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STRONG BARRELLEDNESS OF THE SPACE OF SIMPLE FUNCTIONS

BY

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Abstract. In this paper we prove a strong barrelledness property of the space of simple functions by using measure theory techniques.

1. Introduction. Let us denote by $\ell_0^\infty(X,\mathcal{A})$ the real or complex space generated by the characteristic functions χ_A defined on the elements of a σ-algebra $\mathcal{A}$ on a set X, and endowed with the norm $\|z\| = \sup\{|z(j)| : j \in X\}$. Valdivia [12] proved that $\ell_0^\infty(X,\mathcal{A})$ is suprabarrelled by using duality theory and sliding hump methods. This fact brought out several consequences in measure theory, one of the most relevant ones was to strengthen the Nikodým-Grothendieck fundamental boundedness principle, [12, Theorem 2]. The same kind of consequences was followed up when Ferrando and López Pellicer [2] proved that $\ell_0^\infty(X,\mathcal{A})$ was barrelled of class $\aleph_0$ and when López Pellicer [8] proved that $\ell_0^\infty(X,\mathcal{A})$ was baireled, the former being obtained following Valdivia's techniques and the latter with new ones developed by López Pellicer.

On the other hand, and just using measure theory techniques in [6] we gave a proof of the suprabarrelledness of $\ell_0^\infty(X,\mathcal{A})$ and in [7] of the barrelledness of class $\aleph_0$. The aim of this paper is to show that these techniques also enable to prove that $\ell_0^\infty(X,\mathcal{A})$ is baireled.

2. Background. Throughout this paper the word "space" will stand for "Hausdorff topological vector space over the field of real or complex

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numbers". Let us recall that a space E is *Baire-like* [11] if given an increasing sequence of closed absolutely convex subsets of E covering E, there is one of them that is a neighbourhood of the origin; *suprabarrelled* [13] (*barrelled of class* 1 in [9]) if given an increasing sequence of subspaces of E covering E there is one of them that is dense and barrelled; *barrelled of class* n if given an increasing sequence of subspaces of E covering E, there is one of them which is barrelled of class $n-1$; and *barrelled of class* $\aleph_0$ [9] if E is barrelled of class n for every $n \in \mathbb{N}$, (see also [3, 5]).

We will consider the language defined by the infinite alphabet $\mathbb{N}$ including the empty word, i.e. $W(\mathbb{N}) = \bigcup\{\mathbb{N}^k : k \in \mathbb{N} \cup \{0\}\}$. A *web* in a set Y is a family $\mathcal{W} = \{E_t : t \in W(\mathbb{N})\}$ of subsets of Y such that $Y = E_\emptyset$ and for each $t \in W(\mathbb{N})$ it holds that $E_t = \bigcup\{E_{t,n} : n \in \mathbb{N}\}$, $n \in \mathbb{N}$. A strand of $\mathcal{W}$ is a sequence of subsets $\{E_{n_1 n_2 \ldots n_i} : i \in \mathbb{N}\}$ of $\mathcal{W}$ where $\{n_i : i \in \mathbb{N}\}$ is a sequence of positive integers. Webs and strands were firstly considered by de Wilde [1]. When Y is a linear space, each $E_t \in \mathcal{W}$ is a linear subspace of Y and $E_{t,n} \subseteq E_{t,n+1}$, $n \in \mathbb{N}$, the web is said to be *linear* [4].

A space E is said to be *baireled* [4] (*superbarrelled* in [10]) if each linear web $\mathcal{W}$ in E contains a strand formed by Baire-like spaces.

As usual $ba(\mathcal{A})$ will stand for the space formed by the bounded finitely additive scalar valued measures defined on $\mathcal{A}$. If Π represents the family of all (finite) partitions of X by members of $\mathcal{A}$, then $ba(\mathcal{A})$ becomes a normed space with the variation norm $\|\mu\| = |\mu|(\Omega) = \sup\left\{\sum_{E\in\pi} |\mu(E)| : \pi \in \Pi\right\}$. We may identify $ba(\mathcal{A})$ with $\ell_0^\infty(X,\mathcal{A})'$ by means of the algebraic isomorphism T defined by $\langle T\mu, \chi_E\rangle = \mu(E)$ for each $\mu \in ba(\mathcal{A})$ and $E \in \mathcal{A}$.

The following definitions appeared in [6, 7].

Definition 1. Given $A \in \mathcal{A}$ and $M \subset ba(\mathcal{A})$, M is said to be quasi-bounded with regard to A if there are some $B_1, B_2, \ldots, B_n \in \mathcal{A}$ such that for each $A_0 \subset A$, $A_0 \in \mathcal{A}$, $\sup\left\{|u(A_0)| : u \in M,\ \sum_{j=1}^n |u(B_j)| \leq 1\right\} < \infty$.

Definition 2. If L be a subspace of $\ell_0^\infty(X,\mathcal{A})$ and $M \subset \ell_0^\infty(X,\mathcal{A})'$, M is said to be bounded with regard to L if $\sup\{|u(f)| : u \in M\} < \infty$ for each $f \in L$.

Between boundedness and quasiboundedness we find the following relationship, [6, Propositions 5 and 6].

Proposition 3. *Let L be a non-barrelled subspace of $\ell_0^\infty(X,\mathcal{A})$ (or a non-dense subspace whose closure is infinite-codimensional in $\ell_0^\infty(X,\mathcal{A})$).*

Then there exists some $M \subset ba(\mathcal{A})$ such that M is bounded with regard to L but not quasibounded with regard to X.

3. The space $\ell_0^\infty(X,\mathcal{A})$ is baireled. Given $t=(n_1,...,n_q)\in W(\mathbb{N})$ for each positive integer $i\leq q=|t|$, the length of t, we set $P_i(t)=(n_1,...,n_i)$ and as $P_0(t)$ we set the empty word. A non-void subset T of $W(\mathbb{N})\setminus\{\emptyset\}$ is called a *v-web* [8] if it verifies:
1) If $t\in T$ and $1\leq i\leq |t|$ then $\mathrm{Cd}(\{P_i(s): s\in T,\ P_{i-1}(s)=P_{i-1}(t)\})=\infty$,
2) If $t\in T$ then $\left\{u\in T: |u|>|t|\,,\ P_{|t|-1}(u)=P_{|t|-1}(t)\right\}=\emptyset$,
3) For each sequence $\{t_n\in T: n\in\mathbb{N}, |t_n|\geq n\}$ there is some p with $P_p(t_p)\neq P_p(t_{p+1})$.

The reasoning of [8, Lemma 2.5] and [6, Proposition 2] quite easily enables to obtain the following result.

Proposition 4. *Let us suppose that $A\in\mathcal{A}$ and let T be a v-web such that for each $t\in T$ there is a $M_t\subset ba(\mathcal{A})$ which is not quasibounded with regard to A. Given $\alpha>0$, $C_1,C_2,\ldots,C_n\in\mathcal{A}$, and k elements $t_1,t_2,\ldots,t_k\in T$, then there are k pairwise disjoint subsets $A_i\in\mathcal{A}$, contained in A, and k bounded measures $\mu_i\in M_{t_i}, 1\leq i\leq k$ such that $|\mu_i(A_i)|>\alpha$ and $\sum_{j=1}^n|\mu_i(C_j)|\leq 1$ for $1\leq i\leq k$. Moreover there is a v-web $T^*\subset T$ which contains $t_1,t_2,\ldots,t_k$ such that if $t\in T^*$, then M_t is not quasibounded with regard to $A\setminus\left(\bigcup_{i=1}^k A_i\right)$.*

Proposition 5. *Let $\{E_t: t\in W(\mathbb{N})\}$ be a linear web in $\ell_0^\infty(X,\mathcal{A})$ and let T be a v-web. Then there exists some $t\in T$ such that E_t is barrelled.*

Proof. Let us suppose that no E_t is not barrelled. Then by Proposition 3 nobar for each $t\in T$ there exists some $M_t\subset ba(\mathcal{A})$ such that M_t is bounded with regard to E_t but not quasibounded with regard to X.

Let us apply Proposition 4 with $k=1$ and as element of T consider its first element respect to the lexicographic ordering, which we will denote by 1^*. Proposition 4 enables us to find some $A_{11}\in\mathcal{A}$, $\mu_{11}\in M_{1^*}$ and a v-web $T_1\subseteq T$ such that $|\mu_{11}(A_{11})|>1$, $1^*\in T_1$, and setting $A_1\in\mathcal{A}\setminus A_{11}$, if $t\in T_1$, then M_t is not quasibounded with regard to A_1.

In the second step we single out $|1^*|$ elements t_i, $1\leq i\leq|1^*|$, from T_1 in such a way that for each of them $P_{i-1}(t_i)=P_{i-1}(1^*)$ whilst the ith letter of 1^* is strictly smaller than the i^{th} letter of t_i. We denote these t_i by $2^*,\ldots,r_2^*$ where clearly $r_2=1+|1^*|$. Again Proposition [4], now with

$A = A_1$, $T = T_1$, $C_1 = A_{11}$, and $1^*, 2^*, \ldots, r_2^*$ as elements of T, provides us with r_2 pairwise disjoint elements $A_{l,2} \in \mathcal{A}$, $1 \leq l \leq r_2$ contained in A_1, and r_2 bounded measures $\mu_{l2} \in M_{t^*}$, $1 \leq l \leq r_2$, such that for each of them one has $|\mu_{l2}(A_{l2})| > 2$ and $|\mu_{l2}(A_{11})| \leq 1$. Proposition 4 also provides some v-web $T_2 \subseteq T_1$, which contains $\{1^*, 2^*, \ldots, r_2^*\}$ and such that if $t \in T_2$ and $A_2 := A_1 \setminus \bigcup \{A_{i2} : 1 \leq i \leq r_2\}$ then M_t is not quasibounded with regard to A_2.

By recurrence, as in [8, Proposition 2.6], we are able to obtain a strictly increasing sequence $\{r_j\}_j$ in $\mathbb{N}$, a v-web $\{i^* \in T : i \in \mathbb{N}\}$, a family $\{A_{ij} : 1 \leq i \leq r_j,\ j \in \mathbb{N}\}$ of pairwise disjoint sets belonging to $\mathcal{A}$, and a family of measures $\{\mu_{ij} : 1 \leq i \leq r_j,\ j \in \mathbb{N}\}$, $\mu_{ij} \in M_{i^*}$, such that for $1 \leq i \leq r_j$, $j \in \mathbb{N}$, $|\mu_{ij}(A_{ij})| > j$ and $\sum\{|\mu_{ij}(A_{lk})| : 1 \leq l \leq r_k,\ 1 \leq k \leq j-1\} \leq 1$. And, denoting by $\{\hat{i}\}_i$ the set $\mathbb{N}^2$ endowed with the diagonal ordering, op. cit. methods enable to obtain from these two families some pairwise disjoint elements $B_{i^\wedge} \in \mathcal{A}$, some measures $v_{i^\wedge}$ and families D_i formed by some A_{pj} such that if $i^\wedge = (m, n)$ then

1- $B_{i^\wedge} = A_{m,h(i)}$ with $h(i-1) < h(i)$, and $D_i \subset D_{i-1}$, $i \geq 2$.

2- D_i is the union of infinitely many families B_j, with $j > h(i)$.

3- $v_{i^\wedge} = \mu_{m,h(i)}$ verifies that $|v_{i^\wedge}|(\bigcup D : D \in D_i) < 1$.

Write $B := \bigcup_i B_{i^\wedge}$. Then the fact that $\{t^* : t \in \mathbb{N}\}$ is a v-web yields that there must be some t^* such that $\chi_B \in E_{t^*}$. Let $i^\wedge$ be any element of $\mathbb{N}^2$ whose first coordinate is t, say $i^\wedge = (t, j)$. Then $v_{i^\wedge} = \mu_{th(i)} \in M_{t^*}$, and therefore $|v_{i^\wedge}(B)| < \lambda$ for some $\lambda > 0$, whatever the value of j is. But taking $B = \left(\bigcup_{(m,n)<(t,j)} B_{mn}\right) \cup B_{tj} \cup \left(\bigcup_{(m,n)>(t,j)} B_{mn}\right)$ it should be $|v_{ij}(B)| > j - 2$. Contradiction.

Theorem 6. *The space $\ell_0^\infty(X, \mathcal{A})$ is baireled.*

Proof. Let $\mathcal{W} = \{E_t : t \in W(\mathbb{N})\}$ be a linear web in $\ell_0^\infty(X, \mathcal{A})$. Since $\ell_0^\infty(X, \mathcal{A})$ is suprabarrelled there exists some $b_1 \in \mathbb{N}$ such that for each $n_1 \geq b_1$, E_{n_1} is barrelled and dense. Let $J_1 = \{n \in \mathbb{N} : n_1 \geq b_1\}$. By Amemiya-Kōmura's theorem, for each $a_1 \geq b_1$ either there is a cofinite subset N_{a_1} in $\mathbb{N}$ such that no E_t is barrelled for $t \in C_{a_1} := \{a_1\} \times N_{a_1}$, in which case we set $D_{a_1} := \emptyset$, or there is a cofinite subset M_{a_1} in $\mathbb{N}$ such that E_t is barrelled and dense, and therefore Baire-like in $\ell_0^\infty(X, \mathcal{A})$ for each $t \in D_{a_1} := \{a_1\} \times M_{a_1}$ in which case we set $C_{a_1} := \emptyset$.

Writing $I_2 = \bigcup \{C_{a_1} : a_1 \geq b_1\}$ and $J_2 = \bigcup \{D_{a_1} : a_1 \geq b_1\}$, we proceed in the same way with each $(a_1, a_2) \in J_2$ and form I_3 and J_3. Going on in

this way we build up some I_n and J_n for each $n \in \mathbb{N}$, noting that each $J_n \neq \emptyset$ since $\ell_0^\infty(X, \mathcal{A})$ is barreled of class $\aleph_0$.

Let $T = \bigcup_n I_n$. If $T \neq \emptyset$, then T is a v-web and no E_t, $t \in T$, is barrelled, a contradiction with Proposition 5sequence $\{a_n\}_n$ in $\mathbb{N}$ such that each $t_n := (a_1, ..., a_n) \in J_n$ and each E_{t_n} is barrelled and dense in $\ell_0^\infty(X, \mathcal{A})$, i.e. $\mathcal{W}$ contains a strand formed by Baire-like spaces.

REFERENCES

1. de Wilde, M. – *Closed Graph Theorems and Webbed Spaces*, RNM 19 Pitman 1978.
2. Ferrando, J.C. and López, Pellicer M. – *Strong barrelledness properties in $\ell_0^\infty(X, \mathcal{A})$ and bounded finite additive measures*, Math. Ann. 287 (1990), 727-736.
3. Ferrando, J.C. and López, Pellicer M. – *Barrelled spaces of class n and class $\aleph_0$*, Sem. Mat. Fund. UNED, Fasc. 4 (1992).
4. Ferrando, J.C. and Sánchez, R.L.M. – *A maximal class of spaces with strong barrelledness conditions*, Proc. R. Ir. Acad. 92A, 1 (1992), 69-75.
5. Ferrando, J.C., López Pellicer, M. and Sánchez Ruiz, L.M. – *Metrizable Barrelled Spaces*, Pitman RNMS 332, Longman 1995.
6. Ferrer, J.R., López Pellicer, M. and Sánchez Ruiz, L.M. – *Suprabarrelledness of $\ell_0^\infty(X, \mathcal{A})$ and measure theory"*, An. Şti. Univ. "Al.I.Cuza" Iaşi Matematica 37 (1991), 141-144.
7. Ferrer, J.R. and Sánchez Ruiz, L.M. – *Barrelledness of class $\aleph_0$ of the space $\ell_0^\infty(X, \mathcal{A})$ and measure theory*, An. Şti. Univ. "Al.I.Cuza" Iaşi Matematica 41 (1995), 371-374.
8. López Pellicer, M. – *Webs and Bounded Finitely Additive Measures*, J. Math. Anal. Appl. 210 (1997), 257-267.
9. Rodríguez Salinas, R. – *Sobre el teorema de la gráfica cerrada. Aplicaciones lineales subcontinuas*, Rev. Re. Acad. Cienc. Exact. Fís. Nat. 74 (1980), 811–825.
10. Rodríguez Salinas, R. – *On superbarrelled spaces. Closed graph theorems*, Rev. Re. Acad. Cienc. Exact. Fís. Nat. 89 (1995), 7–10.
11. Saxon, S.A. – *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology*, Math. Ann. 197 (1972), 87-106

12. Valdivia, M. – *On certain barrelled normed spaces.* Ann. Inst. Fourier (Grenoble) 29 (1979), 39–56.

13. Valdivia, M. – *On suprabarrelled spaces, in Func. Anal Holomorphy and Approximation Theory*, Rio de Janeiro 1978, LNM 843, Springer Verlag 1981, 572–580.

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