varepsilon = \{E_\varepsilon, S \setminus E_\varepsilon\}$ such that $\nu(E_\varepsilon) < \dfrac{\varepsilon}{M}$ and $osc(f; S \setminus E_\varepsilon) < \dfrac{\varepsilon}{M}$.

Let us take a partition $P = \{B_i\}_{i=1}^n$ of S with $P \geq P_\varepsilon$. Assume that $B_1, ..., B_k \subseteq E_\varepsilon$ and $\bigcup_{i=1}^{k} B_i = E_\varepsilon$ and $B_{k+1}, ..., B_n \subseteq S \setminus E_\varepsilon$ and $\bigcup_{i=1}^{k} B_i = S \setminus E_\varepsilon$. Let us take $s_i \in E_i, i = 1, 2, ..., n$. Then

$$D\left(\sum_{i=1}^{n} f(s_i)\varphi(B_i), 0\right) = \left|\sum_{i=1}^{k} f(s_i)\varphi(B_i) + \sum_{i=k+1}^{n} f(s_i)\varphi(B_i)\right| =$$
$$= \left|\sum_{i=1}^{k} f(s_i)\varphi(B_i)\right| \leq \sum_{i=1}^{k} |f(s_i)| \cdot |\varphi(B_i)| \leq M \sum_{i=1}^{k} \nu(B_i) =$$
$$= M \cdot \nu\left(\bigcup_{i=1}^{k} B_i\right) = M \cdot \nu(E_\varepsilon) < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Hence for every $\varepsilon > 0$, there exists a partition P_ε of S such that for every partition $P = \{B_i\}_{i=1}^n$ of S with $P \geq P_\varepsilon$ and every choise of $s_i \in B_i, i = 1, 2, ..., n$, we have $D(\sigma(P), 0) < \varepsilon$. Therefore f is φ-integrable and $\int_S f d\varphi = 0$. ■

Theorem 2.9. *If $f : S \to \mathbb{R}$ is a bounded, ν-totally measurable function, then f is φ-integrable.*

Proof. Since f is bounded there exists $M_1 > 0$ such that $|f(s)| \leq M_1$, for every $s \in S$.

Denote $M = \max\{2M_1, \nu(S)\}$.

Since f is ν-totally measurable, for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ of S such that $\nu(E_0) < \dfrac{\varepsilon}{2M}$, for every $i = 1, 2, ..., n$.

Let P_1, P_2 be partitions of S with $P_1 \geq P_\varepsilon, P_2 \geq P_\varepsilon, P_1 = \{A_j\}_{j=1}^m, P_2 = \{B_k\}_{k=1}^p$ and $s_j \in A_j, j = 1, 2, ..., m$, $t_k \in B_k, k = 1, 2, ..., p$.

Let us consider their common refinement $P_1 \wedge P_2 = \{A_j \cap B_k\}_{\substack{j=1,2,...,m \\ k=1,2,...,p}}$; this partition can be rewritten as $\{C_i\}_{i=1}^l$.

Let us assume that $E_0 = \bigcup_{i=1}^{q} C_i$ and $\bigcup_{j=1}^{n} E_j = \bigcup_{i=q+1}^{l} C_i$.

Now we have

$$D\left(\sigma\left(P_1\right),\sigma\left(P_2\right)\right) = D\left(\sum_{j=1}^{m} f\left(s_j\right)\varphi\left(A_j\right), \sum_{k=1}^{p} f\left(t_k\right)\varphi\left(B_k\right)\right) =$$

$$D\left(\sum_{j=1}^{m} f\left(s_j\right)\varphi\left(\bigcup_{k=1}^{p}\left(A_j\cap B_k\right)\right), \sum_{k=1}^{p} f\left(t_k\right)\varphi\left(\bigcup_{j=1}^{m}\left(A_j\cap B_k\right)\right)\right) =$$

$$D\left(\sum_{j=1}^{m}\sum_{k=1}^{p} f\left(s_j\right)\varphi\left(A_j\cap B_k\right), \sum_{j=1}^{m}\sum_{k=1}^{p} f\left(t_k\right)\varphi\left(A_j\cap B_k\right)\right).$$

But since $\bigcup_{j=1}^{m}\bigcup_{k=1}^{p}\left(A_j\cap B_k\right) = \bigcup_{i=1}^{l} C_i = \bigcup_{s=1}^{n} E_s = S$, taking into account that for every $A\in\mathcal{P}_{kc}$ and every $\alpha,\beta\in\mathbb{R}$ we have $D\left(\alpha A,\beta A\right)\leq|\alpha-\beta|\cdot\left|\!\left|A\right|\!\right|$, we obtain

$$\begin{aligned} D\left(\sigma\left(P_1\right),\sigma\left(P_2\right)\right) &\leq \sum_{i=1}^{q}\left|f\left(s_j\right)-f\left(t_k\right)\right|\cdot\left|\!\left|\varphi\left(C_i\right)\right|\!\right| + \\ &+ \sum_{i=q+1}^{l}\left|f\left(s_j\right)-f\left(t_k\right)\right|\cdot\left|\!\left|\varphi\left(C_i\right)\right|\!\right| \leq \\ &\leq M\cdot\sum_{i=1}^{q}\nu\left(C_i\right)+\frac{\varepsilon}{2M}\sum_{i=q+1}^{l}\nu\left(C_i\right) = \\ &= M\cdot\nu\left(\bigcup_{i=1}^{q} C_i\right)+\frac{\varepsilon}{2M}\nu\left(\bigcup_{i=q+1}^{l} C_i\right)\leq \\ &\leq M\cdot\nu\left(E_0\right)+\frac{\varepsilon}{2M}\cdot\nu\left(S\right) < M\cdot\nu\left(E_0\right)+\frac{\varepsilon}{2M}\cdot M<\varepsilon. \end{aligned}$$

Hence the net $\left\{\sigma\left(P\right)\right\}_{P\in\left(\mathcal{P},\leq\right)}$ is Cauchy in the complet space $\left(\mathcal{P}_{kc},D\right)$ and consequently it is convergent, that is f is φ-integrable. ■

Now we point out some properties of φ-integrable functions.

Theorem 2.10. *Let $f:S\to\mathbb{R}$ be a bounded function and the following statements:*

(i) f is φ-integrable.

(ii) *Given $\varepsilon > 0$ there is a partition $P_0 = \{E_i\}_{i=0}^n$ of S such that* $\sum_{i=1}^{n}\sum_{j=1}^{k_i} |\lambda_{ij}| \cdot \left| \varphi(X_{ij}) \right| < \varepsilon$, *where for each $i \in \{0,1,...,n\}$, $\{X_{ij}\}_{j=1}^{k_i} \subset \mathcal{A}$ is an arbitrary partition of E_i and $\lambda_{ij} \in \mathbb{R}$ are such that $|\lambda_{ij}| \leq osc(f, E_i)$.*

(iii) *Given $\varepsilon > 0$, there is a partition P_0 of S such that for any partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and for all $s_i, t_i \in E_i, i = 1,2,...,n$ holds*

$$D\left(\sum_{i=1}^{n} f(s_i)\varphi(E_i), \sum_{i=1}^{n} f(t_i)\varphi(E_i)\right) < \varepsilon.$$

(iv) *f is φ-totally measurable,*

then (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) and if moreover $f \geq 0$ (or $f \leq 0$) then (iii) $\Rightarrow$ (iv).

Proof. **(ii) $\Rightarrow$ (i)**.

Given $\varepsilon > 0$, let us take the partition $P_0 = \{E_i\}_{i=0}^n$ of S as in (ii).

If P_1, P_2 are partitions of S such that $P_1 \geq P_0, P_2 \geq P_0, P_1 = \{A_j\}_{j=1}^m$, $P_2 = \{B_k\}_{k=1}^p$, then the common refinement $P_1 \wedge P_2$ is a family $\{X_{ij}\} \subset \mathcal{A}$ where for each $i \in \{0,1,...,n\}$, $\{X_{ij}\}$ is a subpartition of E_i.

Now we reason as in the proof of Theorem 2.9.

If $s_j \in A_j$ $(j = 1,2,...,m)$ and $t_k \in B_k$ $(k = 1,2,...,p)$, then we have

$$D(\sigma(P_1), \sigma(P_2)) = D\left(\sum_{j=1}^{m} f(s_j)\varphi(A_j), \sum_{k=1}^{p} f(t_k)\varphi(B_k)\right).$$

But s_j, t_k can be regarded as elements $a_{ij}, b_{ij} \in E_i$ and then

$$D(\sigma(P_1), \sigma(P_2)) \leq \sum_{i=1}^{n}\sum_{j=1}^{ki} |f(a_{ij}) - f(b_{ij})| \cdot |\varphi(X_{ij})|.$$

Now we can apply (ii) considering $\lambda_{ij} = |f(a_{ij}) - f(b_{ij})| \leq osc(f, E_i)$, $(\forall)\, i = 0,1...n$ and consequently we obtain

$$D(\sigma(P_1), \sigma(P_2)) < \varepsilon.$$

Since $(\mathcal{P}_{kc}, D)$ is a complete metric space it results that there exists an element $A \in \mathcal{P}_{kc}$ which is the limit of the net $\{\sigma_P\}_{P\in(\mathcal{P},\leq)}$.

Hence f is φ-integrable.

(i) $\Rightarrow$ (iii) From Definition 2.7 it follows that there exists $A \in \mathcal{P}_{kc}$ satisfying the condition: given $\varepsilon > 0$, there is P_0 a partition of S such that for any partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $s_i, t_i \in E_i, i \in \{1, 2, ..., n\}$,

$$D\left(\sum_{i=1}^{n} f(s_i)\varphi(E_i), A\right) < \frac{\varepsilon}{2}. \tag{5}$$

If $P = \{E_i\}_{i=1}^n$ is any partition of S with $P \geq P_0$ and $s_i, t_i \in E_i, i \in \{1, ..., n\}$, from (5) we have

$$D\left(\sum_{i=1}^{n} f(s_i)\varphi(E_i), \sum_{i=1}^{n} f(t_i)\varphi(E_i)\right) \leq D\left(\sum_{i=1}^{n} f(s_i)\varphi(E_i), A\right) + \\ + D\left(A, \sum_{i=1}^{n} f(t_i)\varphi(E_i)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(iii) $\Rightarrow$ (iv) if $f \geq 0$ (or $f \leq 0$).

Suppose $f \geq 0$. Let us consider an arbitrary $\delta > 0$ and denote $\varepsilon = \dfrac{\delta^2}{12} > 0$.

With this $\varepsilon > 0$, from the hypothesis (iii) we obtain a partition $P_0 = \{E_i\}_{i=1}^n$ of S for which (iii) is fulfilled.

We can suppose that the sets E_i are nonempty and are indexed so that the following property $(\star)$ holds for $1 \leq i \leq k$ and fails for $k+1 \leq i \leq n$:

$(\star)$ E_i can be split into $E_i = E_i' \cup E_i''$, $E_i' \cap E_i'' = \phi$, $E_i', E_i'' \in \mathcal{A}$ such that either $E_i' = \phi$ or $osc(f, E_i'') \leq \delta$ and similarly either $E_i'' = \emptyset$, or $osc(f, E_i'') \leq \delta$.

Let P_1 be the partition $\left\{E_1', E_1'', E_2', E_2'', ..., E_k', E_k', \bigcup_{i=k+1}^{n} E_i\right\}$. We shall show that P_1 satisfies the conditions of Definition 2.2 - a) with $E_0 = \bigcup_{i=k+1}^{n} E_i$.

From $(\star)$, $osc(f, E_i') \leq \delta$ or $osc(f, E_i'') \leq \delta$, $i \in \{1, ..., k\}$ so it is only necessary to show that $\widetilde{\varphi}(E_0) \leq \delta$. Let $B_i \subseteq E_i$, $B_i \in \mathcal{A}$, $i \in \{k+1, ..., n\}$.

The index set $\{k+1,...,n\}$ can be partitioned into two disjoint subsets I_1, I_2 so that

(6) $\exists a_i, a_i' \in B_i$ for which $|f(a_i) - f(a_i')| > \delta$, $\forall i \in I_1$, or

(6') $\exists b_i, b_i' \in E_i \setminus B_i$ for which $|f(b_i) - f(b_i')| > \delta$, $\forall i \in I_2$.

This is possible because for each $i \in \{k+1,...,n\}$, $(\star)$ fails.

Let J_1 be any subset of I_1 and let P be the partition $\{E_1, E_2..., E_k, B_{k+1}, C_{k+1}, ..., B_n, C_n\}$, where $C_i = E_i \setminus B_i, (\forall)\, i \in I_2$, which is finer than P_0.

We have two cases:

a) if $f(a_i) - f(a_i') \geq 0$, then since $f(a_i) = [f(a_i) - f(a_i')] + f(a_i')$ we have

$$f(a_i)\varphi(B_i) = [f(a_i) - f(a_i') + f(a_i')] \cdot \varphi(B_i) = [f(a_i) - f(a_i')] \cdot \varphi(B_i) + f(a_i') \cdot \varphi(B_i).$$

In this situation we take $s_i = a_i, t_i = a_i'$ defined as in (6) for every $i \in J_1$.

b) if $f(a_i) - f(a_i') < 0$, then $f(a_i') = [f(a_i') - f(a_i)] + f(a_i)$ which yields

$$f(a_i)\varphi(B_i) = [f(a_i') - f(a_i)] \cdot \varphi(B_i) + f(a_i) \cdot \varphi(B_i).$$

In this situation we take $s_i = a_i', t_i = a_i$ defined as in (6) for $i \in J_1$.

For the others i we consider $s_i = t_i \in B_i$ (arbitrary in B_i).

By virtue of (iii) we have

$$(7) \qquad D\left(\sum_{i=1}^{n} f(s_i)\varphi(V_i), \sum_{i=1}^{n} f(t_i)\varphi(V_i)\right) < \varepsilon,$$

where we denoted by $\{V_i\}_{i=1}^n$ the partition $P = \{E_1, E_2...E_k, B_{k+1}, C_{k+1}, ..., B_n, C_n\}$.

Since $f(s_i) = f(t_i)$, for $i \in I_1 \setminus J_1$ we can rewrite (7) as

$$(7') \qquad D\left(\sum_{i\in J_1} f(s_i)\varphi(B_i), \sum_{i\in J_1} f(t_i)\varphi(B_i)\right) < \varepsilon.$$

Taking into account the choice of s_i, t_i we obtain

$$D\left(\sum_{i\in J_1} f(s_i)\,\varphi(B_i), \sum_{i\in J_1} f(t_i)\,\varphi(B_i)\right) = \left|\sum_{i\in J_1}[f(s_i)-f(t_i)]\cdot\varphi(B_i)\right|.$$

Hence for every $J_1 \subseteq I_1$ we have

$$\left|\sum_{i\in J_1}[f(s_i)-f(t_i)]\cdot\varphi(B_i)\right| < \varepsilon. \tag{8}$$

Applying the proposition 2.1 - (iii) with $c_i = f(s_i) - f(t_i), d_i = \delta$ and $A_i = \varphi(B_i), (\forall)\, i \in I_1$ we obtain

$$\left|\sum_{i\in I_1}\delta\cdot\varphi(B_i)\right| \leq 2\varepsilon.$$

Hence $\left|\delta\sum_{i\in I_1}\varphi(B_i)\right| = \delta\left|\varphi\left(\bigcup_{i\in 1_1} B_i\right)\right| \leq 2\varepsilon$, that is

$$\delta\left|\varphi\left(\bigcup_{i\in 1_1} B_i\right)\right| \leq 2\varepsilon. \tag{9}$$

Similarly, using (6') we obtain

$$\delta\left|\varphi\left(\bigcup_{i\in 1_2} C_i\right)\right| \leq 2\varepsilon. \tag{10}$$

Now we repeat the argument using the fact that for every $i \in I_2$, there exists $x_i, x_i' \in E_i$ such that

$$\left|f(x_i) - f\left(x_i'\right)\right| > \delta. \tag{11}$$

Taking $P = P_0$ in (iii) and reasoning as in the proof of (9) we obtain from (11) that

$$\delta \left| \varphi \left(\bigcup_{i\in 1_2} E_i \right) \right| \leq 2\varepsilon. \tag{12}$$

Using now (10), (12) and taking into account that $E_i = B_i \cup C_i$ and hence $\bigcup_{i\in 1_2} E_i = \left(\bigcup_{i\in 1_2} B_i \right) \cup \left(\bigcup_{i\in 1_2} C_i \right)$, we obtain by virtue of the lemma 1.7

$$\delta \left| \varphi \left(\bigcup_{i\in 1_2} B_i \right) \right| \leq 4\varepsilon. \tag{13}$$

From (9) and (13) we have

$$\delta \left| \varphi \left(\bigcup_{i=k+1}^{n} B_i \right) \right| = \delta \left| \varphi \left(\bigcup_{i\in 1_1} B_i \cup \bigcup_{i\in 1_2} B_i \right) \right| =$$

$$= \delta \left| \varphi \left(\bigcup_{i\in 1} B_i \right) + \varphi \left(\bigcup_{i\in 1_2} B_i \right) \right| \leq$$

$$\leq \delta \left| \varphi \left(\bigcup_{i\in 1} B_i \right) \right| + \delta \left| \varphi \left(\bigcup_{i\in 1_2} B_i \right) \right| \leq 2\varepsilon + 4\varepsilon = 6\varepsilon = \delta^2$$

which yields

$$\left| \varphi \left(\bigcup_{i=k+1}^{n} B_i \right) \right| < \delta. \tag{14}$$

But every measurable subset of $E_0 = \bigcup_{i=k+1}^{n} E_i$ may be written as $\bigcup_{i=k+1}^{n} B_i$ and hence from (14) we have that $\widetilde{\varphi}(E_0) < \delta$. ■

Theorem 2.11. *Let $f : S \to \mathbb{R}$ be a φ-integrable function and $T \in \mathcal{A}$. Then:*

i) f is φ-integrable on T;

ii) f is φ-integrable on T if and only if the function $f \cdot 1_T$ is φ-integrable on S.

In this case

$$\int_S f \cdot 1_T d\varphi = \int_T f d\varphi.$$

Proof. i) Since f is φ-integrable for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=1}^n$ of S such that for every partition P of S with $P \geq P_\varepsilon$ and $(\forall)\, s_i \in E_i, i = 1, 2...n$, we have

$$(15) \quad D\left(\sigma\left(P\right), A\right) < \frac{\varepsilon}{2}, \text{ where } A = \int_S f d\varphi \text{ and } \sigma\left(P\right) = \sum_{i=1}^n f\left(s_i\right)\varphi\left(E_i\right).$$

Let $T \in \mathcal{A}$ and $P_\varepsilon^T = \{T \cap E_i\}_{i=1}^n$. Let $P_1^T = \{D_k\}_{k=1}^m$, $P_2^T = \{C_j\}_{j=1}^p$ be partitions of T with $P_1^T \geq P_\varepsilon^T, P_2^T \geq P_\varepsilon^T$ and $s_k \in D_k, k = 1, 2...m$, $t_j \in C_j, j = 1, 2...p$.

Let $P' = \{D_k\}_{k=1}^m \cup \{E_i \setminus T\}_{i=1}^n$ and $P'' = \{C_j\}_{j=1}^p \cup \{E_i \setminus T\}_{i=1}^n$. We observe that P', P'' are partitions for S. Let $a_i \in E_i \setminus T$, $(\forall)\, i = 1, 2...n$.

Then, using (15) we have

$$\begin{aligned} D\left(\sigma\left(P_1^T\right), \sigma\left(P_2^T\right)\right) &= D\left(\sum_{k=1}^m f\left(s_k\right)\varphi\left(D_k\right), \sum_{j=1}^p f\left(t_j\right)\varphi\left(C_j\right)\right) = \\ &= D\left(\sum_{k=1}^p f\left(s_k\right)\varphi\left(D_k\right) + \sum_{i=1}^n f\left(a_i\right)\varphi\left(E_i \setminus T\right), \right. \\ &\qquad \left. \sum_{j=1}^p f\left(t_j\right)\varphi\left(C_j\right) + \sum_{i=1}^n f\left(a_i\right)\varphi\left(E_i \setminus T\right)\right) = \\ &= D\left(\sigma\left(P'\right), \sigma\left(P''\right)\right) \leq D\left(\sigma\left(P'\right), A\right) + D\left(A, \sigma\left(P''\right)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, for every $\varepsilon > 0$ there exists a partition P_ε^T of T such that for every partitions P_1^T, P_2^T of T with $P_1^T \geq P_\varepsilon^T, P_2^T \geq P_\varepsilon^T$ we have

$$D\left(\sigma\left(P_1^T\right), \sigma\left(P_2^T\right)\right) < \varepsilon,$$

which says that the net $\{\sigma_P\}_{P \in (\mathcal{P}^T, \leq)}$ is Cauchy in $(\mathcal{P}_{kc}, D)$ which is complet; consequently this net is convergent in $(\mathcal{P}_{kc}, D)$, that is f is φ-integrable on T.

ii) Suppose that f is φ-integrable on $T \in \mathcal{A}$. Hence for every $\varepsilon > 0$ there exists a partition $\overline{P}_0^T = \{C_i\}_{i=1}^{n_0}$ of $T \left(\bigcup_{i=1}^{n_0} C_i = T \right)$ such that for every partition $\overline{P}^T = \{E_i\}_{i=1}^{n}$ of T with $\overline{P}^T \geq P_0^T$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$ we have

$$D\left(\sum_{i=1}^{n} f(s_i)\,\varphi(E_i), \int_T f d\varphi\right) < \varepsilon. \tag{16}$$

Let $P_0 = \overline{P}_0 \cup \{cT\}$. We see that P_0 is a partition of S. If $P = \{D_j\}_{j=1}^{m}$ is a partition of S with $P \geq P_0$ and $t_j \in D_j, j = 1, 2...m$, then $(\ni)\, i \in \{1, ..., n_0\}$ such that $D_j \subseteq C_i$ or $D_j \subseteq cT$.

Without any loss of generality we can suppose $D_1, ..., D_k$ included respectively in $C_{i_1}, ..., C_{i_k}$, where $i_1, ..., i_k \in \{1, 2...n_0\}$ and $D_{k+1}, ..., D_m$ included in cT.

Since P is a partition of S we have $\bigcup_{j=1}^{k} D_j = T$ and $\bigcup_{j=k+1}^{m} D_j = cT$.

Now we obtain

$$\begin{aligned} &D\left(\sum_{j=1}^{m} (f \cdot 1_T)(t_j) \cdot \varphi(D_j), \int_T f d\varphi\right) = \\ &= D\left(\sum_{j=1}^{k} f(t_j) \cdot 1_T(t_j) \cdot \varphi(D_j) + \right. \\ &\left. + \sum_{j=k+1}^{m} f(t_j) \cdot 1_T(t_j) \cdot \varphi(D_j), \int_T f d\varphi\right) = \\ &= D\left(\sum_{j=1}^{k} f(t_j) \cdot \varphi(D_j), \int_T f d\varphi\right). \end{aligned} \tag{17}$$

Let $P_1 = \{D_j\}_{j=1}^{k}$. We see that P_1 is a partition of T with $P_1 \geq \overline{P}_0$. Since $t_j \in D_j, j = 1, 2...k$, using (16) we have

$$D\left(\sum_{j=1}^{k} f(t_j) \cdot \varphi(D_j), \int_T f d\varphi\right) < \varepsilon. \tag{18}$$

From (17) and (18) we see that for every $\varepsilon > 0$ there exists the partition P_0 of S such that for every partition $P = \{D_j\}_{j=1}^m$ of S with $P \geq P_0$ and every $t_j \in D_j, j = 1, 2...m$, we have

$$D\left(\sum_{j=1}^{m} (f \cdot 1_T)(t_j) \cdot \varphi(D_j), \int_T f d\varphi\right) < \varepsilon,$$

that is $f \cdot 1_T$ is φ-integrable on S and $\int_S f \cdot 1_T d\varphi = \int_T f d\varphi$.

Conversely, if we suppose that $f \cdot 1_T$ is φ-integrable on S, then there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$, there exists $P_0 = \{C_k\}_{k=1}^m$ partition of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$ we have

$$D\left(\sum_{j=1}^{n} (f \cdot 1_T)(s_i) \cdot \varphi(E_i), A\right) < \varepsilon, \text{ where } A = \int_S f \cdot 1_T d\varphi. \tag{19}$$

Let $\varepsilon > 0$ and $P_0^T = \{C_k \cap T\}_{k=1}^m$, which evidently is a partition of T.

Let $P^T = \{D_j\}_{j=1}^p$ be an arbitrary partition of T with $P^T \geq P_0^T$ and $P = \{D_j\}_{j=1}^p \cup \{C_k \cap cT\}_{k=1}^m$.

Let us denote by $\{E_j\}_{j=1}^p = \{D_j\}_{j=1}^p$ and by $\{E_i\}_{i=p+1}^n = \{C_k \cap cT\}_{k=1}^m$ and let us choice s_i arbitrary in E_i. For $i = p+1, ..., n$ we see that $s_i \in cT$ and hence $(f \cdot 1_T)(s_i) = 0$.

We see that $P = \{E_i\}_{i=1}^n$ is a partition of S with $P \geq P_0$. Now we have

$$D(\sigma(P'), A) = D\left(\sum_{j=1}^{p} f(s_j) \cdot \varphi(D_j), A\right) =$$

$$= D\left(\sum_{j=1}^{p} (f \cdot 1_T)(s_j) \cdot \varphi(D_j) + \sum_{i=p+1}^{n} (f \cdot 1_T)(s_i) \cdot \varphi(E_i), A\right) =$$

$$= D\left(\sum_{i=1}^{n} (f \cdot 1_T)(s_i) \cdot \varphi(E_i), A\right)$$

(the sum $\sum_{i=p+1}^{n} (f \cdot 1_T)(s_i) \cdot \varphi(E_i) = 0$) and by virtue of (19) we obtain

$$D\left(\sigma\left(P'\right),A\right)<\varepsilon.$$

Hence we proved that for every $\varepsilon>0$, there exists a partition $P_0^{'}$ of T such that for every other partition P' of T with $P^{'}=\{D_j\}_{j=1}^{p}$ and $P'\geq P_0'$ and $(\forall)\, s_j\in D_j, j=1,2...,p$ we have $D\left(\sum_{i=1}^{n}\left(f\cdot 1_T\right)(s_i)\cdot\varphi\left(E_i\right),A\right)<\varepsilon$, which says that f is φ-integrable on T. ■

Theorem 2.12. *Let $f,g:S\to\mathbb{R}$ be φ - integrable functions. Then*

$$D\left(\int_S f d\varphi,\int_S g d\varphi\right)\leq\sup_{s\in S}|f(s)-g(s)|\cdot\nu(S).$$

Proof. Let $\varepsilon>0$. Since f is φ - integrable there exists a partition $P_0^{'}$ of S such that for every partition $P'=\{B_i\}_{i=1}^{m}$ of S with $P'\geq P_0^{'}$ and $\forall s_i\in B_i, i=1,2,\cdots,m$

$$D\left(S_f(P'),\int_S f d\varphi\right)<\frac{\varepsilon}{3}\text{ where }S_f(P')=\sum_{i=1}^{m}f(s_i)\cdot\varphi(B_i).$$

Since g is φ - integrable there exists a partition $P_0^{''}$ of S such that for every partition $P''=\{C_j\}_{j=1}^{l}$ of S with $P''\geq P_0^{''}$ and $\forall t_j\in C_j, j=1,2,\cdots,l$

$$D\left(S_g(P''),\int_S g d\varphi\right)<\frac{\varepsilon}{3}\text{ where }S_g(P'')=\sum_{j=1}^{l}g(t_j)\cdot\varphi(C_j).$$

Let $P_0=P_0^{'}\wedge P_0^{''},P=\{E_i\}_{i=1}^{n}$ a partition of S with $P\geq P_0$ and

$s_i \in E_i, i = 1, 2, \cdots, n$. Then

$$D\left(\int_S f d\varphi, \int_S g d\varphi\right) \leq D\left(\int_S f d\varphi, S_f(p)\right) +$$
$$+D(S_f(P), S_g(P)) + D(S_g(P), \int_S g d\varphi) <$$
$$< \frac{2\varepsilon}{3} + D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \sum_{i=1}^{n} g(s_i)\cdot\varphi(E_i)\right) \leq$$
$$\leq \frac{2\varepsilon}{3} + \sum_{i=1}^{n} |f(s_i) - g(s_i)| \cdot ||\varphi(E_i)|| \leq$$
$$\leq \frac{2\varepsilon}{3} + \sum_{i=1}^{n} |f(s_i) - g(s_i)| \cdot \nu(E_i) \leq \frac{2\varepsilon}{3} + \sup_{s\in S} |f(s) - g(s)| \cdot \nu(S).$$

Since $\varepsilon > 0$ is arbitrary the theorem is proved.

Theorem 2.13. *Let $f : S \to \mathbb{R}$ be φ-integrable. Then for every $\alpha \in \mathbb{R}$ the function αf is φ-integrable and*

$$\int_S \alpha f d\varphi = \alpha \int_S f d\varphi.$$

Proof. If $\alpha = 0$ the property is evidently valid.

Suppose $\alpha \neq 0$. Let $\varepsilon > 0$. Since f is φ-integrable there exists a partition P_0 of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$

$$D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \int_S f d\varphi\right) < \frac{\varepsilon}{|\alpha|}. \tag{20}$$

Then, from (20) we have

$$D\left(\sum_{i=1}^{n} \alpha f(s_i)\cdot\varphi(E_i), \alpha\int_S f d\varphi\right) =$$

$$D\left(\alpha\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \alpha\int_S f d\varphi\right) =$$

$$|\alpha|\, D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \int_S f d\varphi\right) < |\alpha|\cdot\frac{\varepsilon}{|\alpha|} = \varepsilon.$$

Hence αf is φ-integrable and

$$\int_S \alpha f d\varphi = \alpha\int_S f d\varphi. \blacksquare$$

Theorem 2.14.

i) Let $f, g : S \to \mathbb{R}$ be φ-integrable functions such that $f(s)\, g(s) \geq 0$ for all $s \in S$. Then $f + g$ is φ-integrable and

$$\int_S (f+g)\, d\varphi = \int_S f d\varphi + \int_S g d\varphi.$$

ii) If $f : S \to \mathbb{R}$ is φ-integrable, then the multifunction $\Gamma : \mathcal{A} \to \mathcal{P}_{kc}$ defined by

$$\Gamma(T) = \int_T f d\varphi,\ (\forall)\, T \in \mathcal{A},$$

is a multimeasure.

Proof. **i)** Since f is φ-integrable for $\varepsilon > 0$ there exists a partition P_1 of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_1$ and every $s_i \in E_i, i = 1, 2..., n$

$$D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \int_S f d\varphi\right) < \frac{\varepsilon}{2}. \tag{21}$$

Similarly since g is φ-integrable, for $\varepsilon > 0$ there exists a partition P_2 of S such that for every partition $P' = \{F_j\}_{j=1}^m$ of S with $P' \geq P_2$ and every $t_j \in F_j, j = 1, 2, ..., m$

$$D\left(\sum_{j=1}^{m} g(t_j)\cdot\varphi(F_j), \int_S g d\varphi\right) < \frac{\varepsilon}{2}. \tag{22}$$

Let $P_0 = P_1 \wedge P_2$. Then for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$, using (21) and (22) and the fact that $(\alpha+\beta)A = \alpha A + \beta A$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$ we have

$$D\left(\sum_{i=1}^{n} [f(s_i)+g(s_i)]\cdot\varphi(E_i), \int_S f d\varphi + \int_S g d\varphi\right) \leq$$

$$D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), \int_S f d\varphi\right) + D\left(\sum_{i=1}^{n} g(s_i)\cdot\varphi(E_i), \int_S g d\varphi\right) <$$

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f+g$ is φ-integrable and

$$\int_S (f+g)\, d\varphi = \int_S f d\varphi + \int_S g d\varphi$$

ii) Let $T_1, T_2 \in \mathcal{A}$ with $T_1 \cap T_2 = \phi$.
Then

$$\Gamma(T_1 \cup T_2) = \int_{T_1\cup T_2} f d\varphi = \int_S f\cdot 1_{T_1\cup T_2} d\varphi = \int_S f\cdot(1_{T_1}+1_{T_2})\, d\varphi =$$

$$= \int_S f\cdot(1_{T_1}+1_{T_2})\, d\varphi = \int_S f\cdot 1_{T_1} d\varphi + \int_S f\cdot 1_{T_2} d\varphi = \int_{T_1} f d\varphi + \int_{T_2} f d\varphi,$$

that is Γ is a multimeasure.

Theorem 2.15.
Let $f : S \to \mathbb{R}$ be φ-integrable and $\Gamma(T) = \int_T f d\varphi, (\forall)\, T \in \mathcal{A}$. Then:

i)

$$\left|\int_S f d\varphi\right| \leq 2\sup_{s\in S}|f(s)|\cdot\nu(S). \tag{23}$$

ii) for every $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for every $E\in\mathcal{A}$ with $\nu(E)<\delta \Rightarrow v(\Gamma,E)<\varepsilon$, that is Γ is absolutely continuous with respect to ν, denoted $\Gamma<<\nu$.

Proof. **i)** Since f is φ-integrable, for every $\varepsilon>0$, there exists a partition P_0 of S such that for every partition $P=\{E_i\}_{i=1}^n$ of S with $P\geq P_0$ and $(\forall)\, s_i\in E_i, i=1,2...,n$ we have

$$D\left(\sum_{i=1}^n f(s_i)\cdot\varphi(E_i), \int_S f d\varphi\right)<\varepsilon. \tag{24}$$

From (24) we obtain

$$\left|\int_S f d\varphi\right| = D\left(\int_S f d\varphi, 0\right) \leq D\left(\int_S f d\varphi, \sum_{i=1}^n f(s_i)\cdot\varphi(E_i)\right) + \left|\sum_{i=1}^n f(s_i)\cdot\varphi(E_i)\right| \tag{25}$$

and then using (24) we have

$$\left|\int_S f d\varphi\right| < \varepsilon + \left|\sum_{i=1}^n f(s_i)\cdot\varphi(E_i)\right|. \tag{26}$$

Now we apply proposition 2.1 - (iii) for $A_i=\varphi(E_i)$, $d_i=f(s_i)$, $c_i=\sup_{s\in S}|f(s)|, (\forall)\, i=1,2...,n$, and $\alpha=\nu(S)\cdot\sup_{s\in S}|f(s)|$.

For $(\forall)\, J\subseteq\{1,2...n\}$ we have

$$\left|\sum_{i\in J} c_i A_i\right| = \left|\sum_{i\in J}\sup_{s\in S}|f(s)|\cdot A_i\right| = \sup_{s\in S}|f(s)|\cdot\left|\sum_{i\in J} A_i\right| \leq$$

$$\leq \sup_{s\in S}|f(s)|\cdot\sum_{i\in J}\mid A_i\mid \leq \sup_{s\in S}|f(s)|\cdot\sum_{i=1}^{n}\mid A_i\mid = \sup_{s\in S}\sqrt{|f(s)|}\cdot\sum_{i=1}^{n}\mid\varphi(E_i)\mid \leq$$

$$\leq \sup_{s\in S}|f(s)|\cdot\sum_{i=1}^{n}\nu(E_i)=\sup_{s\in S}|f(s)|\cdot\nu\left(\bigcup_{i=1}^{n}E_i\right)=\nu(S)\cdot\sup_{s\in S}|f(s)|=\alpha.$$

Hence for every $J\subseteq\{1,2...n\}$ we obtain $\left|\sum_{i\in J}c_iA_i\right|\leq\alpha$ and applying proposition 2.1 - iii) we have $\left|\sum_{i=1}^{n}d_iA_i\right|\leq 2\alpha$, that is

$$\left|\sum_{i=1}^{n}f(s_i)\cdot\varphi(E_i)\right|\leq 2\sup_{s\in S}\mid f(s)\mid\cdot\nu(S) \tag{27}$$

From (26) and (27) we have

$$\left|\int_S f d\varphi\right|<\varepsilon+2\nu(S)\cdot\sup_{s\in S}|f(s)|$$

and because $\varepsilon>0$ is arbitrary we have

$$\left|\int_S f d\varphi\right|\leq 2\nu(S)\cdot\sup_{s\in S}|f(s)|.$$

ii) Let $T\in\mathcal{A}$ be fixed. Replacing S by T in (23) we have

$$\left|\int_T f d\varphi\right|\leq 2\nu(T)\cdot\sup_{s\in T}|f(s)|\leq 2\sup_{s\in S}|f(s)|\cdot\nu(T)=\alpha\nu(T) \tag{28}$$

where $\alpha=2\sup_{s\in S}|f(s)|$.

From (28) we immediately obtain

$$\mid\Gamma(T)\mid\leq\alpha\nu(T). \tag{29}$$

Now let $\{B_i\}_{i=1}^n \subset \mathcal{A}$ be a partition of T.

From (29) we have

$$\left| \Gamma(B_i) \right| \leq \alpha \cdot \nu(B_i), (\forall)\, i = 1, 2..., n$$

and then

$$\sum_{i=1}^{n} \left| \Gamma(B_i) \right| \leq \alpha \nu(T),$$

for every partition $\{B_i\}_{i=1}^n$ of T.

Taking the supremum over all the partitions $\{B_i\}_{i=1}^n$ we obtain

$$\nu(\Gamma, T) \leq \alpha \cdot \nu(T), (\forall)\, T \in \mathcal{A}. \tag{30}$$

Assume $\varepsilon > 0$ and let $\delta(\varepsilon) = \dfrac{\varepsilon}{\alpha} > 0$. From (30), for every $E \in A$ with $\nu(E) < \delta$ we have

$$v(\Gamma, E) \leq \alpha \cdot \nu(E) < \alpha\delta = \varepsilon,$$

that is $\Gamma << \nu$. ■

Proposition 2.16. *Let $f, g : S \to \mathbb{R}$ be φ-integrable and $|f| \leq |g|$.*

If $\sup\limits_{T \in \mathcal{A}} \left| \int_T g d\varphi \right| \leq M$, *then* $\sup\limits_{T \in \mathcal{A}} \left| \int_T f d\varphi \right| \leq 2M$.

Proof. Let $T \in \mathcal{A}$ be fixed.

Since f, g are φ-integrable, for every $\varepsilon > 0$ there exists a partition P_0 of T such that for every partition $P = \{E_i\}_{i=1}^n$ of T with $P \geq P_0$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$ we have

$$D\left(\sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i), \int_T f d\varphi \right) < \varepsilon \tag{31}$$

and

$$D\left(\sum_{i=1}^{n} g(s_i)\cdot\varphi(E_i), \int_T g d\varphi\right) < \varepsilon. \tag{32}$$

Using (32) we have

$$\left|\sum_{i=1}^{n} g(s_i)\cdot\varphi(E_i)\right| \leq D\left(\sum_{i=1}^{n} g(s_i)\cdot\varphi(E_i), \int_T g d\varphi\right) + \left|\int_T g d\varphi\right| \tag{33}$$

$$< \varepsilon + M.$$

For the same partition P, using (31) we have for f

$$\left|\int_T g d\varphi\right| \leq D\left(\int_T f d\varphi, \sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i)\right) + \left|\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i)\right| < \tag{34}$$

$$< \varepsilon + \left|\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i)\right|.$$

Applying proposition 2.1 - iii) for $A_i = \varphi(E_i)$, $c_i = g(s_i)$, $d_i = f(s_i)$, $\alpha = \varepsilon + M$, from (33) we obtain

$$\left|\sum_{i=1}^{n} d_i A_i\right| \leq 2\alpha,$$

that is

$$\left|\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i)\right| \leq 2(\varepsilon + M). \tag{35}$$

Now, from (34) and (35) we obtain

$$\left|\int_T f d\varphi\right| < \varepsilon + 2\varepsilon + 2M = 3\varepsilon + 2M.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\left| \int_T f d\varphi \right| \leq 2M, \, (\forall) \, T \in \mathcal{A},$$

from which we obtain

$$\sup_{T \in \mathcal{A}} \left| \int_T f d\varphi \right| \leq 2M. \, \blacksquare$$

From the proof of proposition 2.16 we also obtain the following result.

Corollary 2.17. *Let $f, g : S \to \mathbb{R}$ be φ-integrable and $|f| \leq |g|$. Then*

$$\left| \int_T f d\varphi \right| \leq 2 \left| \int_T g d\varphi \right|, \, (\forall) \, T \in \mathcal{A}.$$

Remark 2.18. Let $f : S \to \mathbb{R}$ be a bounded function and let $\varphi = \{\mu\}$, where $\mu : \mathcal{A} \to X$ is a vector measure. If f is φ - integrable, then f is Gould μ - integrable and $\int_T f d\varphi = \{(G) \int_T f d\mu\}, (\forall) T \in A$, where $(G) \int_T f d\mu$ is the Gould integral of f with respect to μ.

Let us now analyse the following two examples:

Example 2.19. Let $X = \mathbb{R}$, $\varphi = [\mu_1, \mu_2]$, where μ_1, μ_2 are two positive measures with $0 \leq \mu_1 \leq \mu_2$ and $f : S \to \mathbb{R}_+$ a φ-integrable function.

Then

$$\int_T f d\varphi = \left[(G) \int_T f d\mu_1, (G) \int_T f d\mu_2 \right], (\forall) \, T \in \mathcal{A}. \tag{36}$$

We shall show (36) for $T = S$.

Since f is φ-integrable there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$, $(\exists) \, P_0$ a partition of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $(\forall) \, s_i \in E_i, i = 1, 2..., n$,

$$D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i), A\right) < \varepsilon. \tag{37}$$

Since $A \in \mathcal{P}_{kc}(\mathbb{R})$, we have $A = [a,b]$ with $a < b$, a,b$\in \mathbb{R}$.
Now (37) is equivalent to

$$D\left(\sum_{i=1}^{n} [f(s_i)\cdot\mu_1(E_i), f(s_i)\cdot\mu_2(E_i)], [a,b]\right) < \varepsilon. \tag{38}$$

But

$$\begin{aligned} &D\left(\left[\sum_{i=1}^{n} f(s_i)\cdot\mu_1(E_i), \sum_{i=1}^{n} f(s_i)\cdot\mu_2(E_i)\right], [a,b]\right) = \\ &= \max\left\{\left|\sum_{i=1}^{n} f(s_i)\cdot\mu_1(E_i) - a\right|, \left|\sum_{i=1}^{n} f(s_i)\cdot\mu_2(E_i) - b\right|\right\}. \end{aligned} \tag{39}$$

Since $\max\{\alpha,\beta\} \geq \alpha$ and $\max\{\alpha,\beta\} \geq \beta$, from (38) and (39) we obtain

$$\left|\sum_{i=1}^{n} f(s_i)\cdot\mu_1(E_i) - a\right| < \varepsilon \tag{40}$$

and

$$\left|\sum_{i=1}^{n} f(s_i)\cdot\mu_2(E_i) - b\right| < \varepsilon \tag{41}$$

from which we observe that

$$a = (G)\int_S f d\mu_1 \text{ and } b = (G)\int_S f d\mu_2.$$

Consequently, $\int_S f d\varphi = A = \left[(G)\int_S f d\mu_1, (G)\int_S f d\mu_2\right]$.

Example 2.20. Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a σ-additive multimeasure with the total variation $\nu(S) < \infty$ and let $f : S \to \mathbb{R}_+$ be a bounded, Brooks φ-integrable function [4] such that

(α) $(\exists)\,(f_n)_{n\in N}$ - a defining sequence for f such that $|f_n(s)| \leq M$, $(\forall)\, s \in S$, $(\forall)\, n \in N$ and

(β) for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $n \in \mathbb{N}$, $n \geq 1$, $(\forall)\, x_i, y_i \in \mathbb{R}$ with $|x_i - y_i| < \varepsilon$, $(\forall)\, i = 1, 2..., n$, and $(\forall)\, \{B_i\}_{i=1}^n \subset \mathcal{A}$ we have

$$\sum_{i=1}^{n} |x_i - y_i| \cdot \left| \varphi(B_i) \right| < \varepsilon,$$

then f is φ-integrable and

$$\int_S f d\varphi = (B) \int_S f d\varphi.$$

Proof. Let $f : S \to \mathbb{R}_+$ be a Brooks φ-integrable function and let $(f_n)_{n\in N}$ be a defining sequence for f.

Then $f_n \geq 0$, $(\forall)\, n \in N$, f_n are simple functions with

a) $f_n \overset{\nu}{\to} f$ and

b) $\lim\limits_{n,m\to\infty} \int\limits_S |f_n - f_m|\, d\nu = 0.$

Denote by

$$(B) \int_S f d\varphi = \lim_{n\to\infty} \int_S f_n d\varphi. \tag{42}$$

Given $\varepsilon > 0$, from (β) there exists $\delta > 0$ such that for $n \in N^*$, $x_i, y_i \in \mathbb{R}$ with $|x_i - y_i| < \delta$, $(\forall)\, i = 1, 2..., n$, and for every $\{B_i\}_{i=1}^n \subset \mathcal{A}$

$$\sum_{i=1}^{n} |x_i - y_i| \cdot \left| \varphi(B_i) \right| < \frac{\varepsilon}{3}. \tag{43}$$

From (42) there exists $n_1 \in N$ such that for $n \geq n_1$

$$D\left(\int_S f_n d\varphi, (B)\int_S f d\varphi\right) < \frac{\varepsilon}{3}. \tag{44}$$

From the condition a) there exists a subsequence $\{f_{n_k}\}_{k\in N}$ of $\{f_n\}_{n\in N}$ (denoted again by $\{f_n\}_{n\in N}$) which converges almost uniformly to f.

From $f_n \overset{a.u}{\to} f$ one obtains that for every $\varepsilon > 0$, there exist $T \in \mathcal{A}$ with

$$\nu(T) < \frac{\varepsilon}{6M} \tag{45}$$

such that

$$f_n \overset{u}{\to} f \text{ on } cA. \tag{46}$$

From (46), for $\delta > 0$, $(\exists)\, n_2(\delta) = n_2(\varepsilon) \in N$ such that for $(\forall)\, n \geq n_2 \Rightarrow$

$$|f_n(s) - f(s)| < \delta,\ (\forall)\, s \in cA. \tag{47}$$

Denote $n_0 = \max\{n_1, n_2\}$.

Since f_{n_0} is simple we can write it as $f_{n_0} = \sum_{k=1}^{p} a_k \cdot 1_{B_k}$, $B_k \in \mathcal{A}$, $k = 1, 2...p$, $B_k \cap B_k' = \phi\,(k \neq k')$ and $\bigcup_{k=1}^{p} B_k = S$.

Then

$$\int_S f_{n_0} d\varphi = \sum_{k=1}^{p} a_k \cdot \varphi(B_k).$$

From (44) one obtains

$$D\left(\int_S f_{n_0} d\varphi, (B)\int_S f d\varphi\right) < \frac{\varepsilon}{3}. \tag{48}$$

Let be the partitions $P_1 = \{B_k\}_{k=1}^{p}$ and $P_2 = \{A, cA\}$ of S and let $P_0 = P_1 \wedge P_2$.

If $P = \{E_i\}_{i=1}^{n}$ is a partition of S with $P \geq P_0$ and $s_i \in E_i, i = 1, 2..., n$, then

$$D\left(\sum_{i=1}^{n} f(s_i)\cdot\varphi(E_i),(B)\int_S f d\varphi\right)=D\left(\sum_{i=1}^{m} f(s_i)\cdot\varphi(E_i),\int_S f_{n_0}d\varphi\right)+$$

$$D\left(\int_S f_{n_0}d\varphi,(B)\int_S f d\varphi\right)=D\left(\sum_{i=1}^{m} f(s_i)\cdot\varphi(E_i),\sum_{i=1}^{m} f_{n_0}(s_i)\cdot\varphi(E_i)\right)+$$

$$D\left(\int_S f_{n_0}d\varphi,(B)\int_S f d\varphi\right).$$

If we denote by $\alpha = D\left(\sum_{i=1}^{m} f(s_i)\cdot\varphi(E_i),\sum_{i=1}^{m} f_{n_0}(s_i)\cdot\varphi(E_i)\right)$, by virtue of (48) we have

$$D\left(\sum_{i=1}^{m} f(s_i)\cdot\varphi(E_i),(B)\int_S f d\varphi\right)<\alpha+\frac{\varepsilon}{3}. \tag{49}$$

But

$$\alpha = D\left(\sum_{\substack{E_i\subset T\\ UE_i=T}} f(s_i)\cdot\varphi(E_i)+\sum_{\substack{E_i\subset cT\\ UE_i=cT}} f(s_i)\cdot\varphi(E_i),\right.$$

$$\left.\sum_{\substack{E_i\subset T\\ UE_i=T}} f_{n_0}(s_i)\cdot\varphi(E_i)+\sum_{\substack{E_i\subset cT\\ UE_i=cT}} f_{n_0}(s_i)\cdot\varphi(E_i)\right)\leq$$

$$\leq D\left(\sum_{\substack{E_i\subseteq T\\ UE_i=T}} f(s_i)\cdot\varphi(E_i),\sum_{\substack{E_i\subseteq T\\ UE_i=T}} f_{n_0}(s_i)\cdot\varphi(E_i)\right)+$$

$$+D\left(\sum_{\substack{E_i\subseteq cT\\ UE_i=cT}} f(s_i)\cdot\varphi(E_i),\sum_{\substack{E_i\subseteq cT\\ UE_i=cT}} f_{n_0}(s_i)\cdot\varphi(E_i)\right).$$

Denoting by $\alpha_1 = D\left(\sum\limits_{\substack{E_i\subseteq T\\ UE_i=T}} f(s_i)\cdot\varphi(E_i), \sum\limits_{\substack{E_i\subseteq T\\ UE_i=T}} f_{n_0}(s_i)\cdot\varphi(E_i)\right)$ and by $\alpha_2 = D\left(\sum\limits_{\substack{E_i\subseteq cT\\ UE_i=cT}} f(s_i)\cdot\varphi(E_i), \sum\limits_{\substack{E_i\subseteq cT\\ UE_i=cT}} f_{n_0}(s_i)\cdot\varphi(E_i)\right)$ we have

$$\alpha \leq \alpha_1 + \alpha_2. \tag{50}$$

Now

$$\begin{aligned}\alpha_1 &\leq \left|\sum_{\substack{E_i\subseteq T\\ UE_i=T}} f(s_i)\cdot\varphi(E_i)\right| + \left|\sum_{\substack{E_i\subseteq T\\ UE_i=T}} f_{n_0}(s_i)\cdot\varphi(E_i)\right| \leq \\ &\leq \sum_{\substack{E_i\subset T\\ UE_i=T}} |f(s_i)|\cdot\left|\varphi(E_i)\right| + \sum_{\substack{E_i\subset T\\ UE_i=T}} |f_{n_0}(s_i)|\cdot\left|\varphi(E_i)\right| \leq \\ &\leq 2M\sum_{\substack{E_i\subseteq T\\ UE_i=T}} \nu(E_i) = 2M\cdot\nu(T).\end{aligned}$$

Using now (45) we obtain

$$\alpha_1 < 2M\cdot\frac{\varepsilon}{6M} = \frac{\varepsilon}{3}. \tag{51}$$

On the other hand

$$\begin{aligned}\alpha_2 &\leq \sum_{\substack{E_i\subseteq cT\\ UE_i=cT}} D\left(f(s_i)\cdot\varphi(E_i), f_{n_0}(s_i)\cdot\varphi(E_i)\right) \leq \\ &\leq \sum_{\substack{E_i\subseteq cT\\ UE_i=cT}} |f(s_i) - f_{n_0}(s_i)|\cdot\left|\varphi(E_i)\right|.\end{aligned}$$

Hence

$$\alpha_2 \leq \sum_{\substack{E_i \subseteq cT \\ UE_i = cT}} |f(s_i) - f_{n_0}(s_i)| \cdot \left| \varphi(E_i) \right| . \tag{52}$$

But $s_i \in E_i \subseteq cT$, $(\forall)\, i = 1, 2..., m$, and then, according to (47), we have

$$|f(s_i) - f_{n_0}(s_i)| < \delta,\ (\forall)\, i = 1, 2..., m.$$

From (43) it results that

$$\sum_{\substack{E_i \subseteq cT \\ UE_i = cT}} |f(s_i) - f_{n_0}(s_i)| \cdot \left| \varphi(E_i) \right| < \frac{\varepsilon}{3}. \tag{53}$$

Now, from (52) and (53) we obtain

$$\alpha_2 < \frac{\varepsilon}{3} \tag{54}$$

and, using (49), (50), (51) and (54), we have

$$D\left(\sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i), (B) \int_S f d\varphi \right) < \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists a partition P_0 of S such that for every partition $P = \{E_i\}_{i=1}^{n}$ of S with $P \geq P_0$ and all $s_i \in E_i, i = 1, 2..., n$ we have

$$D\left(\sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i), (B) \int_S f d\varphi \right) < \varepsilon,$$

that is f is φ-integrable and $\int_S f d\varphi = (B) \int_S f d\varphi$.

§3. Strong multivalued Gould integral. In what follows we describe another multivalued Gould integrability, stronger than φ-integrability of Gould type studied in §2.

Instead of Hausdorff distance considered in definition 2.5 we shall use $|\cdot|$.

Definition 3.1.

Let $f : S \to \mathbb{R}$ be a bounded function and $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ a multimeasure with finite variation.

a) We say that f is strong φ-integrable with respect to φ on S (briefly s-φ-integrable) if there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$ there exists a partition P_0 of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$, we have

$$\left| \sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i) - A \right| < \varepsilon.$$

b) The element $A \in \mathcal{P}_{kc}$ defined in a) is said to be **the strong integral** (of Gould type) of f with respect to φ on S. This element A is denoted by $(s) \int_S f d\varphi$.

Remark 3.2

1) If there exists an $A \in \mathcal{P}_{kc}$ as in definition 3.1, then A is uniquely determined.

Indeed, if we suppose that there exist to elements $A_1, A_2 \in \mathcal{P}_{kc}$ as in definition 3.1, then for every $\varepsilon > 0$ there exist two partitions P_1, P_2 of S such that for every partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_1$ and $(\forall)\, s_i \in E_i, i = 1, 2..., n$, we have

$$\left| \sigma(P) - A_1 \right| < \frac{\varepsilon}{2} \tag{55}$$

and every partition $P' = \left\{E'_j\right\}_{j=1}^p$ of S with $P' \geq P_2$ and $(\forall)\, t_j \in E'_j$ we have

$$\left| \sigma\left(P'\right) - A_2 \right| < \frac{\varepsilon}{2}. \tag{56}$$

Denoting by $P_0 = P_1 \wedge P_2$ and considering an arbitrary partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $s_i \in E_i, (\forall)\, i = 1, 2..., n$, we obtain using (55) and (56)

$$D(A_1, A_2) \leq D(A_1, \sigma(P)) + D(\sigma(P), A_2) \leq$$
$$\leq \left| A_1 - \sigma(P) \right| + \left| \sigma(P) - A_2 \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, for every $\varepsilon > 0$ we have

$$0 \leq D(A_1, A_2) < \varepsilon$$

and consequently $A_1 = A_2$.

2) Since for every $A, B \in \mathcal{P}_{kc}$ we have

$$D(A, B) \leq \left| A - B \right|,$$

we easily see that if f is s-φ-integrable then it is at the same time φ-integrable.

3) According to lemma 1.9 we see that A in the definition 3.1 is a singleton.

Definition 3.3. Let $f : S \to \mathbb{R}$ be a bounded function.

a) We say that f is hereditarily strong-φ-integrable (briefly h - s-φ-integrable) if f is s-φ-integrable on every $T \in \mathcal{A}$.

b) If f is h-s-φ-integrable and $T \in \mathcal{A}$ we denote by $(s)\int_T f d\varphi$ the φ-integral of f with respect to $(T, \mathcal{A}_T, \varphi_T)$.

We can easily obtain the following properties

Theorem 3.4

i) If $f, g : S \to \mathbb{R}$ are s-φ-integrable on S, then for every $\alpha, \beta \in \mathbb{R}$ the function $\alpha f + \beta g$ is s-φ-integrable and

$$(s)\int_S (\alpha f + \beta g)\, d\varphi = \alpha \cdot (s)\int_S f d\varphi + \beta \cdot (s)\int_S g d\varphi.$$

ii) If $f : S \to \mathbb{R}$ is s-φ-integrable on $T \in \mathcal{A}$, then the function $f \cdot 1_T$ is s-φ-integrable on S and

$$(s)\int_S f \cdot 1_T d\varphi = (s)\int_T f d\varphi.$$

iii) *If* $f : S \to \mathbb{R}$ *is bounded and h-s-*φ*-integrable, then the multifunction* $\Gamma : \mathcal{A} \to \mathcal{P}_{kc}$, *defined by* $\Gamma(A) = (s)\int_A f d\varphi$, $(\forall)\, A \in \mathcal{A}$, *is a multimeasure and* $\Gamma << \nu$.

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Faculty of Mathematics
University "Al. I. Cuza"
Carol I, nr. 11
R - 6600, Iaşi
ROMÂNIA