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A LOCALLY SYMMETRIC COMPLEX STRUCTURE WITH NORDEN METRIC ON THE TANGENT BUNDLE OF A SPACE FORM

BY

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Abstract. We consider an almost complex structure with Norden metric (TM, G, J) on the tangent bundle TM of an n-dimensional Riemannian manifold M and, denoting by T_0M the nonzero tangent bundle to M, we obtain that (T_0M, G, J) is a locally symmetric complex manifold with Norden metric if and only if M is a space form of positive sectional curvature c and the functions $u(t), v(t)$ which appear in the definition of G are given by $u(t) = \sqrt{2ct} + A, v(t) = u'(t)$, where A is a real constant such that $A \geq 0$ and t denotes the density energy defined on TM by the Riemannian metric g of M. Some important remarks are also presented.

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1. Introduction. The tangent bundle $\tau : TM \longrightarrow M$ of a Riemannian manifold (M, g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways (see [1], [7], [15], [8], [16], [17]).

A slight generalization of the SASAKI metric is a special natural lift G of g to TM considered by OPROIU in [10] (for the definition and the expression of the natural lifts of g to TM see [4], [5], [3]). The considered Riemannian metric G on TM has been defined by using the Levi Civita connection of the Riemannian metric g on M and two smooth real valued functions $u(t), v(t)$ depending on the energy density only defined on TM by the Riemannian metric g on M, such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. He has also considered an almost complex structure J on TM, related to the metric G and has studied the conditions under which (TM, G, J) is

a Kaehler Einstein manifold. Remark that in [10], the author excludes some important cases which appeared, in a certain sense, as singular cases. These singular cases have been studied by Oproiu and the present author in [11], [9], [12]. Note also that one of the cases excluded in [10] is when the Riemannian metric G on TM is defined by using a certain Lagrangian L on the base manifold (M, g) depending on the energy density only (i.e. the case when $v(t) = u'(t)$, see [11]).

In [13], [14] we have considered a special natural 1-st order lift G of g which defines a pseudo-Riemannian metric on TM, so that generally, G is no longer obtained as the complete lift by using a Lagrangian on M (see [8]). This metric G has been defined by using also the Levi Civita connection of the Riemannian metric g and two smooth real valued functions $u(t), v(t)$ such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Next we have studied the necessary and sufficient conditions for the pseudo-Riemannian manifold to be Ricci flat (see [13]), respectively a locally symmetric manifold (see [14]).

Now, we recall that a Norden metric (or an hyperbolic metric) on an almost complex manifold $(\tilde{M}, J)$ is a pseudo-Riemannian metric G on $\tilde{M}$ such that

$$G(JX, JY) = -G(X, Y),$$

for arbitrary vector fields X, Y on $\tilde{M}$. Then G has the signature (n, n) where $2n$ is the (real) dimension of $\tilde{M}$. In [2] the almost complex manifolds with Norden metric have been classified in eight classes.

In the present paper, inspired from [10] and [11], we consider the pseudo-Riemannian metric G of type natural lift on TM defined in [13], [14] and we define an almost complex structure J on TM, related to the pseudo-Riemannian metric G, obtaining an almost complex manifold with Norden metric (TM, G, J). Next, we study necessary and sufficient conditions for the almost complex manifold with Norden metric (TM, G, J) to be a locally symmetric complex manifold with Norden metric. In fact, we obtain a locally symmetric complex structure only on T_0M = the nonzero tangent bundle to M. The obtained main result is given by: The almost complex manifold with Norden metric (T_0M, G, J) is a locally symmetric complex manifold with Norden metric if and only if the base manifold (M, g) is a space form of positive sectional curvature c and the functions $u(t), v(t)$ which appear in the expression of G are given by $u(t) = \sqrt{2ct} + A, v(t) = u'(t)$, where A is a real constant such that $A \geq 0$ (Theorem 6).

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always $\{1, ..., n\}$. We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

2. An almost complex structure with Norden metric on the tangent bundle. Let (M, g) be a smooth $n-$dimensional Riemannian manifold, $n > 1$, and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n-$dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, ..., x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, ...x^n,\ y^1, ..., y^n)$ on TM where the local coordinates $x^i, y^i;\ i = 1, ..., n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau;\ i = 1, ..., n$ of $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of the base point $\tau(y) \in U$ and the last n local coordinates $y^i; i = 1, ..., n$ are the vector space coordinates of the same tangent vector y with respect to the natural local basis $\frac{\partial}{\partial x^i}; i = 1, ..., n$ defined by (U, φ), i.e.

$$y = \Sigma y^i (\frac{\partial}{\partial x^i})_{\tau(y)}.$$

This special structure of TM allows us to introduce the notion of $M-$tensor field on it (see [6]). An $M-$tensor field of type (p, q) on TM is defined by sets of functions

$$T^{i_1...i_p}_{j_1...j_q}(x, y);\ i_1, ..., i_p, j_1, ..., j_q = 1, ..., n$$

assigned to any induced local chart $(\tau^{-1}(U), \Phi)$ on TM, such that the change rule is that of the components of a tensor field of type (p, q) on the base manifold, when a change of local charts on the base manifold is performed. We shall use currently $M-$tensor fields defined on TM. Remark also that any ordinary tensor field on the base manifold may be thought of as an $M-$tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold.

Recall that the Levi Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition

$$TTM = HTM \oplus VTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \text{Ker } \tau_*$ and the horizontal distribution HTM. The vector fields $(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n})$ define a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n})$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0} \frac{\partial}{\partial y^h};\ \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g.

The distributions VTM and HTM are isomorphic each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kaehlerian manifold on TM (see [1], [17]).

Consider now the energy density (kinetic energy)

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k,\ y \in \tau^{-1}(U)$$

of the tangent vector y, defined by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U, φ). Let $u, v : [0, \infty) \longrightarrow \mathbf{R}$ be two real smooth functions such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Then we may consider the following symmetric $M-$tensor field of type (0,2) on TM, defined by the components (see [10], [9]).

$$G_{ij} = u(t) g_{ij} + v(t) g_{0i} g_{0j},$$

where $g_{0i} = g_{hi} y^h$. Then the matrix (G_{ij}) is symmetric and positive definite and has the inverse with the entries

$$H^{kl} = \frac{1}{u} g^{kl} - \frac{v}{u(u + 2tv)} y^k y^l,$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}). The components $H^{kl}(x, y)$ define a symmetric $M-$tensor field of type $(2, 0)$ on TM. We shall use the components $H_{ij}(x, y)$ of a symmetric $M-$ tensor field of type (0,2) obtained from H^{kl} by "lowering" the indices, i.e.

$$H_{ij} = g_{ik} H^{kl} g_{lj} = \frac{1}{u} g_{ij} - \frac{v}{u(u + 2tv)} g_{0i} g_{0j}.$$

We shall use also the $M-$tensor fields on TM defined by the components

$$G^{kl} = g^{ki} G_{ij} g^{jl},\ G^i_k = G^{ih} g_{hk},\ H^i_k = H^{ih} g_{hk}$$

and remark that the matrix (H_k^i) is the inverse of the matrix (G_k^i) (see [10]).

The following pseudo-Riemannian metric may be considered on TM (see also [13], [14]).

$$G = 2G_{ij}\dot{\nabla}y^i dx^j = 2(ug_{ij} + vg_{0i}g_{0j})\dot{\nabla}y^i dx^j, \tag{2}$$

where $\dot{\nabla}y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0,\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0,\ G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = G(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = G_{ij}.$$

Remarks. *(i).* Due to the above conditions satisfied by the functions $u(t), v(t)$, the pseudo-Riemannian metric G defined by (2) is balanced, i.e. has the signature (n, n), and both the distributions VTM, HTM are isotropic.

(ii). In the particular case when $v(t) = u'(t)$, the components G_{ij} from the expression (2) of G there are obtained as in usual Lagrange geometry from the Lagrangian $L : TM \longrightarrow \mathbf{R}$ defined by (see [11]).

$$L = \int u(t)dt, \tag{3}$$

where $u : [0, \infty) \longrightarrow \mathbf{R}$ is a smooth function such that $u(t) > 0,\ u(t) + 2tu'(t) > 0$ for all $t \geq 0$. Because in this case, the usual nonlinear connection determined by the Euler-Lagrange equations associated to the Lagrangian L coincides with the nonlinear connection defined by the Levi Civita connection $\dot{\nabla}$ of g (see Proposition 1 from [11], we obtain that the pseudo-Riemannian metric G coincides with the complete lift of the quadratic form $h = G_{ij}(x, y)dx^i dx^j$ (see [8]), where

$$G_{ij} = \frac{\partial^2 L}{\partial y^i, \partial y^j} = ug_{ij} + u'g_{0i}g_{0j}.$$

(iii). Taking into account remark *(ii)*, it follows that in the case when $u(t) = 1,\ v(t) = 0$ for all $t \in [0, \infty)$, we have that $G = g^c$, where g^c is the complete lift of the Riemannian metric g on M.

Note also that the system of 1-forms $(dx^1, ..., dx^n, \dot{\nabla}y^1, ..., \dot{\nabla}y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n},\ \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition (1).

An almost complex structure J may be defined on TM by

$$J\frac{\delta}{\delta x^i}=G_i^k\frac{\partial}{\partial y^k},\ J\frac{\partial}{\partial y^i}=-H_i^k\frac{\delta}{\delta x^k}.$$

Then we obtain

Proposition 1. *(TM,G,J) is an almost complex manifold with Norden metric.*

Proof. First of all we may check easily that

$$J^2\frac{\delta}{\delta x^i}=-\frac{\delta}{\delta x^i};\ J^2\frac{\partial}{\partial y^i}=-\frac{\partial}{\partial y^i},$$

thus J really defines an almost complex structure on TM. Next, we have

$$G(J\frac{\delta}{\delta x^i},J\frac{\delta}{\delta x^j})=G(J\frac{\partial}{\partial y^i},J\frac{\partial}{\partial y^j})=0,$$

$$G(J\frac{\partial}{\partial y^i},J\frac{\delta}{\delta x^j})=\ -G(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})=-G_{ij},$$

i.e. we have

$$G(JX,JY)=-G(X,Y),\forall X,Y\in\Gamma(TM),$$

thus G is a Norden metric for J.

In the following we present the expression of the Levi Civita connection ∇ of the pseudo-Riemannian metric G on TM, where G is defined by (2) (see also [13], [14]). To do this we use the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^i};\ i=1,...,n$:

$$[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}]=0;\ [\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}]=-\Gamma_{ij}^h\frac{\partial}{\partial y^h};\ [\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}]=-R_{0ij}^h\frac{\partial}{\partial y^h},$$

where $R_{0ij}^h=R_{kij}^hy^k$ and R_{kij}^h are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M. We have

Proposition 2, [13], [14]. *Let (M,g) be a Riemannian manifold. Then the Levi Civita connection ∇ of the pseudo-Riemannian manifold (TM,G)*

has the following expression in the local adapted frame $(\frac{\partial}{\partial y^1}, \ ..., \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n})$*:*

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \ Q^h_{ij} \frac{\partial}{\partial y^h}; \ \nabla_{\frac{\delta}{\delta x^i}} \ \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h} + \ P^h_{ji} \frac{\delta}{\delta x^h};$$

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = \ P^h_{ij} \frac{\delta}{\delta x^h}; \ \nabla_{\frac{\delta}{\delta x^i}} \ \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + \ S^h_{ij} \frac{\partial}{\partial y^h},$$

where the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ *are given by:*

$$P^h_{ij} = \frac{u' - v}{2u}(g_{0i}\delta^h_j - \frac{u}{u + 2tv} g_{ij} y^h - \ \frac{v}{u + 2tv} g_{0i} g_{0j} y^h),$$

$$Q^h_{ij} = \frac{u' + v}{2u}(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{v}{u + 2tv} g_{ij} y^h + \frac{v'u - u'v - v^2}{u(u + 2tv)} g_{0i} g_{0j} y^h,$$

$$S^h_{ij} = g^{hk} R_{0ikj} + \frac{v}{u + 2tv} R_{0ij0} y^h,$$

R_{likj} *denoting the local coordinate components of the Riemann-Christoffel tensor of* $\dot{\nabla}$ *on* M *and* $R_{0ikj} = R_{likj} y^l, \ R_{0ij0} = R_{lijk} y^l y^k$.

3. A locally symmetric complex structure with Norden metric on TM. Throughout this section we assume that the Riemannian manifold (M, g) is not flat. If ∇ is the Levi Civita connection of G on the almost complex manifold with Norden metric (TM, G, J), then the following tensor field F, of type (0,3) may be considered (see [2])

$$F(X, Y, Z) = G((\nabla_X J)Y, Z); \ X, Y, Z \in \Gamma(TM).$$

The tensor field F has the following properties

$$F(X, Y, Z) = F(X, JY, JZ) = F(X, Z, Y); \ X, Y, Z \in \Gamma(TM).$$

According to the terminology and the classification from [2], we have

Definition. The almost complex manifold with Norden metric (TM, G, J) is called a complex manifold with Norden metric, or an $\omega_1 \oplus \omega_2$-manifold if

$$(4) \quad F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0; \ X, Y, Z \in \Gamma(TM).$$

Remark that the condition (4) is equivalent to the condition $N_J = 0$, where N_J denotes the Nijenhuis tensor field of the almost complex structure J.

We get by a straightforward computation

Proposition 3. *The expressions of F in the case of (TM, G, J) are given in the adapted local frame $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i})$ by*

$$F(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = \frac{u' + v}{u} g_{0i} g_{jk} +$$

$$+\frac{uu' + 2tu'v + 3uv + 2tv^2}{2u(u + 2tv)}(g_{0j} g_{ik} + g_{0k} g_{ij}) + \frac{2(v'u - u'v - v^2)}{u(u + 2tv)} g_{0i} g_{0j} g_{0k},$$

$$F(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}) = F(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}) = 0,$$

$$F(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) = u(u' + v) g_{0i} g_{jk} +$$

$$\frac{uu' + 2tu'v + 3uv + 2tv^2}{2} (g_{0j} g_{ik} + g_{0k} g_{ij}) + (u'v + 2uv' + 4tvv' + 3v^2) g_{0i} g_{0j} g_{0k},$$

$$F(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}) = F(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}) =$$

$$= R_{0ijk} + \frac{v}{u + 2tv} g_{0j} R_{0ik0} - \frac{u' - v}{2}[u g_{0j} g_{ik} - (u + 2tv) g_{0k} g_{ij} + v g_{0i} g_{0j} g_{0k}],$$

$$F(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = F(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) = 0.$$

By using Proposition 3, the condition (4) which must be fulfilled in order to (TM, G, J) be a complex manifold with Norden metric is reduced, after a long but straightforward computation to the condition

$$R_{0kij} - [v(2tu' - u) + uu'](g_{0i} g_{kj} - g_{0j} g_{ki}) = 0. \tag{5}$$

Introduce, for convenience, the following notation:

$$z(t) = v(2tu' - u) + uu'$$

and assume that the function $z(t)$ satisfies the condition $z(0) \neq 0$. We obtain that the condition (5) is equivalent to the relations

$$v(2tu' - u) = c - uu', \; R_{hkij} = c(g_{hi} g_{kj} - g_{hj} g_{ki}),$$

where $c = z(0)$ is a nonzero real constant.

It follows that the base manifold (M, g) must have constant sectional curvature $c \neq 0$ and the functions $u(t), v(t)$ must satisfy the relation

$$(2tu' - u)v = c - uu'. \tag{6}$$

Hence, in the case when $z(0) = c \neq 0$, we may state

Theorem 4. *The almost complex manifold with Norden metric (TM, G, J) is a complex manifold with Norden metric if and only if the base manifold (M, g) has constant sectional curvature c and the functions $u(t), v(t)$ are related by the relation (6), such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$.*

Remarks. *(i).* In the particular case when $v(t) = 0$ for all $t \in [0, \infty)$, the condition (6) becomes

$$uu' = c, \tag{7}$$

where $u(0)u'(0) = c \neq 0$. The general solution of the differential equation (7) is given by

$$u(t) = \sqrt{2ct + A}, \tag{8}$$

where A is an arbitrary real constant. From (8) it follows that $u(t) > 0$ for all $t \in [0, \infty)$ if and only if we have $c > 0$ and $A > 0$. Hence we have

Corollary 5. *Assuming that $v(t) = 0$ for all $t \in [0, \infty)$, then (TM, G, J) is a complex manifold with Norden metric if and only if (M, g) has constant positive sectional curvature c and the function $u(t)$ is given by (8) where $A > 0$.*

(ii). Assuming that $v(t) = 0$ for all $t \in [0, \infty)$, (M, g) has constant negative sectional curvature c and $u(t)$ is given by (8) where $A > 0$, we obtain a complex structure with Norden metric only on the tube around the zero section in TM defined by the conditions:

$$0 \leq t < -\frac{A}{2c}.$$

(iii). Assuming that $v(t) = 0$ for all $t \in [0, \infty)$, (M, g) has constant positive sectional curvature c and $u(t)$ is given by (8) where $A \leq 0$, we

obtain a complex structure with Norden metric only on the manifold in TM defined by the condition:

$$t > -\frac{A}{2c}.$$

In the following we suppose that (TM, G, J) is a complex manifold with Norden metric, i.e. (M, g) is a space form of constant sectional curvature $c \neq 0$ and the functions $u(t), v(t)$ are related by (6), and satisfy the conditions $u(t) > 0, u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$ and $z(0) = c \neq 0$. We study the conditions which must be fulfilled in order to (TM, G, J) be a locally symmetric space, i.e. $\nabla K = 0$, where K denotes the curvature tensor field of the Levi Civita connection ∇ of the pseudo-Riemannian metric G on TM. To do this we use the following result obtained in [14].

Theorem 6, [14]. *Let (M, g) be a Riemannian manifold and consider the pseudo-Riemannian manifold (TM, G), where G is given by* (2). *Then (TM, G) is a locally symmetric space if and only if the base manifold (M, g) is a locally symmetric space and the functions $u(t), v(t)$ satisfy the condition $v = u'$.*

By using Theorems 4 and 6 and taking into account the above assumptions, we get that the almost complex manifold (TM, G, J) is a locally symmetric complex manifold with Norden metric if and only if the base manifold has positive constant sectional curvature c and the functions $u(t), v(t)$ satisfy the following two relations

$$(i)\ 2t(u')^2 = c,\ (ii)\ v = u'. \tag{9}$$

Since we look for a solution u defined for $t > 0$, for which $u > 0$, $u + 2tu' > 0$, we may take

$$u(t) = \sqrt{2ct} + A,\ v(t) = u'(t) = \frac{c}{\sqrt{2ct}}, \tag{10}$$

where A is a nonnegative real constant, i.e. $A \geq 0$. Remark that the function $u(t)$ is smooth only on the nonzero tangent vectors of M, hence we obtain, in fact, a locally symmetric complex structure with Norden metric only on the manifold $T_0M =$ the tangent bundle to M minus the null section. We have

Theorem 7. *The almost complex manifold with Norden metric (T_0M, G, J) is a locally symmetric complex manifold with Norden metric*

if and only if the base manifold (M, g) *is a space form of positive sectional curvature* c *and the functions* $u(t), v(t)$ *are given by* (10)*, where* $A \geq 0$.

Taking into account remark *(ii)* from section 2, it follows that Theorem 7 can be formulated by

Corollary 8. *The almost complex manifold with Norden metric* (T_0M, G, J) *is a locally symmetric complex manifold with Norden metric if and only if the base manifold* (M, g) *is a space form of positive sectional curvature* c *and the pseudo-Riemannian metric* G *is the complete lift of the quadratic form* $h = G_{ij}dx^i dx^j$*, where the components* G_{ij} *are obtained as in usual Lagrange geometry considering on the base manifold the Lagrangian* L *defined by* (3) *with the function* $u(t)$ *defined by* $u(t) = \sqrt{2ct} + A$, $\forall A \geq 0$.

Remarks. *(i).* In the case where (M, g) has a positive constant sectional curvature c and the functions $u(t), v(t)$ are given by (10) where $A = -B^2$ is an arbitrary negative real constant, we obtain a locally symmetric complex structure with Norden metric only on the manifold in TM defined by the condition:

$$t > \frac{A^2}{2c}.$$

(ii). Assuming that (T_0M, G, J) is a locally symmetric complex manifold with Norden metric and computing the Ricci tensor we get that (T_0M, G, J) cannot be an Einstein manifold (see [13]).

(iii). In the particular case when $u(t) = 1, v(t) = 0$, i.e. $G = g^c$ where g^c is the complete lift of the Riemannian metric g on M, we get that (TM, g^c, J) is a complex manifold with Norden metric if and only if the base manifold is flat. Moreover, in this case (TM, g^c, J) is a flat Kähler manifold with Norden metric, i.e. $F(X, Y, Z) = 0$, or equivalently, $\nabla J = 0$.

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