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DENSITY TOPOLOGY IN A BITOPOLOGICAL SPACE

BY

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Abstract. The notion of density of sets has been introduced in a bitopological space and a density topology has been generated. Several separation properties are derived involving the density topology.

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1. Introduction. The idea of Romanovski space has been widely used in developing Denjoy integration in abstract spaces [11], in separation and density properties [12] and in further closer developments of density of sets [10] in metric spaces. In this paper we attempt to define density of sets in a bitopological space $(X, \mathcal{P}, Q)$ (without any metric structure) and investigate some of its consequences with the help of a few of the analogous axioms instead of all the axioms needed to define a Romanovski space.

In section 2, with the idea of pairwise closure we generate a topology which is found to be helpful in deriving certain separation properties. Introducing the idea of pairwise fundamental sets as a subclass of pairwise Borel sets, we define in section 3 density of sets and eventually density topology in a bitopological space. In section 4 we establish several separation properties in bitopological spaces involving the density topology.

2. Pairwise Closure.

Definition 1[5]. A set X on which are defined two arbitrary topologies $\mathcal{P}$ and Q is called a *bitopological space* and is denoted by $(X, \mathcal{P}, Q)$.

Let $(X, \mathcal{P}, Q)$ be a bitopological space. For any $A \subset X$, we define

$\overline{A} = \cap\{F_1 \cup F_2$; where $F_1 \cup F_2 \supset A$ and F_1 and F_2 are respectively $\mathcal{P}$ and Q closed$\}$.

Lemma 1. *The function from $\mathcal{P}(X)$ to itself given by $A \to \overline{A}$ for any $A \subset X$, satisfies the properties of Kuratowski closure operator.*

Definition 2. For any $A \subset X, \overline{A}$ is called the *pairwise closure* of A and the symbol 'bar' will be reserved throughout for pairwise closure.

Theorem 1. *Let $\mathcal{R} = \{U; U \subset X$ and $\overline{(X-U)} = X - U\}$. Then $(X, \mathcal{R})$ is a topological space.*

Note 1. We observe that if F is $\mathcal{P}$ or Q closed then $\overline{F} \subset F \cup \phi = F$, so that $\overline{F} = F$. Hence F is $\mathcal{R}$-closed. This shows that $\mathcal{R}$ is finer than both $\mathcal{P}$ and Q.

The following lemma is vital in our analysis.

Lemma 2. *The collection of sets $\{P \cap Q; P \in \mathcal{P}$ and $\mathbf{Q} \in Q\}$ form a base for $\mathcal{R}$.*

Proof. Evidently the sets $P \cap Q$ belong to $\mathcal{R}$. If $U \in \mathcal{R}$ then $X - U = \overline{(X-U)} = \cap\{F_1 \cup F_2; F_1 \cup F_2 \supset X - U$ and F_1, F_2 are respectively $\mathcal{P}$ and Q closed $\}$. Therefore $U = X - (X - U) = \cup\{P \cap Q; P \cap Q \subset U,\ P \in \mathcal{P}$ and $Q \in Q\}$ and this proves the lemma.

3. Density of Sets.

Definition 3. The σ-algebra generated by the class of all sets of the form $P \cup Q, P \in \mathcal{P}$ and $\mathbf{Q} \in Q$ is called the class of *pairwise Borel sets.*

Definition 4. $\mathcal{B} \subset \mathcal{P} \cup Q$ is said to be *pairwise open base* of $(X, \mathcal{P}, Q)$ if $\mathcal{B} \cap \mathcal{P}$ form a base for $\mathcal{P}$ and $\mathcal{B} \cap Q$ form a base for Q.

Definition 5 [1]. $\mathcal{B}$ is said to be a *pairwise open cover* of $(X, \mathcal{P}, Q)$ if $\mathcal{B} \subset \mathcal{P} \cup Q, \mathcal{B}$ is a cover of X and $\mathcal{B} \cap \mathcal{P}$ and $\mathcal{B} \cap Q$ contain at least one nonempty set.

Definition 6 [1]. $(X, \mathcal{P}, Q)$ is called *pairwise compact* if every pairwise open cover of $(X, \mathcal{P}, Q)$ has a finite subcover.

In $(X, \mathcal{P}, Q)$, let S_1 be the class of all pairwise Borel sets. Let μ be a measure on S_1 such that $\mu(X)$ is finite. We also assume μ to be non-zero for all non-void sets of the form $P \cap Q, P \in \mathcal{P}$ and $\mathbf{Q} \in Q$. Let μ^* be the outer measure on $\mathcal{P}(X)$ generated by μ. Let S be the class of all μ^*-measurable sets.

Let $\mathcal{A}$ be a class of sets from S_1.

Definition 7. By a *decomposition* { cf. [8]} S_U of $U \in S_1$ we mean a finite disjoint family $A_1, \ldots, A_n$ from $\mathcal{A}$ such that

(i) $\overset{n}{\underset{i=1}{\cup}} A_i \subset U$ and

(ii) $\mu(U - \overset{n}{\underset{i=1}{\cup}} A_i) = 0$.

The class $\mathcal{A}$ is called a *pairwise fundamental sets* { cf. [8]} if the following axioms hold.

Axiom I. $\mathcal{A}$ form a pairwise open base of $(X, \mathcal{P}, Q)$ (and hence also $\mathcal{A} \subset \mathcal{P} \cup Q$).

Axiom II. For any $A \in \mathcal{A}$ and $\epsilon > 0$ there is a decomposition S_A of A such that $A' \in S_A$ implies $\mu(A') < \epsilon$.

Axiom III. For each pairwise compact set W and for each $\mathcal{P}$ or Q open set $U \supset W$, there is an $\epsilon > 0$ such that if $\mu(A) < \epsilon$ and $\overline{A} \cap W \neq \phi$ for $A \in \mathcal{A}$ then $A \subset U$.

Axiom IV. Given $A \in \mathcal{A}$ and $\epsilon > 0$ there is a $A' \in \mathcal{A}$ such that $\overline{A} \subset A'$ and $\mu(A' - A) < \epsilon$.

The above axioms resemblance closely to some of the axioms needed to define the Romanovski space {[8], see also [6], [7], [9]}. This Romanovski space has been extensively used in [7] to study closely the properties of density functions, density topology etc. defined on such a space. In our present study we do not require all the axioms (analogous) as given in [8]. We only need the above axioms and we shall assume that $(X, \mathcal{P}, Q)$ is fitted only with these axioms.

We already observed that $\mathcal{R}$ is finer than both $\mathcal{P}$ and Q. Let S_U be a decomposition of $U \in \mathcal{P}$ or Q consisting of a finite disjoint family $A_1, \ldots, A_n$ from $\mathcal{A}$. Then

$$\mu(U - \overset{n}{\underset{i=1}{\cup}} \overline{A}_i) \leq \mu(U - \overset{n}{\underset{i=1}{\cup}} A_i) = 0.$$

Now $U - \overset{n}{\underset{i=1}{\cup}} \overline{A}_i$ is $\mathcal{R}$-open. If it is non-void it contains a non-void set of the form $P \cap Q, P \in \mathcal{P}$ and $\mathbf{Q} \in Q$, by Lemma 2. But since μ is assumed to be positive on non-void sets $P \cap Q$, $P \in \mathcal{P}, \mathbf{Q} \in Q$, this implies therefore

that $U - \overset{n}{\underset{i=1}{\cup}} \overline{A}_i = \phi$. From Axiom I it follows that for $x \in X$ there is a $A \in \mathcal{A}$ such that $x \in A$. The above and the Axiom II guarantee the existence of a sequence of pairwise fundamental sets $\{A_{n,x}\}$ corresponding to each $x \in X$ such that $x \in \overline{A}_{n,x}, \mu(A_{n,x}) < 1/n \forall\ n$.

Definition 8. For $E \subset X$ and $x \in X$ the *upper and lower outer density* of E at x denoted respectively by $\overline{\phi}^*(E,x)$, $\underline{\phi}^*(E,x)$ are defined by

$$\begin{aligned} \overline{\phi}^*(E,x) &= \lim_{n\to\infty} \overline{\phi}^*_n(E,x), \\ \underline{\phi}^*(E,x) &= \lim_{n\to\infty} \underline{\phi}^*_n(E,x) \end{aligned}$$

where

$$\begin{aligned} \overline{\phi}^*_n(E,x) &= \sup\{m^*(E,A);\ X \in \overline{A},\ \mu(A) < 1/n,\ A \in \mathcal{A}\} \\ \underline{\phi}^*_n(E,x) &= \inf\{m^*(E,A);\ X \in \overline{A},\ \mu(A) < 1/n,\ A \in \mathcal{A}\} \end{aligned}$$

where

$$m^*(E,A) = \mu^*(E \cap A)/\mu(A).$$

Clearly $0 \le \underline{\phi}^*(E,x) \le \overline{\phi}^*(E,x) \le 1$. If they are equal, we denote the common value by $\phi^*(E,x)$ and say the *outer density* of E exists at x. If $E \in S$ we write $\overline{\phi}^*(E,x) = \overline{\phi}(E,x)$ and $\underline{\phi}^*(E,x) = \underline{\phi}(E,x)$. If they are equal we write $\underline{\phi}(E,x) = \overline{\phi}(E,x) = \phi(E,x)$. We call x, an *outer density point* or an *outer dispersion point* of E according as $\underline{\phi}^*(E,x) = 1$ or $\overline{\phi}^*(E,x) = 0$.

Theorem 2. *If $E, F \in S$, $\phi(E,x)$, $\phi(F,x)$ exist and $E \subset F$ then $\phi(F-E,x)$ exists and*

$$\phi(F-E,x) = \phi(F,x) - \phi(E,x).$$

The proof is similar to Theorem 3 of [7] and so omitted.

Definition 9. Let $\mathcal{D} = \{U, U \subset X$ and $\phi^*(X-U,x) = 0$ for all $x \in U\}$.

As in Martin [5] one can verify that $\mathcal{D}$ is a topology on X. $\mathcal{D}$ is called the *density topology* (d-topology in short) on $(x, \mathcal{P}, Q)$.

In a Romanovski space it is known [7] that if U is open then $\overline{\phi}(U,x) = 1 \forall\ x \in U$. We noted earlier that $\mathcal{R}$ is finer than both $\mathcal{P}$ and Q. In the next theorem we observe that the above property is true even if $U \in \mathcal{R}$. However, if $\mathcal{P} = Q$ then $\mathcal{P} = Q = \mathcal{R}$ and the theorem coincides with Theorem 11 [7].

Theorem 3. *If U is $\mathcal{R}$-open then for all $x \in U$ the outer density of U exists at x and $\phi^*(U,x) = 1$.*

Proof. Let $x \in U$. By Lemma 2 there is a set $P \cap Q,\ P \in \mathcal{P}, \mathbf{Q} \in Q$ such that $x \in P \cap Q \subset U$. Since $\{x\}$ is pairwise compact and $\{x\} \subset P$, by Axiom III there is $\epsilon > 0$ such that $\{x\} \cap \overline{A} \neq \phi,\ A \in \mathcal{A}$ and $\mu(A) < \epsilon \Rightarrow A \subset P$. Choose $n_0 \in N$ such that $1/n_0 < \epsilon$. Then for all $n \geq n_0,\ x \in \overline{A},\ A \in \mathcal{A}$ and $\mu(A) < 1/n$ will imply $A \subset P$. Similarly, we can choose $n_1 \in N$ such that for all $n \geq n_1, x \in \overline{A},\ A \in \mathcal{A}$ and $\mu(A) < 1/n \Rightarrow A \subset Q$. Then for all $n \geq m = \max\{n_0, n_1\}, x \in \overline{A}, A \in \mathcal{A}$ and $\mu(A) < 1/n \Rightarrow A \subset P \cap Q \subset U$. From the definition of $\underline{\phi}^*_n(U,x)$ it follows that for all $n \geq m,\ \underline{\phi}^*_n(U,x) = 1$. Hence $\underline{\phi}^*(U,x) = 1$. Therefore, $\phi^*(U,x) = 1$.

Theorem 4. *The d-topology $\mathcal{D}$ is finer than $\mathcal{R}$.*

Proof. If U is $\mathcal{P}$ or Q open then $U \in S_1$ and so $U \in S$. By Theorems 2 and 3 we have $\overline{\phi}(X - U, x) = 0 \forall\ x \in U$. Hence $U \in \mathcal{D}$. Thus all the sets of the form $P \cap Q,\ P \in \mathcal{P}$ and $\mathbf{Q} \in Q$ belong to $\mathcal{D}$. From Lemma 2 it follows therefore that all $\mathcal{R}$-open sets are d-open. This completes the proof.

4. Separation Properties. We consider now the bitopological space $(X, \mathcal{D}, \mathcal{R})$. In the next theorems we obtain some separation properties on $(X, \mathcal{D}, \mathcal{R})$.

Theorem 5. $(X, \mathcal{R})$ *is regular.*

Proof. Let E be $\mathcal{R}$-closed and $x \notin E$. Then $x \in X - E \in \mathcal{R}$ and so by Lemma 2 there are $P \in \mathcal{P}$ and $\mathbf{Q} \in Q$ such that $x \in P \cap Q \subset X - E$. We associate with x, a sequence of pairwise fundamental sets $\{A_{n,x}\}$ such that $x \in \overline{A}_{n,x},\ \mu(A_{n,x}) < 1/n\ \forall\ n$. By Axiom IV, we can find a $B_{n,x} \in \mathcal{A}$ such that $x \in \overline{A}_{2n,x} \subset B_{n,x}$ and $\mu(B_{n,x} - A_{2n,x}) < 1/2n$. Then $\mu(B_{n,x}) \leq \mu(A_{2n,x}) + \mu(B_{n,x} - A_{2n,x}) < 1/2n + 1/2n = 1/n$. Thus we obtain a sequence $\{B_{n,x}\}$ from $\mathcal{A}$ such that $x \in B_{n,x}, \mu(B_{n,x}) < 1/n \forall\ n$. Again proceeding as above we get a sequence $\{C_{n,x}\}$ from $\mathcal{A}$ satisfying $x \in B_{2n,x} \subset \overline{B}_{2n,x} \subset C_{n,x}$ and $\mu(C_{n,x}) < 1/n \forall\ n$. Since $\{x\}$ is pairwise compact and $x \in P$, by Axiom III, there is $\epsilon > 0$ such that $x \in \overline{A}, A \in \mathcal{A}, \mu(A) < \epsilon \Rightarrow A \subset P$. Choose $n_0 \in N$ such that $1/n_0 < \epsilon$. Then $C_{n_0,x} \subset P$ and we have

$$x \in B_{2n_0,x} \subset \overline{B}_{2n_0,x} \subset C_{n_0,x} \subset P.$$

Similarly, we can find $n_1 \in N$ such that

$$x \in B_{2n_1,x} \subset \overline{B}_{2n_1,x} \subset C_{n_1,x} \subset Q.$$

Then $x \in B_{2n_0,x} \cap B_{2n_1,x} = U$ (say) $\subset \overline{B}_{2n_0,x} \cap \overline{B}_{2n_1,x} = F$ (say) $\subset C_{n_0,x} \cap C_{n_1,x} \subset P \cap Q \subset X - E$. Hence we have $U \in \mathcal{R}$ (by Axiom I), $V = X - F \in \mathcal{R}$ satisfying $x \in U, E \subset V$ and $U \cap V = \phi$. This proves the theorem.

Definition 10 [4]. In $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is said to be *regular* with respect to Q if for any $\mathcal{P}$-closed set F and $x \in X$ such that $x \notin F$, there are $U \in \mathcal{P}$, $V \in Q$ such that $x \in U, F \subset V$ and $U \cap V = \phi$. $(X, \mathcal{P}, \mathcal{Q})$ is called *pairwise regular* if both $\mathcal{P}$ and Q are regular with respect to each other.

Corollary 1. *In $(X, \mathcal{D}, \mathcal{R})$, $\mathcal{R}$ is regular with respect to $\mathcal{D}$.*
The Corollary follows from Theorems 4 and 5.

Theorem 6. *$(X, \mathcal{D}, \mathcal{R})$ is pairwise regular if the following condition holds.*

(A) For any d-closed set E, if $\{A_n\}$ is a sequence of pairwise fundamental sets such that $\mu^*(E \cap A_n) \to 0$ as $n \to \infty$, there is at least one $k \in N$ such that $E \cap A_k = \phi$.

Proof. In view of the Corollary 1, we have only to show that $\mathcal{D}$ is regular with respect to $\mathcal{R}$. Let E be d-closed and $x \notin E$. As in Theorem 5 we construct two sequences $\{B_{n,x}\}$, $\{C_{n,x}\}$ from $\mathcal{A}$ such that $x \in B_{2n,x} \subset \overline{B}_{2n,x} \subset C_{n,x}$ and $\mu(C_{n,x}) < 1/n$ for all n. Since $X - E$ is d-open and $x \in X - E$, $\overline{\phi}^*(E, x) = 0$. Let $\epsilon > 0$ be arbitrary. There is $n_0 \in N$ such that $1/n < \epsilon$ and $\overline{\phi}^*_n(E, x) < \epsilon \forall\ n \geq n_0$. Since $x \in C_{n,x} \subset \overline{C}_{n,x}$, $\mu(C_{n,x}) < 1/n$, $m^*(E, C_{n,x}) < \epsilon \forall\ n \geq n_0$ i.e. $\mu^*(E \cap C_{n,x})/\mu(C_{n,x}) < \epsilon$, i.e. $\mu^*(E \cap C_{n,x}) < \epsilon.\mu(C_{n,x}) < \epsilon\ \forall\ n \geq n_0$. Thus $\mu^*(E \cap C_{n,x}) \to 0$ as $n \to \infty$. By the condition (A) there is $k \in N$ such that $E \cap C_{k,x} = \phi$. Hence using Axiom I, $x \in B_{2k,x} \in \mathcal{R} \subset \mathcal{D}$, $E \subset X - C_{k,x} \subset X - \overline{B}_{2k,x} \in \mathcal{R}$ and $B_{2k,x} \cap (X - \overline{B}_{2k,x}) = \phi$. This proves the theorem.

Definition 11 [4]. $(X, \mathcal{P}, Q)$ is said to be *pairwise Hausdorff* if for arbitrary $x, y \in X, x \neq y$ there are $U \in P$, $V \in Q$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Theorem 7. *If (X, P) or (X, Q) is T_1, then $(X, \mathcal{D}, \mathcal{R})$ is pairwise Hausdorff.*

Proof. Let (X, P) be T_1. Let $x, y \in X$ and $x \neq y$. Then there exist $U \in P$ such that $x \in U, y \notin U$. From Note 1, $x \notin \mathcal{R} - cl\{y\}$. This holds for all $x \in X, x \neq y$. Hence $\mathcal{R} - cl\{y\} = \{y\}$. Since $x \notin \{y\}$, by Corollary 2 there are $U \in \mathcal{R}, V \in \mathcal{D}$ satisfying $x \in U, y \in V$ and $U \cap V = \phi$. So $(X, \mathcal{D}, \mathcal{R})$ is pairwise Hausdorff.

Definition 12 [4]. $(X, \mathcal{P}, Q)$, is called *pairwise normal* if for any P-closed set A and Q-closed set B with $A \cap B = \phi$, there exist $U \in P, V \in Q$ such that $A \subset V, B \subset U$ and $U \cap V = \phi$.

Theorem 8 : *If $(X, \mathcal{D})$ is compact then $(X, \mathcal{D}, \mathcal{R})$ is pairwise normal.*

Proof : Let A and B be respectively $\mathcal{R}$-closed and d-closed sets such that $A \cap B = \phi$. Let $x \in B$. By Theorem 5 there exist $U_x \in \mathcal{R}$, $V_x \in \mathcal{R}$ such that $x \in U_x$, $A \subset V_x$ and $U_x \cap V_x = \phi$. Now $\{U_x; x \in B\}$ form a d-open cover of B (in view of Theorem 4). Since B being a d-closed subset of the compact space $(X, \mathcal{D})$, is compact, there are $x_1, \ldots, x_n \in B$ such that $B \subset \overset{n}{\underset{i=1}{\cup}} U_{x_i} = U$ (say). Let $V = \overset{n}{\underset{i=1}{\cap}} V_{x_i}$. Then $A \subset V \in \mathcal{R} \subset \mathcal{D}$, $B \subset U \in \mathcal{R}$ and $U \cap V = \phi$. Therefore, $(X, \mathcal{D}, \mathcal{R})$ is pairwise normal.

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