ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.1.

A VIABILITY APPROACH FOR OPTIMAL CONTROL WITH INFIMUM COST

BY

MARC QUINCAMPOIX and OANA-SILVIA SEREA

Abstract. We characterize the discontinuous value function of an optimal control problem with infimum cost with finite and respectively infinite horizon. This characterization is made using the fact that the Epigraph of the value function is a viability kernel. A consequence of this is also that the value function is the smallest lower semicontinuous supersolution to a suitable Hamilton-Jacobi -Bellman equation. The main tool is viability theory.

Key words and phrases: value function, viability kernel, viscosity solutions.

1. Introduction. We consider a control system which dynamics is given by:

$$x'(t) = f(x(t), u(t)) \tag{1}$$

where the state variable x belongs to $\mathbb{R}^N$, the control $u(\cdot) : [0, \infty) \to U$ is a measurable function and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$.

The infimum control problem consists in

$$\begin{array}{c} \text{Minimize } \inf_{\tau \in [t_0, \infty)} \{ g\left(x(\tau; t_0, x_0, u(\cdot))\right) + \\ + \displaystyle\int_{t_0}^{\tau} L\left(x(s; t_0, x_0, u(\cdot)), u(s)\right) ds \} \end{array} \tag{2}$$

over all absolutely continuous solutions of (1), where $x(\cdot; t_0, x_0, u(\cdot))$ denotes the solution of (1) starting from $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^N$.

Here $g : \mathbb{R}^N \to \mathbb{R}_+$ and $L : \mathbb{R}^N \times U \to \mathbb{R}$ are given bounded functions. If we denote by $\mathcal{U}(t_0)$ the set of measurable controls on $[t_0, \infty)$ with values

in U, then the value function corresponding to the optimal control problem (1) and (2) is given by:

$$V(t_0,x_0)=\inf_{u\in\mathcal{U}(t_0)}\inf_{\tau\in[t_0,\infty)}\{g\left(x(\tau;t_0,x_0,u(\cdot))\right)+ \\ +\int_{t_0}^{\tau}L\left(x(s;t_0,x_0,u(\cdot)),u(s)\right)ds\} \tag{3}$$

The main question we address here is to characterize the value function (3) through a viability kernel or an equation of Hamilton-Jacobi type.

For viability characterization, for a different context the Epigraph of the minimal time to reach a set appear to be a viability kernel (cf. [9]).

Some results for the well-known case of a Lipschitz function g are already known (cf. [19], [6]). Namely the value is the unique Lipschitz viscosity solution for the following equation:

$$\begin{cases}\min\left[\left(\frac{\partial V}{\partial t}(t,x)+H(x,V(t,x),\frac{\partial V}{\partial x}(t,x))\right);(g(x)-V(t,x))\right]=0\\ (t,x)\in[0,\infty)\times\mathbb{R}^N;\end{cases} \tag{4}$$

where the Hamiltonian $H:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\bar{\mathbb{R}}$ is:

$$H(x,r,p)=\begin{cases}\min\limits_{u\in U}(p\cdot f(x,u)+L(x,u)), & g(x)\le r\ ;\\ +\infty, & g(x)>r\ .\end{cases}$$

Before studying the infinite horizon control problem (1) and (2), we will consider the finite horizon problem, i. e. :

$$\text{Minimize }\inf_{\tau\in[t_0,T]}\{g\left(x(\tau;t_0,x_0,u(\cdot))\right)+ \\ +\int_{t_0}^{\tau}L\left(x(s;t_0,x_0,u(\cdot)),u(s)\right)ds\} \tag{5}$$

over all absolutely continuous solutions of (1), with $0<T<\infty$.

The value function corresponding to the optimal control problem (1) and (5) is:

$$V^T(t_0,x_0)=\inf_{u\in\mathcal{U}(t_0)}\inf_{\tau\in[t_0,T]}\{g\left(x(\tau;t_0,x_0,u(\cdot))\right)+ \\ +\int_{t_0}^{\tau}L\left(x(s;t_0,x_0,u(\cdot)),u(s)\right)ds\} \tag{6}$$

and it is well known (see for instance [19]) that when g is Lipschitz, V^T is the unique Lipschitz viscosity solution to the following PDE:

$$\begin{cases}\min\left[\left(\frac{\partial V}{\partial t}(t,x)+H(x,V(t,x),\frac{\partial V}{\partial x}(t,x))\right);(g(x)-V(t,x))\right]=0\\ (t,x)\in[0,T)\times\mathbb{R}^N;\\ \text{with the final condition } V(T,x)=g(x),\text{ for all } x\in\mathbb{R}^N\end{cases} \tag{7}$$

The main goal of the present paper is to study the case when g is lower semicontinuous. In this context the value function may be discontinuous and standard uniqueness results for viscosity solutions of PDE cannot be used. We obtain a characterization of the semicontinuous value function using viability techniques.
Our main result says that the value function can be characterized through a viability kernel of a suitable dynamics. From this result we deduce that the value function is the smallest lower semicontinuous supersolution to (4) (see Definition 3).

Our proofs are based on the fact that the definition of viscosity supersolution, that we will give below, gives some invariance properties for the epigraph of the supersolution(see [2]). Semicontinuous viscosity solutions and semicontinuous value function have been introduced by Barron-Jensen [7] and Frankowska [14]. This problem was also considered in [19] for a Lipschitz function g. For the finite horizon case but for the problem with Supremum cost, the value function was characterized as generalized viscosity solution (see [20]), where g was supposed only bounded. Another characterization of this type was made in [15] for Mayer control problem.

This paper is organized as follows: First, we introduce some preliminaries. In the second section we present our main results concerning the finite horizon case with g lower semicontinuous. In the third section we will extend our study to the infinite horizon problem which is not analogous with the previous case.

2. Preliminaries.

2.1. Definitions, assumptions and notations.

We assume that:
The function $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous and satisfies :

$$\begin{cases} ||f(x,u)|| \leq c_1(1+\|x\|) \\ ||f(x,u)-f(y,u)|| \leq c_1\|x-y\| \quad \forall x,y \in \mathbb{R}^N, u \in U \end{cases} \tag{8}$$

where $c_1 > 0$ is constant; U is a compact set of some metric space.

$$\forall x \in \mathbb{R}^N,\ f(x,U) = \{f(x,u), u \in U\} \text{ is convex.} \tag{9}$$

It is well known that under (8), for every $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^N$, the Cauchy Problem (PC):

$$(PC) \qquad \begin{cases} x'(t) = f(x(t),u(t)) \text{ for almost every } t \in [t_0,\infty)\ ; \\ x(t_0) = x_0 \end{cases}$$

has an unique absolutely continuous solution denoted by $x(\cdot; t_0, x_0, u(\cdot))$.

We also assume that $L : \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous and satisfies :

$$\left\{ \begin{array}{c} ||L(x,u)|| \leq c_1(1 + \|x\|) \\ ||L(x,u) - L(y,u)|| \leq c_1 \|x - y\| \quad \forall x, y \in \mathbb{R}^N, u \in U \end{array} \right. \tag{10}$$

and

$$\forall x \in \mathbb{R}^N,\ L(x,U) = \{L(x,u), u \in U\} \text{ is convex.} \tag{11}$$

2.2. Viability theory. Here we will use the notion of viability domain with target introduced in [18].

Let us give a characterization of such sets, recalling the following known Viability Theorem (in a version adapted from [10] Theorem 2.3):

Proposition 1. *We assume that D and E are closed sets. Let us suppose that $\chi : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function, Lipschitz in the first variable, with furthermore $x \to \chi(x,U) = \{\chi(x,u); u \in U\}$ is Lipschitz continuous set valued map with convex, compact, nonempty values.*

Then the two following assertions are equivalent :*

i)

$$\begin{array}{c} \forall x \in D \backslash E, \forall p \in NP_D(x), \min_u < \chi(x,u), p > \leq 0 \\ \textit{(respectively } \max_u < \chi(x,u), p > \leq 0\textit{)} \end{array} \tag{12}$$

ii) there exists $u \in \mathcal{U}(t_0)$, such that the solution of (respectively, for all $u \in \mathcal{U}(t_0)$, the solution of)

$$\left\{ \begin{array}{c} x'(t) = \chi(x(t), u(t)) \textit{ for almost every } t \in [t_0, T] \ ; \\ x(t_0) = x_0 \end{array} \right.$$

remains in D as long as it does not reach E.

When i) or ii) holds true we say that D is a viability domain with target E (respectively, D is an invariance domain with target E) for the dynamics χ. When $E = \emptyset$, this is the classical notion of viability (respectively invariance) domain cf. [2].

*Here $NP_D(x)$ denotes the set of proximal normal to D at x i.e., the set of $p \in \mathbb{R}^N$ such that the distance of $x + p$ to D is equal to $||p||$.

Definition 2. Let K be a closed set. We call viability kernel in K with target E, for a dynamics Φ, denoted $Viab_\Phi(K,\ E)$, the largest closed subset of K, which is viability domain with target E for Φ.

It is proved (see Theorem 3.2.3. in [1] concerning the case $E = \emptyset$) that $Viab_\Phi(K,\ E)$ is also the set of x_0 such that there exists $x(\cdot)$ solution of

$$x'(t) \in \Phi(x(t))$$

starting from x_0 which is either defined on $[0, \infty)$ and $x(t) \in K$ for all $t \geq 0$, or is defined on a finite interval $[0, \tau]$, satisfies $x(t) \in K$ for all $t \in [0, \tau]$ and $x(\tau) \in E$.

Our proofs are based on the fact that the definition of solutions for a PDE of the type (4) with g replaced by h and for a another Hamiltonian $\bar{H}$ gives some invariance properties for the Epigraph and the Hypograph[†] of the solution (see Definition 4 and Proposition 1). More precisely the Epigraph of a supersolution is viability domain in $[0,T] \times \mathbb{R}^{N+2}$ with target $[0,T] \times Epi(h)$, for the dynamics:

$$(t,x,z,r) \to \Phi(t,x,z,r) = (1, f(x,U); L(x,U), 0)$$

and the Hypograph of a subsolution is invariance domain in $[0,T] \times \mathbb{R}^{N+2}$ with target $[0,T] \times Epi(h)$, for the dynamics :

$$(t,x,z,r) \to -\Phi(t,x,z,r) = -(1, f(x,U); L(x,U), 0).$$

Namely we will use specially the fact that the Epigraph the value function is viability kernel in $[0,T] \times \mathbb{R}^{N+2}$ with target $[0,T] \times Epi(h)$, i. e. the largest closed viability domain with target for Φ.

Here *Epi* stands for the epigraph and *Hypo* stands for the hypograph.

2.3. Viscosity Solutions. Let us define viscosity solution to first order Hamilton-Jacobi equation (cf. [3] for instance):

Definition 3. A *viscosity supersolution* for (7) is a lower semicontinuous function $\psi : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ such that:

$$\text{for any } \phi \in C^1 \text{ and } (t_0,x_0) \in \arg Min\,(\psi - \phi)\,,$$
$$\min\left\{(\tfrac{\partial \phi}{\partial t}(t_0,x_0) + H(x_0, \phi(t_0,x_0), \tfrac{\partial \phi}{\partial x}(t_0,x_0)))\ ;\ (g(x_0) - \phi(t_0,x_0))\right\} \leq 0$$

[†]For $w : [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ we have:
$Epi(w) := \{(t,x,r) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \mid w(t,x) \leq r\}$;
$Hypo(w) := \{(t,x,r) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \mid w(t,x) \geq r\}$.

and a *viscosity subsolution* for (7) is an upper semicontinuous function $\varphi : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that:

$$\text{for any } \phi \in C^1 \text{ and } (t_0,x_0) \in \arg Max\,(\varphi - \phi)\,,$$
$$\min\left\{(\tfrac{\partial\phi}{\partial t}(t_0,x_0) + H(x_0,\phi(t_0,x_0),\tfrac{\partial\phi}{\partial x}(t_0,x_0)))\ ;\ (g(x_0)-\phi(t_0,x_0))\right\} \geq 0.$$

A *viscosity solution* of (7) is a function which is both subsolution and supersolution (so, it is in particular continuous).

The definition of the solutions for (4) is similar to the previous definition and we must consider that a supersolution is defined on $[0,\infty)\times\mathbb{R}^N$ and a subsolution is defined on $(0,\infty)\times\mathbb{R}^N$.

For the different definitions of discontinuous viscosity solutions we refer the reader to Ishii solutions (cf. [3]) based on semicontinuous envelopes of functions, to Barron-Jensen-Frankowska semicontinuous solutions ([3], [7] for convex Hamiltonians), to Subbotin minimax solution [21] (called bilateral solutions in [4]) see also [16].

Now in order to establish some links between our definition of viscosity solutions and the interpretation in terms of viability notions we give an equivalent formulation of the definition of super- and subsolutions to (4) in terms of proximal normals (see [15], [17]):

Proposition 4. *(cf. Proposition 3.3 in [17])A viscosity supersolution for (4) is a l.s.c. function $\psi : [0,T)\times\mathbb{R}^N \to \mathbb{R}$ such that:*

$$\text{for any } (p_t,p_x,p_r) \in NP_{Epi(\psi)}(t_0,x_0,\psi(t_0,x_0)),$$
$$\min\{(p_t + H(x_0,\psi(t_0,x_0),p_x))\ ;\ (g(x_0)-\psi(t_0,x_0))\} \leq 0$$

A viscosity subsolution for (4) is an u.s.c. function $\varphi : [0,T)\times\mathbb{R}^N \to \mathbb{R}$ such that:

$$\text{for any } (p_t,p_x,p_r) \in NP_{Hypo(\varphi)}(t_0,x_0,\varphi(t_0,x_0)),$$
$$\min\{(p_t + H(x_0,\varphi(t_0,x_0),p_x))\ ;\ (g(x_0)-\varphi(t_0,x_0))\} \geq 0.$$

3. Discontinuous Infimum Control Problem with Finite Horizon.

We suppose :

$$\left\{\begin{array}{c} |g(x)| \leq c_2 \text{ for all } x \in \mathbb{R}^N, \\ g \text{ is lower-semicontinuous in } \mathbb{R}^N. \end{array}\right. \tag{13}$$

and $L : \mathbb{R}^N \times U \to \mathbb{R}$ satisfies (10), (11). If g is discontinuous then so is in general the value function. This section is devoted to the characterization of the value function through the Hamilton-Jacobi Equation (4).

In order to do this, we will consider the following modified dynamics:

$$y'(t) = (f(x(t), u(t)); L(x(t), u(t)). \tag{14}$$

Here $y(\cdot; t_0, y_0, u(\cdot)) := (x(\cdot; t_0, x_0, u(\cdot)); z(\cdot; t_0, z_0, u(\cdot)))$ is the solution of (14) starting to $(t_0, x_0, z_0) := (t_0, y_0) \in [0, T] \times \mathbb{R}^{N+1}$ and $S(t_0, y_0) := \{y(\cdot; t_0, y_0, u(\cdot));\ u \in \mathcal{U}(t_0)\}$.

Let $h : \mathbb{R}^{N+1} \to \mathbb{R}$, given by $h(y) := g(x) + z$ with $y := (x, z)$. The control problem that we will study is:

$$\text{Minimize} \inf_{\tau \in [t_0, T]} h\left(y(\tau; t_0, y_0, u(\cdot))\right) \tag{15}$$

over all absolutely continuous solutions of (14).

With the above notations, we define the following value function:

$$W^T(t_0, y_0) = \inf_{u \in \mathcal{U}(t_0)} \inf_{\tau \in [t_0, T]} h\left(y(\tau; t_0, y_0, u(\cdot))\right). \tag{16}$$

Note that we have the relation: $W^T(t_0, y_0) = V^T(t_0, x_0) + z_0$ and we will study the properties of W^T in order to obtain as a consequence the characterization for V^T.

Before giving the main result of this section, we prove some classical results and we recall well known results when the function h is Lipschitz.

3.1. Regularity and Dynamic Programming Principle. We first prove some results concerning the regularity of W^T.

Lemma 5. *Suppose that (8), (9), (10), (11), hold true. Assume that h is lower semicontinuous. Then we have:*

i) (Existence of Optimal control) There exists an optimal trajectory starting from each point $(t_0, y_0) \in [0, T] \times \mathbb{R}^{N+1}$ i.e. there exists $\bar{y}(\cdot) \in S(t_0, y_0)$ such that

$$W^T(t_0, y_0) = \inf_{\tau \in [t_0, T]} h(\bar{y}(\tau; t_0, y_0, \bar{u}(\cdot))) \text{ for all } (t_0, y_0) \in [0, T] \times \mathbb{R}^{N+1}$$

ii) W^T *is lower semicontinuous.*

Proof. *i)* By definition of W^T there exists $y_n(\cdot) \in S(t_0, y_0)$ which is the set of solutions starting to (t_0, y_0) of (14) for all $u \in \mathcal{U}(t_0)$, such that

$$W^T(t_0, y_0) = \lim_{n\to\infty} \inf_{\tau\in[t_0,T]} h(y_n(\tau; t_0, y_0, \bar{u}(\cdot)).$$

The compactness arguments (see for instance [2]) provide the existence of $\bar{y}(\cdot) \in S(t_0, y_0)$, the limit of $y_n(\cdot)$ in $W^{1,1}[t_0, T]$.

Using the semicontinuity of h we obtain that there exists $\tau_n \in [t_0, T]$ such that

$$\inf_{\tau\in[t_0,T]} h(y_n(\tau)) = h(y_n(\tau_n)) \text{ and}$$

$$W^T(t_0, y_0) = \lim_{n\to\infty} \inf_{\tau\in[t_0,T]} h(y_n(\tau)) = \lim_{n\to\infty} h(y_n(\tau_n)).$$

So there exists a subsequence τ_{n_k} of τ_n and $\bar{\tau} \in [t_0, T]$ such that $\tau_{n_k} \to \bar{\tau}$ and letting $n \to \infty$ we have

$$W^T(t_0, y_0) = \lim_{k\to\infty} \inf h(y_{n_k}(\tau_{n_k})) \geq h(\bar{y}(\bar{\tau})) \geq \inf_{\tau\in[t_0,T]} h(\bar{y}(\tau)) \geq W^T(t_0, y_0)$$

and *i)* follows.

ii) We consider $(t_n, y_n) \in [0, T] \times \mathbb{R}^{N+1}$ such that $(t_n, y_n) \to (t_0, y_0)$. We want to obtain that

$$\liminf_{n\to\infty} W^T(t_n, y_n) \geq W^T(t_0, y_0)$$

By i) there exists $y_n(\cdot) \in S(t_n, y_n)$ with $W^T(t_n, y_n) = \inf_{\tau\in[t_0,T]} h(y_n(\tau))$ and by the compactness properties of the solution map of the differential inclusion associated with (14) there exists $\bar{y}(\cdot) \in S(t_0, y_0)$ and $y_n(\cdot) \to \bar{y}(\cdot)$ uniformly on compacts.

So there exists a subsequence n_k and $\bar{\tau} \in [t_0, T]$ such that $\tau_{n_k} \to \bar{\tau}$ and because h is l.s.c. we have

$$\begin{aligned}\liminf_{k\to\infty} W^T(t_n, y_n) &= \lim \inf_{k\to\infty} \inf_{\tau\in[t_{n_k},T]} h(y_{n_k}(\tau)) = \lim \inf_{k\to\infty} h(y_{n_k}(\tau_{n_k})) \\ &\geq h(\bar{y}(\bar{\tau})) \geq W^T(t_0, y_0)\end{aligned}$$

and the proof is done.

Next we prove Bellman Dynamic Programming from which is derived the PDE satisfied by the value function.

Proposition 6. *(Dynamic Programming Principle)*
Let $g : \mathbb{R}^N \to \mathbb{R}$ be a bounded function and suppose that (8), (9), (10), (11) hold true. Then for all $(t_0, y_0) \in [0, T] \times \mathbb{R}^{N+1}$ and $\alpha > 0$ such that $t_0 + \alpha \leq T$:

$$W^T(t_0, y_0) = \min\{\inf_{u(\cdot)\in\mathcal{U}(t_0)} W^T(t_0 + \alpha, y(t_0 + \alpha)); \inf_{\tau\in[t_0,t_0+\alpha]} h(y(\tau; t_0, y_0, u(\cdot))\}. \tag{17}$$

Proof. From one hand, for all $u(\cdot) \in \mathcal{U}(t_0)$, the very definition of W^T yields:

$$W^T(t_0, y_0) \leq \min\{\inf_{\tau\in[t_0,t_0+\alpha]} h(y(\tau; t_0, y_0, u(\cdot)); \inf_{\tau\in[t_0+\alpha,T]} h(y(\tau; t_0, y_0, u(\cdot))\}.$$

For $\bar{u}(\cdot) \in \mathcal{U}(t_0 + \alpha)$ such that

$$W^T(t_0 + \alpha, y(t_0 + \alpha)) = \inf_{\tau\in[t_0+\alpha,T]} h(\bar{y}(\tau; t_0 + \alpha, y(t_0 + \alpha), \bar{u}(\cdot)))$$

we define the measurable function $\tilde{u}(\cdot) \in \mathcal{U}(t_0)$

$$\tilde{u}(t) = \begin{cases} u(t) \text{ if } t \in [t_0, t_0 + \alpha] \\ \bar{u}(t) \text{ if } t \in [t_0 + \alpha, T] \end{cases}$$

and we have for all $u(\cdot) \in \mathcal{U}(t_0)$,

$$W^T(t_0, y_0) \leq \min\{\inf_{\tau\in[t_0,t_0+\alpha]} h(y(\tau; t_0, y_0, u(\cdot)); W^T(t_0 + \alpha, y(t_0 + \alpha))\}$$

so

$$W^T(t_0, y_0) \leq$$
$$\leq \inf_{u(\cdot)\in\mathcal{U}(t_0)} \min\{\inf_{\tau\in[t_0,t_0+\alpha]} h(y(\tau; t_0, y_0, u(\cdot)); W^T(t_0 + \alpha, y(t_0 + \alpha))\}.$$

From the other hand, for the reverse inequality for $\bar{u}(\cdot) \in \mathcal{U}(t_0)$, such that:

$$W^T(t_0, y_0) = \inf_{\tau\in[t_0,T]} h(\bar{y}(\tau; t_0, y_0, \bar{u}(\cdot))$$

we have

$$W^T(t_0,y_0)=\min\{\inf_{\tau\in[t_0,t_0+\alpha]}h(\bar{y}(\tau;t_0,y_0,\bar{u}(\cdot));\inf_{\tau\in[t_0+\alpha,T]}h(\bar{y}(\tau;t_0,y_0,\bar{u}(\cdot))\}$$

and by definition

$$W^T(t_0,y_0)\geq\min\{\inf_{\tau\in[t_0,t_0+\alpha]}h(\bar{y}(\tau;t_0,y_0,\bar{u}(\cdot));W^T(t_0+\alpha,y(t_0+\alpha))$$

and the proof is complete. ■

3.2. The Hamilton-Jacobi Partial Differential Equation with finite horizon. Using Dynamic Programming Principle we prove that for the control problem with finite horizon, when the value function W^T is regular enough then it is the viscosity solution in the sense of Definition 3 for the following PDE:

(18)
$$\begin{cases}\min\Big[\Big(\frac{\partial W^T}{\partial t}(t,y)+\bar{H}(y,W^T(t,y),\frac{\partial W^T}{\partial y}(t,y))\Big);\big(h(y)-W^T(t,y)\big)\Big]=0\\(t,y)\in[0,T)\times\mathbb{R}^{N+1}\\W^T(T,\cdot)=h(\cdot).\end{cases}$$

where the Hamiltonian $\bar{H}:\mathbb{R}^{N+1}\times\mathbb{R}\times\mathbb{R}^{N+1}\to\bar{\mathbb{R}}$ is:

$$\bar{H}(y,r,q)=\begin{cases}\min\limits_{u\in U}(q\cdot(f(x,u),L(x,u))), & h(x)\leq r\ ;\\+\infty, & h(x)>r\ .\end{cases}$$

Proposition 7. *If h is a Lipschitz application then W^T is the unique Lipschitz viscosity solution to (18) with the final condition $W^T(T,\cdot)=h(\cdot)$.*

This result is classical and is based on the Dynamic Programming Principle and the regularity results for the value function, so we omit the proof.

Remark 8. We can verify that $\frac{\partial W^T}{\partial z}=1$ for almost all $(t,y)\in[0,T]\times\mathbb{R}^{N+1}$ and as a consequence V^T is the unique Lipschitz viscosity solution to (4) with the final condition $V^T(T,\cdot)=g(\cdot)$.

3.3. The lower semicontinuous case for the finite horizon control problem. In this section we suppose that the application h is lower semicontinuous (in short l.s.c.).

We have already proved that in this case the value function W^T is also lower semicontinuous. Next we will prove that the value is the smallest l.s.c. supersolution of (7).

Theorem 9. *If (8), (9) (10), (11) hold true then $EpiW^T$ is viability kernel in $[0,T]\times\mathbb{R}^{N+2}$ with target $[0,T]\times Epi(h)$ for the dynamics $(t,x,z,r)\to\Phi(t,x,z,r)=(1,f(x,U),L(x,U),0)$:*

$$EpiV^T = Viab_\Phi([0,T]\times\mathbb{R}^{N+2},[0,T]\times Epi(h))$$

As a consequence the value function is the smallest lower semicontinuous supersolution to (18) and it verifies furthermore the final condition $W^T(T,\cdot)=h(\cdot)$.

Proof. First we will prove that the value is a l.s.c. viscosity supersolution to (18). We will use a inf-convolution method. We define a sequence h_n

$$h_n(y) = \inf_{w\in\mathbb{R}^N}(h(w)+n\,\|w-y\|)\ ;\ n\in\mathbb{N} \tag{19}$$

The functions h_n are Lipschitz, $h_n(y)\le h_{n+1}(y)$ and $\lim_{n\to\infty} h_n(y)=h(y)$ for every $y\in\mathbb{R}^{N+1}$. Using the results for the Lipschitz continuous case we obtain that $W^T_{h_n}$ is a non-decreasing sequence of Lipschitz viscosity solutions for (18) which limit is denoted by $U:=\lim_{n\to\infty} W^T_{h_n}$.

From one hand $h_n\le h$ yields $U\le W^T_h$. From the other hand, U is a non-decreasing limit of l.s.c. solutions. Stability arguments (see for instance Theorem 4.1 in [3]), allows us to conclude that U is a lower semicontinuous viscosity solution to (7).

Now we want to prove that $U\ge W^T_h$. To do this we will use the following result which proof is postponed:

Lemma 10. *Assume that (8), (9) (10), (11) hold true. If φ is an lower semicontinuous supersolution for (7) with $\varphi(t,y)\le h(y)$ for all $(t,y)\in[0,T]\times\mathbb{R}^{N+1}$ then $Epi(\varphi)$ is viability domain in $[0,T]\times\mathbb{R}^{N+2}$ with target $[0,T]\times Epi(h)$, for the dynamics Φ of Theorem 9.*

An interpretation of the above Lemma is:

for every $(t_0,y_0)\in(0,T)\times\mathbb{R}^{N+1}$, there exists $u\in\mathcal{U}(t_0)$ such that:

the function $t \to (t, y(t), \varphi(t_0, y_0))$ with $y(\cdot) := y(\cdot; t_0, y_0, u)$, remains in the epigraph of φ until it (possibly) reaches the set $[0,T] \times Epi(h)$. Or equivalently :

$$\text{for every } t \in [0,T], \text{ if } \varphi(s, y(s))) < \ h(y(s)) \text{ for all } s \in [0,t], \text{ then:}$$
$$\varphi(t, y(t))) \leq \varphi(t_0, y_0).$$

This property and the definition of the value function gives the following inequality:

Under the assumption of Lemma 10:

$$\varphi(t,y) \geq W_h^T(t,y) \text{ for all } (t,y) \in [0,T] \times \mathbb{R}^{N+1} \tag{20}$$

if $\varphi(T,y) = h(y)$ for all $y \in \mathbb{R}^{N+1}$

Now because $U(t,y) \leq h(y)$ on $[0,T] \times \mathbb{R}^{N+1}$ and $U(T,y) = h(y)$ for all $y \in \mathbb{R}^{N+1}$ the inequality $U \geq W_h^T$ follows from (20). So we obtain that $W^T = U$ and W^T is a l.s.c. supersolution for (7).

Because W^T is a l.s.c. supersolution for (7) and using Lemma 10 we obtain that $Epi(W^T)$ is viability domain in $[0,T] \times \mathbb{R}^{N+2}$ with target $[0,T] \times Epi(h)$, for the dynamics Φ.

Now we want to prove that

$$Viab_\Phi([0,T] \times \mathbb{R}^{N+2}, [0,T] \times Epi(h)) \subset EpiW^T.$$

Let us consider $(t_0, y_0, r_0) \in [0,T] \times \mathbb{R}^{N+2}$ such that there exists $y(\cdot)$ a solution to (14) denoted by:

$$t \to (t, y(t), \varphi(t_0, y_0)) \text{ with } y(\cdot) := y(\cdot; t_0, y_0, u),$$

which remains in $[0,T] \times \mathbb{R}^{N+2}$ until it (possibly) reaches the set $[0,T] \times Epi(h)$.

So there exists $s \in [0,T]$ with $h(y(s)) \leq r_0$. By the very definition of the value function $W^T(t_0, y_0) \leq r_0$ so $(t_0, y_0, r_0) \in EpiW^T$ and the proof of Theorem 9 is complete.

Now we will prove Lemma 10:

Proof. of Lemma 10. Fix $t_0 \in (0,T)$. We set

$$D_\varphi = cl(\{(t,y,r) : t \in (0,T], y \in \mathbb{R}^{N+1}, r \geq \varphi(t,y)\}) \cup [T, \infty) \times \mathbb{R}^{N+1} \times \mathbb{R}$$

$$\tilde{f}(t,x,u,r) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0}(1, \bar{f}(x,u), 0) \text{ if } t \in [0, t_0] \\ (1, f(x,u), 0) \text{ if } t \in [t_0, T] \\ (1, f(x,u), 0) \text{ if } t > T \end{cases}$$

where $(f(x,u);L(x,u)) = \bar{f}(x,u)$, for $(x,u) \in \mathbb{R}^N \times U$.

We show that the viability condition (12), holds true for $\tilde{f}$, D_φ and $[0,T] \times Epi(h)$. Let $w_0 = (s_0, y_0, r_0 := \varphi(t_0, y_0)) \in D_\varphi \backslash \{[0,T] \times Epi(h)\}$. If $s_0 = 0$ then $\tilde{f} = 0$. Obviously (12), holds true.

If $s_0 \geq T$ and $(p_s, p_y, p_r) \in N_{D_\varphi}(s_0, y_0, r_0)$, then $p_s \leq 0$, $p_y = 0$, $p_r = 0$. Hence the viability condition (12), holds true.

It remains to consider the case $s_0 \in (0,T)$. We have

$$N_{D_\varphi}(s_0, y_0, r_0) \subset N_{D_\varphi}(s_0, y_0, \varphi(s_0, y_0))$$

Let $(p_s, p_y, p_r) \in N_{D_\varphi}(s_0, y_0, \varphi(s_0, y_0))$.

Since φ is a *supersolution* of (7) we have that

$$\min\left\{(p_s + \bar{H}(y_0, \varphi(t_0, y_0), p_y)\ ;\ (h(y_0) - \varphi(t_0, y_0))\right\} \leq 0.$$

and using the fact that $\varphi(s_0, y_0) < h(y_0)$ we obtain

$$p_s + \min_u < \bar{f}(x,u),\ p_y >\leq 0.$$

Hence

$$\min_u < \tilde{f}(x,u), (p_s, p_y, p_r) >\leq 0.$$

In view of Proposition 1, we have that there exists $u \in \mathcal{U}(t_0)$ such that the solution $w(\cdot; t_0, y_0, u)$ to the Cauchy problem

$$w'(s) \in \bar{f}(w(s)),\ w(t_0) = w_0$$

$w(s) = (t(s), y(s), r(s))$ with

$$t(s) = s,\ r(s) = r_0 = \varphi(t_0, x_0)$$

stays in D_φ until it reaches $[0,T] \times Epi(h)$ and the proof of Lemma 10 is complete.

4. Discontinuous Infimum Control Problem with Infinite Horizon. From now on we study the following value function[‡] :

$$W(t_0, y_0) = \inf_{u \in \mathcal{U}(t_0)} \inf_{\tau \in [t_0, \infty)} h\left(y(\tau; t_0, y_0, u(\cdot))\right) \tag{21}$$

[‡]We would like to thanks A. Rappaport who brought our attention on this problem.

over absolutely continuous solutions of the modified dynamics (14).

We note that

$$W(t_0, y_0) = V(t_0, x_0) + z_0 \text{ for all } (t_0, y_0) = (t_0, x_0, z_0) \in [0, \infty) \times \mathbb{R}^{N+1}$$

so it is sufficient to study W in order to obtain a characterization for V.

In this section we suppose that the application L satisfies furthermore the following assumption:

(H_L) L is uniformely bounded from below by $L_0 > 0$.

We will see that this hypothesis allows us to obtain Proposition 11 (see [19]).

We will prove that the value function W is viscosity solution for the following variational inequality

$$(22) \quad \begin{cases} \min\left[\left(\frac{\partial W}{\partial t}(t,y) + H(y, W(t,y), \frac{\partial W}{\partial y}(t,y))\right); (h(y) - W(t,y))\right] = 0 \\ (t,y) \in [0,\infty) \times \mathbb{R}^{N+1} \end{cases}$$

without final condition.

The main result of this section consist in characterizing the value through a viability kernel. Before giving the proof we will give some classical results.

4.1. The Lipschitz case. Here the function h is supposed to be Lipschitz continuous and we suppose also that (H_L) holds true.

The infinite horizon problem with such hypotheses was considered in [19] for differential games. The results of [19] can be adapted for the control problem, when h is supposed to be Lipschitz.

So, the Hamilton-Jacobi Partial Differential Equation can be deduced of [19], using the Dynamic Programming Principle. More precisely, we prove that for the control problem with infinite horizon, the value function is viscosity Lipschitz solution in the sense of Definition 3 for (22):

Proposition 11. *(cf. Proposition 2.2. in [19]) If h is a Lipschitz application and (H_L) holds, then W is the unique Lipschitz viscosity solution to (22).*

Remark 12. We can verify that $\frac{\partial W}{\partial z} = 1$ for almost all $(t, y) \in [0, T] \times \mathbb{R}^{N+1}$ and as a consequence V is the unique Lipschitz viscosity solution to (4) with the final condition $V(T, \cdot) = g(\cdot)$.

Hypothesis (H_L) is crucial for the previous result. Indeed in [12], [19] is a counterexample which shows that we may not expect continuity when one deals with infinite horizon without hypothesis (H_L).

4.1.1. The discontinuous value function for infinite horizon control problem. In this section we suppose that h is lower semicontinuous and (H_l) holds true.

We have the following regularity result:

Lemma 13. *Suppose that (H_L), (8), (9), (10), (11) holds true and that h is lower semicontinuous. Then we have:*

i) (Existence of Optimal control) There exists an optimal trajectory starting from each point $(t_0, y_0) \in [0, \infty) \times \mathbb{R}^{N+1}$ i.e. there exists $\bar{y}(\cdot) \in S_F(t_0, y_0)$ such that

$$W(t_0, y_0) = \inf_{\tau \in [t_0, \infty)} h(\bar{y}(\tau; t_0, y_0, \bar{u}(\cdot))) \text{ for all } (t_0, y_0) \in [0, \infty) \times \mathbb{R}^{N+1}$$

ii) W is lower semicontinuous.

Proof. The proof is omitted because it is similar with the proof of Lemma 5. We must underline that hypothesis (10) gives the following estimation :

$$M + z_0 \geq W(t_0, y_0) \geq \inf_{u \in \mathcal{U}(t_0)} \inf_{\tau \in [t_0, \infty)} \{g\left(y(\tau; t_0, y_0, u(\cdot))\right) +$$
$$+ z_0 + \int_{t_0}^{\tau} L\left(x(s; t_0, x_0, u(\cdot)), u(s)\right) ds\} \geq$$
$$\inf_{\tau \in [t_0, \infty)} \{-M + z_0 + \int_{t_0}^{\tau} L_0 ds\} \geq \inf_{\tau \in [t_0, \infty)} \{-M + z_0 + L_0(\tau - \tau_0)\}$$

and in fact is sufficient to consider the infimum only in the case when $\tau \in [t_0, t_0 + \frac{2M}{L_0}]$ and because of this fact we can adapt the proof Lemma 5 to the infinite horizon case.

We note that the above estimation gives the following:

$$\lim_{T \to \infty} W^T(t, y) = W(t, y) \text{ for all } (t, y) \in [0, \infty) \times \mathbb{R}^{N+1}.$$

The main result of this section is:

Proposition 14. *Suppose that (H_L), (8), (9), (10), (11), (13) hold true. Then the value function W is characterized by*

$$W(t,y) = \inf \left\{ \begin{array}{c} r \in \mathbb{R} | (t,0,y,r) \in \\ Viab_{\bar{\Phi}}([0,\infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}, [0,\infty) \times [0,\infty) \times Epi(h)) \end{array} \right\}$$

for all $(t,y) \in [0,\infty) \times \mathbb{R}^{N+1}$.

Proof. The proof is a direct consequence of the following relation

$$W(t_0,y_0) = \bar{W}(t_0,0,y_0) \text{ for all } (t_0,y_0) \in [0,\infty) \times \mathbb{R}^{N+1}.$$

where $\bar{W} : [0,\infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+1} \to \mathbb{R}$ and $Epi\bar{W}$ is viability kernel in $[0,\infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}$ with target $[0,\infty) \times [0,\infty) \times Epi(h)$ for the dynamics $(t,l,y,r) \to \bar{\Phi}(t,l,y,r) = (1,1,f(x,U),L(x,U),0)$:

$$Epi\bar{W} = Viab_{\bar{\Phi}}([0,\infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}, [0,\infty) \times [0,\infty) \times Epi(h)).$$

Consider $(t_0,y_0) \in [0,\infty) \times \mathbb{R}^{N+1}$. Obviously $(t_0,0,y_0,\bar{W}(t_0,0,y_0)) \in Epi\bar{W}$.

By the definition of the viability kernel there exists a solution $t \to (t_0+t,t,y(t),\bar{W}(t_0,0,y_0))$ starting to $(t_0,0,y_0,\bar{W}(t_0,0,y_0))$ for the dynamics $\bar{\Phi}$ and there exists a time $\tau \in [0, \frac{2M}{L_0}]$ such that

$$(t_0+\tau,\tau,y(\tau),\bar{W}(t_0,0,y_0)) \in [0,\infty) \times [0,\infty) \times Epi(h))$$

so

$$h(y(\tau)) \leq \bar{W}(t_0,0,y_0)$$

and by the very definition of the value function W we have

$$W(t_0,y_0) \leq \bar{W}(t_0,0,y_0).$$

For the reverse inequality we have that for each point $(t_0,y_0) \in [0,\infty) \times \mathbb{R}^{N+1}$, there exists $\bar{y}(\cdot) \in S_F(t_0,y_0)$ such that

$$W(t_0,y_0) = \inf_{\tau \in [t_0,\infty)} h(\bar{y}(\tau;t_0,y_0,\bar{u}(\cdot)))$$

$$= \inf_{\tau \in [t_0,t_0+\frac{2M}{L_0}]} h(\bar{y}(\tau;t_0,y_0,\bar{u}(\cdot))) = h(\bar{y}(\bar{\tau};t_0,y_0,\bar{u}(\cdot)))$$

with $\bar{\tau} \in [0, \frac{2M}{L_0}]$.
So there exists $t \to (t_0 + t, t, \bar{y}(t), W(t_0, y_0))$ starting to $(t_0, 0, y_0, W(t_0, y_0))$ for the dynamics $\bar{\bar{\Phi}}$ staying in $[0, \infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}$ until it reaches the target in a time $\bar{\tau} \in [0, \frac{2M}{L_0}]$:

$$(t_0 + \bar{\tau}, \bar{\tau}, \bar{y}(\bar{\tau}), W(t_0, y_0)) \in [0, \infty) \times [0, \infty) \times Epi(h))$$

By the definition of the viability kernel with target:

$$(t_0, 0, y_0, W(t_0, y_0)) \in Viab_{\bar{\bar{\Phi}}}([0, \infty) \times \\ \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}, [0, \infty) \times [0, \infty) \times Epi(h)).$$

so

$$(t_0, 0, y_0, W(t_0, y_0)) \in Epi\bar{W}$$

which gives the following inequality:

$$\bar{W}(t_0, 0, y_0) \leq W(t_0, y_0)$$

and the proof of is complete.

For all $(t, y) \in [0, \infty) \times \mathbb{R}^{N+1}$ we have

$$W(t, y) = \inf \left\{ \begin{array}{c} r \in \mathbb{R} \mid (t, 0, y, r) \in \\ Viab_{\bar{\bar{\Phi}}}([0, \infty) \times [0, \frac{2M}{L_0}] \times \mathbb{R}^{N+2}, [0, \infty) \times [0, \infty) \times Epi(h)) \end{array} \right\}$$

Remark 15. The above result allows us to get the value function using numerical methods in order to calculate the viability kernel (see [10]).

Proposition 16. *Suppose that (H_L), (8), (9), (10), (11), (13) hold true. Then the value function W is a lower semicontinuous supersolution to (22).*

Proof. First we will prove that the value is a l.s.c. viscosity solution. We will use a inf-convolution method. We define a sequence h_n

$$h_n(y) = \inf_{w \in \mathbb{R}^N} (h(w) + n \left\| w - y \right\|) \ ; \ n \in \mathbb{N} \tag{23}$$

The functions h_n are Lipschitz, $h_n(y) \leq h_{n+1}(y)$ and $\lim_{n \to \infty} h_n(y) = h(y)$ for every $y \in \mathbb{R}^{N+1}$. Using the results for the Lipschitz continuous case we

obtain that W_{h_n} is a non-decreasing sequence of Lipschitz viscosity solutions for (4) which limit is denoted by $U := \lim_{n\to\infty} W_{h_n}$.

From one hand $h_n \leq h$ yields $U \leq W_h$. From the other hand, U is a non-decreasing limit of l.s.c. solutions. Stability arguments (see for instance Theorem 4.1 in [3]), allows us to conclude that U is a lower semicontinuous viscosity solution to (4).

Now we will use the fact that h_n are Lipschitz and that there exists an optimal control and we have for all $(t_0, y_0) \in [0,\infty) \times \mathbb{R}^{N+1}$ i.e. there exists $\bar{y}_n(\cdot) \in S_F(t_0, y_0)$ such that

$$W_{h_n}(t_0, y_0) = h_n(\bar{y}_n(\tau_n; t_0, y_0, \bar{u}_n(\cdot)))$$

with $\tau_n \in [t_0, t_0 + \frac{2M}{L_0}]$.

Furthermore using compactness arguments there exists $\bar{y}(\cdot) \in S_F(t_0, y_0)$ which is the limit of $\bar{y}_n(\cdot)$ in $W^{1,1}[t_0, t_0 + \frac{2M}{L_0}]$. We can choose a subsequence τ_{n_k} of τ_n with $\tau_{n_k} \to \tau \in [t_0, t_0 + \frac{2M}{L_0}]$ and $||y_{n_k}(\tau_{n_k}) - \bar{y}(\tau)|| \leq \frac{1}{n_k^2}$. So we have the following estimation:

$$W_{h_{n_k}}(t_0, y_0) = h_{n_k}(\bar{y}_{n_k}(\tau_{n_k})) - h_{n_k}(\bar{y}(\tau)) + h_{n_k}(\bar{y}(\tau))$$

$$\leq n||y_{n_k}(\tau_{n_k}) - \bar{y}(\tau)|| + h_{n_k}(\bar{y}(\tau))$$

and passing to the limit we obtain that

$$U(t_0, y_0) = h(\bar{y}(\tau)) \geq W_h(t_0, y_0)$$

and the proof is complete.

Remark 17. The definition of supersolution for (22) can be interpreted using viability arguments. So if φ is an lower semicontinuous supersolution for (22) then for every $(t_0, y_0) \in (0, T) \times \mathbb{R}^{N+1}$, there exists $u \in \mathcal{U}(t_0)$ such that the function $t \to (t, y(t), \varphi(t_0, y_0))$ with $y(\cdot) := y(\cdot; t_0, y_0, u)$, solution for the dynamics given by Φ of Theorem 9, remains in the epigraph of φ until it (possibly) reaches the set $[0, T] \times Epi(h)$. Or equivalently :

$$\text{for every } t \in [0,\infty), \text{ if } \varphi(s, y(s))) < \ h(y(s)) \text{ for all } s \in [0, t], \text{ then:}$$

$$\varphi(t, y(t))) \leq \varphi(t_0, y_0).$$

If there exists $\tau \in [0,\infty)$, such that $\varphi(\tau, y(\tau))) = h(y(\tau))$ then by definition of the value function we get that $W \leq \varphi$.

If $\varphi(s, y(s))) < h(y(s))$ for all $s \in [0, \infty)$, then:

$$\varphi(t, y(t))) \leq \varphi(t_0, y_0) \text{ for all } t \in [0, \infty)$$

and in general, we can not obtain a comparison result between W and φ.

REFERENCES

1. Aubin, J.-P. – *Viability Theory*, Birkhäuser, 1992.
2. Aubin, J.-P. – *Viability kernels and capture basins of sets under differential inclusions,* SIAM J. Control Optimization, Vol. 40, No.3, pp. 853-881, 2001
3. Barles, G. – *Solutions de viscosité des équations de Hamilton-Jacobi,* Springer, Paris, 1994
4. Bardi, M. and Capuzzo–Dolcetta, I. – *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations,* Birkhäuser, Boston, 1997.
5. Barron, E.N. and Liu, W. – *Semicontinuous Solutions for Hamilton-Jacobi Equations and the L^∞-Control Problem,* Appl.Math.Optim,Vol. 34, pp. 325-360, 1996.
6. Barron, E.N. – *Differential Games with Maximum Cost,* Nonlinear Analysis, Theory, Methods & Applications, Vol. 14, No. 11, pp. 971-989, 1989.
7. Barron, E.N. and Jensen, R. – *Semicontinuous Viscosity Solutions of Hamilton-Jacobi Equations with Convex Hamiltonian,* Comm. PDE., vol. 15, pp. 1713–1742, 1990.
8. Barron, E.N. and Ishii, H. – *The Bellman equation for minimizing the maximum cost,* Nonlinear Analysis, vol.3, pp.1067-1090, 1989.
9. Cardaliaguet, P., Quincampoix, M. and Saint-Pierre, P. – *Optimal times for constrained nonlinear control problems without local controllability,* Appl. Math. Optimization, Vol. 36, No.1, pp. 21–42, 1997.
10. Cardaliaguet, P., Quincampoix, M. and Saint-Pierre, P.– *Set valued numerical analysis for optimal control and differential games,* in Stochastic and differential Games: Theory and Numerical Methods, Ann. Internat. Soc. Dynam. Games 4, Bardi M. , Raghavan T.E.S. and Parthasarathy T. , eds. , Birkhäuser, Boston, Boston, MA, pp. 177–274, 1999.
11. Crandall, M.G., Ishii, H. and Lions, P.L. – *User's guide to the viscosity solutions of Hamilton-Jacobi equations,* Trans. Amer. Math. Soc., Vol. 282, pp. 487-502, 1992
12. Di Marco, S.C. and Gonzales, R.L.V. – *Supersolutions and subsolutions techniques in a minmax optimal control problem with infinite horison,* Indian J. pure appl. Math., Vol. 29, pp. 1083-1098, 1998.

13. Evans, L.C. – *Partial Differential Equations*, Graduate studies in mathematics, Vol. 19, A.M.S., Providence, Rhode Island, 1998.

14. Frankowska, H. – *Lower Semicontinuous Solutions of the Bellman Equation,* SIAM J. Control Optim., Vol. 31, pp. 257–272, 1993.

15. Plaskacz, S. and Quincampoix, M. – *Value Function for Differential Games and Control Systems with Discontinuous Terminal Cost,* SIAM J. Control Optim. Vol. 39, No. 5, pp. 1485-1498, 2001.

16. Plaskacz, S. and Quincampoix, M. – *Discontinuous Mayer Problem Under State-Constraints,* Topological Methods in Nonlinear Analysis, Vol. 15, pp. 91-100, 2000.

17. Plaskacz, S. and Quincampoix, M. – *On Representation Formulas for Hamilton Jacobi Equations related to Calculus of Variation Problems,* Topological Methods in Nonlinear Analysis,2002, (to appear).

18. Quincampoix, M. and Veliov, V. – *Viability with a target: Theory and Applications,* in Applications of Mathematical Engineering, Cheshankov B. and Todorov M., eds. ,pp. 47-58, Heron Press, Sofia, 1998.

19. Rapaport, A. – *A Characterisation of barriers of Differential Games,* J.Optimisation Theory Appl. 97, No. 1, pp. 151-179, 1998.

20. Serea, O.-S. – *Discontinuous Differential Games and Control Systems with Supremum Cost,* Prépublication du Département de Mathématiques de L'Université de Bretagne Occidentale, Brest, 2001.

21. Subbotin, A.I. – *Generalized Solutions of First-Order PDEs The Dynamical Optimization Perspective,* Birkhäuser, Boston, 1995.

Received: 27.VI.2002

Laboratoire de Mathématiques
Unité CNRS, FRE-2218
Université de Bretagne Occidentale
6 avenue Victor Le Gorgeu
BP 809, 29285 Brest cedex
FRANCE
Marc-Quincampoix@univ-brest.fr
Oana-Silvia.Serea@univ-brest.fr