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EFFICIENT POINTS AND VECTOR OPTIMIZATION

BY

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It is well-known that efficient points are proved to be useful tools for the study of vectorial optimization problems. Between these, the Pareto efficient points enjoy a powerful study by looking to find some sufficient conditions for the set and for the ordering cone which can ensure the existence of these points. Also, the applications of these points are pursued in the theory of vectorial subdifferentiability and to duality theory. We mention here the papers of Borwein [1], Jahn [10], Tanino and Sawaragi [19], Luc [15], Malivert [5], [6], Isac and Postolică [8], Zălinescu [24], [25].

This paper intends to prepare an approach of these problems by using the proximal infimum points, which were introduced and used by Isac and Postolică [8] in the study of vectorial subdifferentiability for multifunctions. In fact, this notion is a generalization of the "weak infimum" defined by Gross and used by Tanino and Sawaragi in R^n (see [19]). The advantage of this type of efficient points consists in the fact that they are nonempty for a large class of sets, even if the Pareto minimum points set is empty. The main result of the paper, i.e. the theorem which provides sufficient conditions for a set to have proximal infimum, will be systematically used for developing a duality theory and for the study of the classical dual problems. This study is made by using the saddle points for a Lagrangian defined in this case with the proximal infimum points.

The paper presents the general properties of the efficient points which will be utilized, i.e. algebraic and topological properties, and introduces a new concept of domination which will be used in the duality theory with multifunction. In the end of the paper we present some sufficient conditions which ensure that the domination property is fulfilled, as well as its applications.

1. Notations and general properties for efficient points. In what follows, we will consider Z a vectorial space and $Z_+ \subset Z$ a nonempty, convex cone ($\lambda Z_+ \subset Z_+, \forall \lambda \in R_+,\ Z_+ + Z_+ = Z_+$) which is pointed ($Z_+ \cap -Z_+ = \{0\}$). For $x_1, x_2 \in Z$, we will write $x_2 \leq_{Z_+} x_1$ or $x_1 \geq_{Z_+} x_2$ if $x_1 - x_2 \in Z_+$, $x_2 <_{Z_+} x_1$ or $x_1 >_{Z_+} x_2$ if $x_1 - x_2 \in Z_+ \setminus \{0\}$, $x_1 \not\leq_{Z_+} x_2$ or $x_2 \not\geq_{Z_+} x_1$ if $x_2 - x_1 \notin Z_+$ and $x_1 \not> _{Z_+} x_2$ or $x_2 \not< _{Z_+} x_1$ if $x_1 - x_2 \notin Z_+ \setminus \{0\}$. If no confusion can happen, we will renounce to index the cone. We add to Z a smallest element denoted $-\infty$ and a biggest element denoted $+\infty$; we denote $\overline{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$. We make the following conventions: $\lambda \cdot (\pm\infty) = \pm\infty$, $x + (+\infty) = +\infty$ for all $\lambda > 0$ and $x \in \overline{Z}$.

We begin this section with the definitions of the different efficiencies which will be used in this paper.

Let $A \subset \overline{Z}, A \neq \emptyset, \varepsilon \in Z_+$.

Definition 1.1.

1. *We say that $p \in \overline{Z}$ is an $\varepsilon-$ infimum of A if there is no $a \in A$ such that $a <_{Z_+} p - \varepsilon$. The set of $\varepsilon-$infimum points of A will be denoted by ${}^{\varepsilon}INF\,(A \mid Z_+)$.*

2. *We say that $p \in \overline{Z}$ is an $\varepsilon-$minimum of A if p is $\varepsilon-$infimum of A and $p \in A$. The set of the $\varepsilon-$minimum points of A is denoted by ${}^{\varepsilon}MIN\,(A \mid Z_+)$.*

3. *We say that $p \in \overline{Z}$ is an $\varepsilon-$proximal infimum of A if p is $\varepsilon-$infimum of A and $\forall p' \in \overline{Z}, p' >_{Z_+} p$ there exists $a' \in A$ such that $a' <_{Z_+} p'$. The set of $\varepsilon-$proximal infimum points of A is denoted by ${}^{\varepsilon}INF_1(A \mid Z_+)$.*

If no confusion is possible, we will renounce to the cone index and we denote these sets simply by ${}^{\varepsilon}INF\,A$, ${}^{\varepsilon}MIN\,A$, ${}^{\varepsilon}INF_1 A$.
In a similar manner, for a set $A \subseteq \overline{Z}$, we define the sets ${}^{\varepsilon}SUP\,A$, ${}^{\varepsilon}MAX\,A$ and ${}^{\varepsilon}SUP_1 A$.
For $\varepsilon = 0$, we find the above notions within the work of Isac and Postolică

[8]. On the other hand the minimum point for $\varepsilon = 0$ is the Pareto minimum points well-known in vectorial optimization. We mention that we re-find the notion $SUP_1 A$ in the papers of DOLECKI and MALIVERT [5], [6] defined with the aid of the superior closure of A. The notion of $\varepsilon-$minimum is also used in [8], where we find a general notion of $H-$efficiency with H a subset of Z_+.

Definition 1.2. *[16] Let H be a non-void set of $Z_+\backslash\{0\}$. A point $p \in A$ is $H-$efficient point for A if $(p - H - Z_+ \setminus \{0\}) \cap A = \emptyset$. The set of H-efficient points is denoted by ${}^H Eff\, A$.*

Remark 1.1. Let $A \subset \overline{Z}$, $A \neq \emptyset$.
If $A = \{+\infty\}$, then

$$SUP\, A = MAX\, A = +\infty,\ INF_1 A = MIN\, A = +\infty,\ INF\, A = \overline{Z}.$$

If $A \neq \{+\infty\}$ but $+\infty \in A$, then

$$SUP\, A = MAX\, A = +\infty,\ INF\, A = INF\,(A \setminus \{+\infty\}).$$

If $A = \{-\infty\}$ then

$$SUP\, A = \overline{Z},\ INF\, A = MIN\, A = \{-\infty\}.$$

If $A \neq \{-\infty\}$ but $-\infty \in A$ then

$$INF\, A = MIN\, A = \{-\infty\},\ SUP\, A = SUP(A \setminus \{-\infty\}).$$

We can observe that for a nonempty set $A \subset \overline{Z}$, with $A \neq \{+\infty\}, \{-\infty\}$, $\{+\infty, -\infty\}$ the study of the efficient points is reduced to the study of efficient points for the set $A\setminus\{+\infty, -\infty\} \subset Z$. For this reason, in what follows we will consider $\emptyset \neq A \subset Z$.

Remark 1.2.

1. ${}^\varepsilon INF\, A = -\,{}^\varepsilon SUP(-A)$, ${}^\varepsilon INF_1\, A = -\,{}^\varepsilon SUP_1(-A)$,
$$ {}^\varepsilon MIN\, A = -\,{}^\varepsilon MAX(-A);$$

2. ${}^\varepsilon INF\, A = INF(A + \varepsilon) = INF\, A + \varepsilon = INF(A + Z_+ + \varepsilon)$
${}^\varepsilon SUP\, A = SUP(A - \varepsilon) = SUP\, A - \varepsilon = SUP(A - Z_+ - \varepsilon)$;

3. ${}^{\varepsilon}INF_1\, A = {}^{\varepsilon}INF_1\,(A + Z_+) = {}^{\varepsilon}INF_1\,(A + Z_+ \setminus \{0\})$, ${}^{\varepsilon}SUP_1 A = {}^{\varepsilon}SUP_1(A - Z_+) = {}^{\varepsilon}SUP_1(A - Z_+ \setminus \{0\})$;

4. $B, C \subset Z$, $B \neq \emptyset$, $B \subseteq C \Rightarrow {}^{\varepsilon}INF\, C \subseteq {}^{\varepsilon}INF\, B$; ${}^{\varepsilon}SUP\, C \subseteq {}^{\varepsilon}SUP\, B$;

5. $K_1 \subset K_2 \subset Z, K_i$ convex cones $\Rightarrow INF(A \mid K_1) \supseteq INF(A \mid K_2)$;

6. $\varepsilon_1 \leq_{Z_+} \varepsilon_2 \Rightarrow {}^{\varepsilon_1}INF\, A \subseteq {}^{\varepsilon_2}INF\, A$.

Proposition 1.1. *If $A \subset Z, A \neq \emptyset, \varepsilon \in Z_+$ then*

$$ {}^{\varepsilon}MIN\, A = A \cap {}^{\varepsilon}INF\, A = A \cap {}^{\varepsilon}INF_1\, A;$$

$$ {}^{\varepsilon}MAX\, A = A \cap {}^{\varepsilon}SUP\, A = A \cap {}^{\varepsilon}SUP_1\, A.$$

Proof. Obviously, $A \cap {}^{\varepsilon}INF_1\, A \subseteq A \cap {}^{\varepsilon}INF\, A$.
Now let $a \in A \cap {}^{\varepsilon}INF\, A$ and $p' >_{Z_+} a$. We will find $a' \in A$, that is $a' = a <_{Z_+} p'$ which implies that

$$A \cap {}^{\varepsilon}INF\, A \subseteq A \cap {}^{\varepsilon}INF_1\, A$$

and the first part is proved. The proof for the second part is similar. □

Proposition 1.2.

1. *If A is a nonempty subset of Z with $INF_1\, A \neq \emptyset, \{-\infty\}$ then ${}^{\varepsilon}MIN\, A \neq \emptyset$ for all $\varepsilon \in Z_+ \setminus \{0\}$.*

2. *If A is a nonempty subset of Z with $SUP_1 A \neq \emptyset, \{+\infty\}$ then ${}^{\varepsilon}MAX\, A \neq \emptyset$ for all $\varepsilon \in Z_+ \setminus \{0\}$.*

Proof. Let $\alpha \in INF_1\, A$ and $\varepsilon \in Z_+ \setminus \{0\}$. From definition, we will get $a \in A$ such that $\alpha + \varepsilon >_{Z_+} a$. Suppose that $a \notin {}^{\varepsilon}MIN\, A$. This implies that there exists $b \in A$ such that $b <_{Z_+} a - \varepsilon$ and finally $b <_{Z_+} \alpha$, contradiction. The second part follows similarly. □

Proposition 1.3. *Let $A, B \subset Z$, be two nonempty sets.*

1. *If $A \subseteq B \subseteq A + Z_+$ then ${}^{\varepsilon}INF\, A = {}^{\varepsilon}INF\, B$, ${}^{\varepsilon}INF_1 A = {}^{\varepsilon}INF_1 B$ and $MIN\, A = MIN\, B$.*

2. *If* $A \subseteq B \subseteq A - Z_+$ *then* ${}^{\varepsilon}SUP\, A = {}^{\varepsilon}SUP\, B$, ${}^{\varepsilon}SUP_1 A = {}^{\varepsilon}SUP_1 B$ *and* $MAX\ A = MAX\ B$.

In what follows we intend to present some properties for the superior and inferior closure of a set A defined in [5] by

$$A_i = \bigcap_{\varepsilon >_{Z_+} 0} (A - \varepsilon + Z_+), \qquad A_s = \bigcap_{\varepsilon >_{Z_+} 0} (A + \varepsilon - Z_+).$$

Proposition 1.4. *Let* $A, B \subset Z$ *be two nonempty sets and* $Z_+ \subset Z$ *a convex pointed cone. The following relations hold:*

1. $A_s = -(-A)_i$;
2. $A \subseteq B \Rightarrow A_i \subseteq B_i$, $A_s \subseteq B_s$;
3. $A \subseteq B \subseteq A + Z_+ \Rightarrow A_i = B_i$; $A \subseteq B \subseteq A - Z_+ \Rightarrow A_s = B_s$;
4. $A + Z_+ \subseteq A_i = A_i + Z_+ = (A_i)_i$, $A - Z_+ \subseteq A_s = A_s - Z_+ = (A_s)_s$;
5. $A + Z_+ \backslash \{0\} = A_i + Z_+ \backslash \{0\}$, $A - Z_+ \backslash \{0\} = A_s - Z_+ \backslash \{0\}$;
   particularly ${}^{\varepsilon}INF\, A = {}^{\varepsilon}INF\, A_i$ *and* ${}^{\varepsilon}SUP\, A = {}^{\varepsilon}SUP\, A_s$;
6. $({}^{\varepsilon}INF\, A)_s = {}^{\varepsilon}INF\, A$, $({}^{\varepsilon}SUP\, A)_i = {}^{\varepsilon}SUP\, A$;
7. *If* ${}^{\varepsilon}INF_1\, A \neq \{-\infty\}$, *then* ${}^{\varepsilon}INF_1\, A = A_i \cap {}^{\varepsilon}INF\, A$,
   If ${}^{\varepsilon}SUP_1\, A \neq \{+\infty\}$ *then* ${}^{\varepsilon}SUP_1\, A = A_s \cap {}^{\varepsilon}SUP\, A$.

Proof. The assertions 1. and 2. are obvious by definition. The third assertion follows from the relation

$$A - \varepsilon + Z_+ \subseteq B - \varepsilon + Z_+ \subseteq A + Z_+ - \varepsilon + Z_+ = A - \varepsilon + Z_+.$$

4. For all $\varepsilon > 0$ we have $A + Z_+ \subset A - \varepsilon + Z_+$ and thus $A + Z_+ \subseteq A_i$ which implies $A_i \subseteq A_i + Z_+ \subseteq (A_i)_i$. Let $x \in (A_i)_i$ and $\varepsilon > 0$. We will have

$$x \in A_i - \frac{\varepsilon}{2} + Z_+ \subseteq A - \frac{\varepsilon}{2} + Z_+ - \frac{\varepsilon}{2} + Z_+ = A - \varepsilon + Z_+.$$

So, $x \in A_i$ and finally $(A_i)_i = A_i$.

5. Let $x \in A_i$ and $\varepsilon > 0$. Then $x \in A - \frac{\varepsilon}{2} + Z_+$ which implies that

$$x + \varepsilon \in A + \frac{\varepsilon}{2} + Z_+ \subseteq A + Z_+ \backslash \{0\}.$$

We obtain that $A_i + Z_+ \backslash \{0\} \subseteq A + Z_+ \backslash \{0\}$. Since the converse inclusion follows from 4., the relations are proved. The facts that ${}^{\varepsilon}INF\, A = {}^{\varepsilon}INF\, A_i$ and ${}^{\varepsilon}SUP\, A = {}^{\varepsilon}SUP\, A_i$ follow from the following relation

$${}^{\varepsilon}INF\, A = Z \backslash (A + Z_+ \backslash \{0\} + \varepsilon)$$

and from the equalities which have been proved above.

6. Since ${}^{\varepsilon}INF\, A = INF(A + \varepsilon)$ and ${}^{\varepsilon}SUP\, A = SUP(A + \varepsilon)$ it will be sufficient to prove the relations for $\varepsilon = 0$. From 3) we get $SUP\, A \subseteq (SUP\, A)_i$. Suppose that there exists $x \in (SUP\, A)_i \backslash SUP\, A$. This implies that there exists $a \in A$ such that $x = a - \mu, \mu > 0$. Since

$$x \in (SUP\, A)_i, x = a - \mu \in SUP\, A - \frac{\mu}{2} + Z_+$$

we get that $a - \frac{\mu}{2} - k \in SUP\, A$ for some $k \in Z_+$. Obviously, this is contradicting the definition of $SUP\, A$.

7. Let $p \in A_i$ and $p < p'$; thus $p' = p + \mu$ for some $\mu > 0$. Since $p \in A_i$ we have that $p \in A - \frac{\mu}{2} + Z_+$ and thus $p' \in A + \frac{\mu}{2} + Z_+ \subset A + Z_+ \backslash \{0\}$.

By this way, there exists $a \in A$ such that $a < p'$. Now, let $p \in Z$ with the property that for all $p' \in Z, p' > p$ there is $a \in A$ such that $a < p'$. Let $\mu > 0$; since $p' = p + \mu > p$ there exists $a \in A$ such that $a < p + \mu$, which leads to $p \in A - \mu + Z_+$. Thus $p \in A_i$ and so the equality $INF_1 A = A_i \cap INF\, A$ is obvious. □

In the following propositions we present some formulas for A_i and A_s.

Proposition 1.5. *Let $A \subset Z$ be a nonempty set such that $INF_1 A \neq \{-\infty\}, SUP_1 A \neq \{+\infty\}$. Then:*

$$A_i = SUP(INF\, A) = (A + Z_+) \cup INF_1 A = (A + Z_+ \backslash \{0\}) \cup INF_1 A,$$

$$A_s = INF(SUP\, A) = (A - Z_+) \cup SUP_1 A = (A - Z_+ \backslash \{0\}) \cup SUP_1 A.$$

Proof. Obviously, the following inclusion does hold:

$$(A + Z_+ \backslash \{0\}) \cup INF_1 A \subseteq (A + Z_+) \cup INF_1 A. \tag{1}$$

Since $A + Z_+ \subseteq A_i$ and $INF_1 A \subseteq A_i$ we have that

$$(A + Z_+) \cup INF_1 A \subseteq A_i. \tag{2}$$

Let $x \in A_i$ and suppose that $x \notin SUP(INF\, A)$; thus $x \in INF\, A - Z_+\backslash\{0\}$ and so, there exists $\varepsilon > 0$ such that $x + \varepsilon \in INF\, A$. Since $x \in A_i$, we have $x \in A - \frac{\varepsilon}{2} + Z_+$ which implies that $x + \varepsilon \in A + \frac{\varepsilon}{2} + Z_+ \subseteq A + Z_+\backslash\{0\}$ contradiction with $x + \varepsilon \in INF\, A$. Thus

$$A_i \subseteq SUP(INF\, A). \tag{3}$$

Now, let $x \in SUP(INF\, A)$ and suppose that $x \notin A + Z_+\backslash\{0\}$. This implies that $x \in INF\, A$. By the contrary, suppose that $x \notin A_i$. Then there exists $\varepsilon > 0$ such that $x \notin A - \varepsilon + Z_+$ which implies that $x + \varepsilon \notin A + Z_+ \supset A + Z_+\backslash\{0\}$. Thus we have $x+\varepsilon \in INF\, A$ and $x = (x+\varepsilon)-\varepsilon \in INF\, A - Z_+\backslash\{0\}$ which involves that $x \notin SUP(INF\, A)$, false. Now, $x \in A_i \cap INF\, A$ and so,

$$SUP(INF\, A) \subseteq (A + Z_+\backslash\{0\}) \cup INF_1 A = (A + Z_+\backslash\{0\}) \cup (A_i \cap INF\, A). \tag{4}$$

From the relations (1)-(4) we get the expression of A_i. For A_s we obtain the result in the same manner. □

Definition 1.3. We will say that a set A is inferior closed (in respect with Z_+) if $A_i = A + Z_+$ and A is superior closed (in rapport with Z_+) if $A_s = A - Z_+$.

Relations 4. and 6. from the Proposition 1.5 imply that $A_i, SUP\, A$ are inferior closed and $A_s, INF\, A$ are superior closed (in respect with Z_+).

Proposition 1.6. *Let $A \subset Z$ be a nonempty set. If A is inferior closed and $INF_1 A \neq \{-\infty\}$ then $INF_1 A = MIN\, A$ and if A is superior closed and $SUP_1 A \neq \{+\infty\}$ then $SUP_1 A = MAX\, A$, respectively.*
Particularly,

$$INF_1 A_i = MIN\, A_i, \qquad SUP_1 A_s = MAX\, A_s,$$

$$INF_1(SUP\, A) = MIN(SUP\, A), \qquad SUP_1(INF\, A) = MAX(INF\, A).$$

Proof. Suppose that A is inferior closed, i.e. $A_i = A + Z_+$. Then we have the following relations:

$$INF_1 A = A_i \cap INF\, A = (A + Z_+) \cap INF\, A =$$
$$= (A + Z_+) \backslash (A + Z_+ \backslash \{0\}) = MIN\, A.$$

If A is superior closed, we obtain the relation by the same manner. If we apply these formulas for $A_i, SUP\, A, A_s, INF\, A$ we obtain the other relations.

Proposition 1.7. *Let $A \subset Z$ be a nonempty set and suppose $INF_1 A \neq \{-\infty\}, SUP_1 A \neq \{+\infty\}$. In this hypothesis,*

$$INF_1 A = MIN\, A_i = MAX(INF\, A) = SUP_1 INF\, A;$$

$$SUP_1 A = MAX\, A_s = MIN(SUP\, A) = INF_1 SUP\, A.$$

Proof. By Proposition 1.4 we have

$$INF_1 A = A_i \cap INF\, A = A_i \cap INF\, A_i = MIN\, A_i.$$

Let $x \in INF_1 A = A_i \cap INF\, A$. If we suppose that $x \notin MAX(INF\, A)$ then there exists $x' \in INF\, A, x' >_{Z_+} x$. Let $\varepsilon = x' - x$; since $x \in A_i$ it follows that $x \in A - \frac{\varepsilon}{2} + Z_+$. Thus we obtain that $x' \in A + \frac{\varepsilon}{2} + Z_+ \subset A + Z_+ \backslash \{0\} = X \backslash INF\, A$, contradiction.

For the converse implication, let $x \in MAX(INF\, A)$ and suppose that $x \notin A_i$. We will find $\varepsilon >_{Z_+} 0$ such that $x \notin A - \varepsilon + Z_+$. This implies that $x + \varepsilon \notin A + Z_+ \backslash \{0\}$ and thus $x + \varepsilon \in INF\, A$ which contradicts that $x \in MAX(INF\, A)$. □

The last result is proved also in [6].

2. New types of domination properties. In what follows, we intend to present some properties for the sets of iterated efficient points using some new types of domination properties which will be introduced here. In the conventional approach, one says that a nonempty set $A \subset Z$ has the domination property if:

$$A \subseteq MIN(A \mid Z_+) + Z_+. \qquad (DP)$$

We will say that A has (DP') if

$$A \subseteq MAX(A \mid Z_+) - Z_+, \qquad (DP')$$

We present now the new types of dominations.

Definition 2.1. For $A \subseteq Z$, $\delta \in Z_+$ we say that A has the property:

1. (P^δ) if ${}^\delta INF_1 A \neq \emptyset, \{-\infty\}$ and $A \subseteq {}^\delta INF_1 A + Z_+$;
2. (P'^δ) if ${}^\delta SUP_1 A \neq \emptyset, \{+\infty\}$ and $A \subseteq {}^\delta SUP_1 A - Z_+$;
3. (P_1^δ) if ${}^\delta INF_1 A \neq \emptyset$ and $INF\, A \subseteq ({}^\delta INF_1 A - Z_+) \cup \{-\infty\}$;
4. $(P_1'^\delta)$ if ${}^\delta SUP_1 A \neq \emptyset$ and $SUP\, A \subseteq ({}^\delta SUP_1 A + Z_+) \cup \{+\infty\}$.

For $\delta = 0$ we denote these properties simply by P, P', P_1, P_1'.

Remark 2.1. The property (P) is equivalent with the domination property for A_i.

Indeed, if the set A_i has the domination property, then $MIN\ (A_i|Z_+)$ is nonempty and $A_i \subseteq MIN\ (A_i|Z_+) + Z_+$. But $MIN\ A_i = INF_1 A$ and thus $INF_1 A \neq \emptyset, \{-\infty\}$, so $MIN\ A_i = INF_1 A$ and from Proposition 1.5

$$A_i = (A + Z_+) \cup INF_1 A \subseteq INF_1 A + Z_+.$$

Particularly, we get $A \subseteq INF_1 A + Z_+$ and thus, (P) is fully filed.

Now, if (P) is satisfied, then $INF_1 A \neq \emptyset, \{-\infty\}$ and $A \subseteq INF_1 A + Z_+$. Using the definition of $INF_1 A$, we have $A + Z_+ \setminus \{0\} = INF_1 A + Z_+ \setminus \{0\}$. Using again Proposition 1.5, we obtain that

$$A_i = (A + Z_+ \setminus \{0\}) \cup INF_1 A = (INF_1 A + Z_+ \setminus \{0\}) \cup INF_1 A =$$

$$= INF_1 A + Z_+ = MIN\ A_i + Z_+$$

and the domination property does hold for A_i.

In fact, we get that if the domination property does hold for A_i, then it happens with equality, i.e. $A_i = MIN\ A_i + Z_+$.

Proposition 2.1. *Let $A \subseteq Z$ be a nonempty set and $\varepsilon, \delta \in Z_+$. We have*

1. ${}^{\varepsilon}SUP^{\delta}SUP_1A \supseteq {}^{\varepsilon}SUP_1A$;

2. *If A has the property (P'^{δ}) then* ${}^{\varepsilon}SUP_1{}^{\delta}SUP_1A = {}^{\varepsilon}SUP_1A$.

3. ${}^{\varepsilon}INF^{\delta}INF_1A \supseteq {}^{\varepsilon}INF_1A$;

4. *If A has property (P^{δ}) then* ${}^{\varepsilon}INF_1{}^{\delta}INF_1A = {}^{\varepsilon}INF_1A$.

Proof. Using the notations from Proposition 1.4 for $\delta \in Z_+$, we have that: $A_1 = {}^{\delta}INF_1A \subseteq A_i$ which implies $(A_1)_i \subseteq (A_i)_i = A_i$ and thus

$$^{\varepsilon}INF\,A_1 \supseteq {}^{\varepsilon}INF\,A_i = {}^{\varepsilon}INF\,A \supseteq {}^{\varepsilon}INF_1A$$

If $A \subseteq A_1 + Z_+$ then $A_i \subseteq (A_1 + Z_+)_i = (A_1)_i$ and so, $A_i = (A_1)_i$. Finally,

$$^{\varepsilon}INF_1A_1 = {}^{\varepsilon}INF\,A_1 \cap (A_1)_i = {}^{\varepsilon}INF\,A \cap A_i = {}^{\varepsilon}INF_1A.$$

The proof for 1. and 2. is similar. □

For $\delta = 0$ we re-find the Proposition 1.3 from [6].

Proposition 2.2. *If $A \subseteq Z$ is a nonempty set, we have the following inclusions:*

1.
$$^{\varepsilon}INF_1A \subseteq {}^{\varepsilon}SUP_1{}^{\varepsilon}INF_1A \subseteq {}^{\varepsilon}INF\,A.$$
If A has (P_1), then for $\varepsilon = 0$, we have equality in the first inclusion.

2.
$$^{\varepsilon}SUP_1A \subseteq {}^{\varepsilon}INF_1{}^{\varepsilon}SUP_1A \subseteq {}^{\varepsilon}SUP\,A.$$
If A has (P_1'), then for $\varepsilon = 0$ we have equality in the first inclusion.

Proof. We shall prove that we have the following inclusion:

$$^{\varepsilon}INF_1A \subseteq {}^{\varepsilon}SUP\,{}^{\varepsilon}INF_1A.$$

Suppose by the contrary that there exists $\alpha \in {}^{\varepsilon}INF_1A$ such that we get $\beta \in {}^{\varepsilon}INF_1A$ with $\beta < \alpha - \varepsilon$. Since $\beta \in {}^{\varepsilon}INF_1A$, the previous relation implies that there exists an element $a \in A, a < \alpha - \varepsilon$, which is in contradiction with $\alpha \in {}^{\varepsilon}INF_1A$.

Now, since ${}^{\varepsilon}INF_1A \subseteq ({}^{\varepsilon}INF_1A)_s$, it follows,

$$ {}^{\varepsilon}INF_1A \subseteq ({}^{\varepsilon}INF_1A)_s \cap {}^{\varepsilon}SUP\,{}^{\varepsilon}INF_1A = {}^{\varepsilon}SUP_1\,{}^{\varepsilon}INF_1A. $$

For the other inclusion, since ${}^{\varepsilon}INF_1A \subseteq {}^{\varepsilon}INF\,A$, it follows that

$$ ({}^{\varepsilon}INF_1A)_s \subseteq ({}^{\varepsilon}INF\,A)_s = {}^{\varepsilon}INF\,A. $$

Thus,

$$ \begin{aligned} ({}^{\varepsilon}INF_1A)_s \cap {}^{\varepsilon}SUP\,{}^{\varepsilon}INF_1A &= {}^{\varepsilon}SUP_1\,{}^{\varepsilon}INF_1A \\ &= {}^{\varepsilon}INF\,A \cap {}^{\varepsilon}SUP\,{}^{\varepsilon}INF_1A \subseteq {}^{\varepsilon}INF\,A \end{aligned} $$

and the first assertion is proved. For $\varepsilon = 0$, if A has property (P_1), then $INF_1A \neq \{-\infty\}$ and

$$ INF_1A \subseteq SUP_1\,INF_1A \subseteq INF\,A \setminus \{-\infty\} \subseteq INF_1A - Z_+. $$

Thus, it is necessary that $INF_1A = SUP_1INF_1A$.
For the second part, the proof is similar. □

Proposition 2.3. *Let $A \subset Z$ be a nonempty set.*

1. *If A has property (P') then $SUP\,SUP_1A = SUP\,A$.*
2. *If A has property (P), then $INF\,INF_1A = INF\,A$.*

Proof. Since A has property (P'), we have

$$ SUP_1A - Z_+ \setminus \{0\} = A - Z_+ \setminus \{0\}. $$

From Remark 1.2 we deduce that

$$ SUP\,A = SUP\,(A - Z_+ \setminus \{0\}) = SUP\,(SUP_1A - Z_+ \setminus \{0\}) = SUP\,(SUP_1A). $$

In a similar way, we obtain the relation for the set of infimum points. □

Proposition 2.4.

1. *Let $A, B \subset Z$, $A, B \neq \emptyset$ with $A \subseteq B - Z_+$ and B has property (P'^{δ}), for $\delta \in Z_+$. Under this hypothesis, the following inclusion does hold:*

$$^{\delta}SUP_1A \subseteq \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} (^{\delta}SUP_1B - Z_+ + \varepsilon);$$

2. *Let $A, B \subset Z$, $A, B \neq \emptyset$ with $A \subseteq B + Z_+$ and B has property (P^{δ}), for $\delta \in Z_+$. Under this hypothesis, the following relation does hold:*

$$^{\delta}INF_1A \subseteq \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} (^{\delta}INF_1B - Z_+ + \varepsilon), \forall \delta \in Z_+.$$

Proof. Since B has the property (P'^{δ}) we have

$$B \subseteq B_1 - Z_+ \text{ where } B_1 = {}^{\delta}SUP_1B.$$

Thus,

$$A \subseteq B - Z_+ \subseteq B_1 - Z_+ - Z_+ = B_1 - Z_+.$$

Using this inclusions we get

$$A_s \subseteq (B_1 - Z_+)_s = (B_1)_s,$$

and so, $^{\delta}SUP_1A \subseteq A_s \subseteq (B_1)_s$.
In the same way we can prove the second part. □

3. Topological properties for efficient points. In what follows, we are interested to find some topological properties for the sets of efficient points.

Everywhere within this section we will consider (Z, τ) to be a locally convex space ordered by a convex pointed normal cone with nonempty interior.

We recall that a convex cone is normal if there is a basis of neighborhoods of the origin $\mathcal{V}_\tau(0)$ such that

$$(V - Z_+) \cap (V + Z_+) = V, \forall V \in \mathcal{V}_\tau(0).$$

We denote by Int A, cl A the interior, respectively the closure of the set A, by $C(A)$ the complementary set of A and let $K = \{0\} \cup \text{Int } Z_+$,

$$[a,b]_{Z_+} = (a + Z_+) \cap (b - Z_+)$$

and

$$(a,b)_{Z_+} = (a + Z_+ \setminus \{0\}) \cap (b - Z_+ \setminus \{0\}).$$

Based on these hypotheses,

$$U(a) = \{(a - \varepsilon, a + \varepsilon), \varepsilon \in \text{Int } Z_+\}$$

forms a fundamental system of neighborhoods for $a \in Z$.

We say that $V \subseteq \overline{Z}$ is a neighborhood for $+\infty$ if there exists $a \in Z$ such that $\{+\infty\} \cup (a + Z_+) \subseteq V$, respectively, $V \subseteq \overline{Z}$ is a neighborhood for $-\infty$ if there exists $a \in Z$ with $\{-\infty\} \cup (a - Z_+) \subseteq V$.

The sets of efficient points with respect to the cone K will be called weak efficient points and are denoted by $w^{\varepsilon}INF\, A$, $w^{\varepsilon}INF_1 A$, $w^{\varepsilon}MIN\, A$, $w^{\varepsilon}SUP\, A$,$w^{\varepsilon}SUP_1 A$, $w^{\varepsilon}MAX\, A$ and the domination properties for $\delta = 0$ will be denoted by (wP_1), (wP_1'), (wP), (wP'), (wDP), (wDP').

Proposition 3.1. *Let $A \subseteq Z$ be a nonempty set, $\varepsilon \in \text{Int } Z_+$. The following relations do hold:*

1. $w^{\varepsilon}INF\, A = w^{\varepsilon}INF(\text{cl } A) = \text{cl}(w^{\varepsilon}INF\, A)$.
2. $w^{\varepsilon}INF_1\, A = w^{\varepsilon}INF_1\, (\text{cl } A) = \text{cl}(w^{\varepsilon}INF_1\, A)$.

Proof. First we prove that if D is an open set, then for any set A we have the equality $A + D = \text{cl } A + D$.
For that, let $x = \overline{a} + d, \overline{a} \in cl\, A, d \in D$. We will find $(a_i)_{i \in I} \subset A, a_i \to \overline{a}$. Thus $x = a_i + (\overline{a} - a_i + d)$. Since $-a_i + \overline{a} + d \to d \in D = \text{Int } D$ it follows that there exists $i_0, \forall i > i_0 : \overline{a} - a_i + d \in D$ so $x \in A + D$.
Thus

$$\begin{aligned} wINF\, A &= Z\setminus(A + K\setminus\{0\}) \cup \{-\infty\} \\ &= Z\setminus(A + \text{Int } Z_+) \cup \{-\infty\} \\ &= Z\setminus(\text{cl } A + \text{Int } Z_+) \cup \{-\infty\} \\ &= wINF\, (\text{cl } A) \end{aligned}$$

and since $A + \text{Int } Z_+$ is open, it follows that $wINF\, A$ is closed.
Now,

$$\begin{aligned}\bigcap_{\varepsilon >_K 0}(A-\varepsilon+K) &= \bigcap_{\varepsilon >_K 0}(A-\varepsilon+K\backslash\{0\})\\ &= \bigcap_{\varepsilon >_K 0}(A-\varepsilon+\text{Int } Z_+)\\ &= \bigcap_{\varepsilon >_K 0}(\text{cl}\, A-\varepsilon+\text{Int } Z_+)\\ &= \bigcap_{\varepsilon >_K 0}(\text{cl}\, A-\varepsilon+\text{Int } Z_+\cup\{0\}).\end{aligned}$$

If $wINF_1A = \{-\infty\}$ then $w^{\varepsilon}INF\, A = \{-\infty\}$ and thus $w^{\varepsilon}INF_1A = wINF_1A = \{-\infty\}$ and the equality from 2. is obvious.
Now, let $wINF\, A \neq \{-\infty\}$ and thus $wINF_1A \neq \{-\infty\}$. We have

$$\begin{aligned}w^{\varepsilon}INF_1A &= (w^{\varepsilon}INF\, A)\bigcap\bigcap_{\varepsilon >_K 0}(A-\varepsilon+K)\\ &= (w^{\varepsilon}INF\,(\text{cl}\, A))\bigcap\bigcap_{\varepsilon >_K 0}(\text{cl}\, A-\varepsilon+K)\\ &= w^{\varepsilon}INF_1(\text{cl}\, A).\end{aligned}$$

We can prove that

$$\bigcap_{\varepsilon >_K 0}(A-\varepsilon+K\backslash\{0\}) = \text{cl}(A+Z_+). \tag{5}$$

Indeed, since $\{[-\varepsilon,\varepsilon]_{Z_+}, \varepsilon >_K 0\}$ is a fundamental system of neighborhoods for 0 and

$$A+K-\varepsilon = A+K+[-\varepsilon,\varepsilon]_{Z_+} = A+Z_+ +[-\varepsilon,\varepsilon]_{Z_+},$$

we get

$$\bigcap_{\varepsilon >_K 0}(A-\varepsilon+K\backslash\{0\}) = \text{cl}(A+Z_+) = \text{cl}(A+K).$$

Because $w^{\varepsilon}INF\, A = w\, INF(A+\varepsilon)$ is a closed set and we have just proved that the set
$\bigcap_{\varepsilon >_K 0}(A-\varepsilon+K\backslash\{0\})$ is closed, it follows that $w^{\varepsilon}INF_1A$ is closed. □

Similarly to the previous proposition we can prove:

Proposition 3.2. *Let $A \subseteq Z$ be a nonempty set, $\varepsilon \in \text{Int } Z_+$. The following relations do hold:*

1. $w^{\varepsilon}SUP\, A = w^{\varepsilon}SUP\,(cl\, A) = cl(w^{\varepsilon}SUP\, A)$.

2. $w^{\varepsilon}SUP_1\, A = w^{\varepsilon}SUP_1\,(cl\, A) = cl(w^{\varepsilon}SUP_1\, A)$.

Corollary 3.1. *Let $A \subset Z$ be a nonempty set with $wINF_1 A \neq +\infty$. We have that*

$$w\, INF_1 A = w\, MIN(\mathrm{cl}(A + Z_+))$$

$$w\, SUP_1 A = w\, MAX(\mathrm{cl}(A - Z_+))$$

Proof. From Proposition 1.4 and the previous proof (relation (5)) we have

$$w\, INF_1 A = w\, MIN\,(\bigcap_{\varepsilon >_K 0} (A - \varepsilon + K)) = w\, MIN(\mathrm{cl}(A + Z_+))$$

and

$$w\, SUP_1 A = w\, MAX\,(\bigcap_{\varepsilon >_K 0} (A + \varepsilon - K)) = w\, MAX(\mathrm{cl}(A - Z_+)).\square$$

Corollary 3.2. *Let $A \subset Z$ be a nonempty set. The following relations do hold:*

$$w\, SUP_1 A = Fr(A - Z_+);$$

$$w\, INF_1 A = Fr(A + Z_+).$$

(where $Fr\ B$ denotes the set of boundary points of B).

Proof. Let $\alpha \in w\, INF_1 A = w\, MIN(\mathrm{cl}(A + Z_+)) \subset \mathrm{cl}(A + Z_+)$. We shall prove that $\alpha \notin \mathrm{Int}\,(A + Z_+)$. If we suppose that $\alpha \in \mathrm{Int}\,(A + Z_+)$ it follows that there exists $\varepsilon \in \mathrm{Int}\, Z_+$ such that $(\alpha - \varepsilon, \alpha + \varepsilon)_{Z_+} \subset A + Z_+$. Thus we get $a \in \alpha - \mathrm{Int}\, Z_+$, $a \in A$, false. It follows that $\alpha \in Fr(A + Z_+)$. Now, let $\alpha \in Fr(A + Z_+)$. Suppose that there exists $\beta \in \mathrm{cl}(A + Z_+)$ with $\beta <_K \alpha$. Thus, $\alpha \in \mathrm{cl}(A + Z_+) + \mathrm{Int}\, Z_+ = A + Z_+ + \mathrm{Int}\, Z_+ = A + \mathrm{Int}\, Z_+ = \mathrm{Int}\,(A + Z_+)$. But $\alpha \notin \mathrm{Int}\,(A + Z_+)$, and thus our supposition is false, so the equality is proved.

The other relations follow similarly. $\square$

Proposition 3.3. *Let $A \subseteq Z$ be a nonempty set and for $a, d \in Z$, let us denote*

$$B_{a,d} = [d, a]_K \cap wINF\, A$$

$$B'_{a,d} = [a, d]_K \cap wSUP\, A.$$

A has the property (wP) if and only if $\forall a \in A, \exists d <_K a$ such that $wMAX\, B_{a,d} \neq \emptyset$.

A has the property (wP') if and only if $\forall a \in A, \exists d >_K a$ such that $wMIN\, B'_{a,d} \neq \emptyset$.

Proof. Suppose that

$$A \subseteq wINF_1 A + K \tag{6}$$

and we shall prove that

$$\forall a \in A, \exists d <_K a \,\text{such that}\; wMAX([d, a] \cap wINF\, A) \neq \emptyset. \tag{7}$$

Let $a \in A$. We shall find $a_1 \in wINF_1 A = wMAX\,(wINF\, A)\,, y \in K$ with $a = a_1 + y$. If $y = 0$, let d be such that $d <_K a$, chosen arbitrarily. Then $[d, a]_K \cap wINF\, A = [d, a]_K$ and thus $wMAX([d, a]_K \cap wINF\, A) = \{a\}$.
If $y \neq 0$, let $d = a_1$. In this case, since $a_1 \in wMAX(wINF\, A)$ we get

$$[d, a]_K \cap wINF\, A = \{a_1\}.$$

We obtain thus $wMAX\, B_d \neq \emptyset$.

Let us prove now that (7) implies (6).
Let $a \in A$; for this $a, \exists d <_K a$ and $a' \in wMAX([d, a]_K \cap wINF\, A$.
If $a' = a$ then $a \in A \cap wINF\, A \subseteq wINF_1 A \subseteq wINF_1 A + K$. Now, let $a' <_K a$. and suppose that $a' \notin wINF_1 A$. We shall get $\varepsilon >_K 0$ such that $a' \notin A - \varepsilon + K$. Since K has nonempty interior we can find $\varepsilon \leq_K a - a'$. (in this case the space has the property that $\forall \varepsilon_1, \varepsilon_2 >_K 0, \exists \varepsilon_3 >_K 0$ with $\varepsilon_3 \leq_K \varepsilon_1, \varepsilon_2$). This implies that $a' + \frac{\varepsilon}{2} <_K a$ and $a' + \frac{\varepsilon}{2} \notin A - \frac{\varepsilon}{2} + K \supseteq A + K \backslash \{0\}$. Thus $a' + \frac{\varepsilon}{2} \in wINF\, A$ and $a >_K a' + \frac{\varepsilon}{2} >_K a'$. We deduce that $a' + \frac{\varepsilon}{2} \in B_{a,d}$, which is contradicting the fact that a' is a weak maximal element for $B_{a,d}$.

The proof of the second assertion is similar.

Remark 3.1. In the previous proposition, (6) $\Longrightarrow$ (7) holds for an arbitrarily convex pointed cone Z_+ which replaces K (and the sets of efficient

points are chosen with respect to this new cone) and (7) $\Longrightarrow$ (6), for a cone which satisfies the condition $\forall \varepsilon_1, \varepsilon_2 > 0, \exists \varepsilon_3 > 0$ such that $\varepsilon_3 \leq \varepsilon_1, \varepsilon_2$.

4. An existence result for efficient points. We recall here some notions and two important results for the theory of efficient points which will be used in what follows.

Definition 4.1.

1. Let $A \subset Z$ a nonempty set. We say that a point $p \in Z$ is strong supremum for A if $p \geq a$, $\forall a \in A$ and for all $p' \in Z$ with $p' \geq a$, $\forall a \in A$, we have $p' \geq p$.

2. The cone Z_+ is called Daniell if every decreasing net which has strong infimum is convergent to its infimum.

3. The cone Z_+ is well based if there exists a convex set B such that $0 \notin B$ and for all $z \in Z_+$ there is $\lambda > 0, b \in B$ such that $z = \lambda b$.

4. The cone Z_+ is locally compact if there exists a compact set A such that for all $z \in Z_+$ there is $\lambda > 0, a \in A$ such that $z = \lambda a$.

5. A set $A \in Z$ is Z_+-bounded if there is a topological bounded set $M \in Z$ such that $A \subset M + Z_+$ and it is Z_+-closed if $A + Z_+$ is a closed set.

Theorem 4.1. *[8]*

1. *If A is a nonempty bounded closed set from Z and Z_+ is a convex pointed closed and locally compact cone then $MIN\, A \neq \emptyset$ and (DP) hold;*

2. *If A is a nonempty bounded closed set from Z and Z_+ is a convex pointed closed well based by a complete set then $MIN\, A \neq \emptyset$ and (DP) hold;*

3. *If A is a Z_+-bounded and Z_+-closed set from Z and Z_+ is a convex pointed closed supernormal cone and Z is qvasicomplete, then $MIN\, A \neq \emptyset$ and (DP) hold.*

Theorem 4.2. *[1]*

1. *If A is a nonempty set from Z such that A has a closed lower bounded section and Z_+ is a convex closed Daniell cone then $MIN\, A \neq \emptyset$.*

2. *If A is a nonempty set from Z such that A is bounded and closed and Z_+ is a convex closed Daniell cone and any decreasing net from Z has strong infimum then $MIN\, A \neq \emptyset$.*

3. *If A has a compact section and Z_+ is a convex closed pointed cone, then we have that $MIN\, A \neq \emptyset$.*

We are able now to present the main result of the paper.

Theorem 4.3. *Let (Z, τ) be a locally convex space ordered by a convex pointed normal cone with nonempty interior Z_+ such that $\operatorname{cl} Z_+$ is Daniell. Then*

every nonempty set $A \subseteq Z$ with $wINF\, A \neq \{-\infty\}$ has property (wP) and

$$wINF\, A = (wINF_1 A - \operatorname{Int} Z_+ \cup \{0\}) \cup \{-\infty\}; \tag{8}$$

every nonempty set $A \subseteq Z$ with $wSUP\, A \neq \{-\infty\}$ has property (wP') and

$$wSUP\, A = (wSUP_1 A + \operatorname{Int} Z_+ \cup \{0\}) \cup \{+\infty\}. \tag{9}$$

Remark 3.2. Relation (8) says that $wINF\, A \setminus \{-\infty\}$ has the property (DP) and (9) says that the set $wSUP\, A \setminus \{+\infty\}$ has the property (DP$'$).

Proof. We denote $K = \operatorname{Int} Z_+ \cup \{0\}$ and we observe that $\operatorname{cl} Z_+ = \operatorname{cl} K$. Since $wSUP\, A \neq \{+\infty\}$ it follows that $wSUP_1 A \neq \{+\infty\}$. Let us consider $a \in A$ and $c \in wSUP\, A, c \neq +\infty$. Since $\operatorname{Int} Z_+ \neq \emptyset$, there exists $d \in Z$, $d >_K a$, $d >_K c$. The set $B = (a + \operatorname{cl} K) \cap wSUP\, A \neq \emptyset$, $(d \in B)$, is closed and $a \leq_{\operatorname{cl} K} b, \forall b \in B$. Thus from Theorem 4.2 we have that $MIN(B \mid \operatorname{cl} Z_+) \neq \emptyset$.
Now let $\alpha \in MIN(B \mid \operatorname{cl} Z_+)$. If $\alpha = a$, then $a \in wMAX\, A$, so

$$a \in wSUP_1 A - \operatorname{cl} Z_+.$$

If $\alpha \neq a$, let $C = (a + \operatorname{cl} Z_+) \cap (\alpha - \operatorname{cl} Z_+)$ which is nonempty $(a, \alpha \in C)$, closed and has the property that $C \neq \{\alpha\}$.
The set $(wSUP\, A) \cap C$ does not contain any element different by α because α belongs to $MIN(B \mid \operatorname{cl} Z_+)$.

Let $\varepsilon \in K\backslash\{0\}$ and $D = \{\lambda a + (1-\lambda)\alpha, \lambda \in [0,1]\} \subseteq C$. Since $(\alpha - \varepsilon, \alpha + \varepsilon)$ is a convex neighborhood for α we find $\mu \in D \cap (\alpha - \varepsilon, \alpha + \varepsilon), \mu \neq \alpha$. Thus there exists $\mu \in C, \mu \neq \alpha, \mu >_K \alpha - \varepsilon$ which implies that $\alpha - \varepsilon \notin wSUP\, A$ and so $\alpha \in wSUP_1 A$. We have

$$
\begin{aligned}
A &\subseteq wSUP_1 A - \operatorname{cl} Z_+ \\
&\subseteq \bigcap_{\varepsilon \in K\backslash\{0\}} (wSUP_1 A - K + \varepsilon) \\
&\subseteq \operatorname{cl}(wSUP_1 A - K)\backslash\{+\infty\} \\
&= wSUP_1 A \cup \operatorname{cl}(wSUP_1 A - K\backslash\{0\})\backslash\{+\infty\}.
\end{aligned} \tag{10}
$$

Obviously $A - K\backslash\{0\} \supseteq wSUP_1 A - K\backslash\{0\}$ and the first inclusion from the previous relation provides the equality, $A - K\backslash\{0\} = wSUP_1 A - K\backslash\{0\}$. We can show that the following relation does hold:

$$wSUP_1 A - K = \operatorname{cl}(A - K\backslash\{0\})\backslash\{+\infty\}$$

which implies $A \subseteq wSUP_1 A - K$.
Since $wSUP\, A \neq \{+\infty\}$ we have that $+\infty \notin wSUP_1 A$.

Let $\alpha \in wSUP_1 A - K$; then $\alpha \in wSUP_1 A$ or $\alpha \in wSUP_1 A - K\backslash\{0\}$.
If $\alpha \in wSUP_1 A - K\backslash\{0\}$ then $\alpha \in A - K\backslash\{0\} \subseteq \operatorname{cl}(A - K\backslash\{0\})$.
If $\alpha \in wSUP_1 A$, then for all $\varepsilon \in K\backslash\{0\}$ there exists $a \in A$ with $a >_K \alpha - \varepsilon$ and thus

$$\alpha \in \bigcap_{\varepsilon \in K\backslash\{0\}} (A - K\backslash\{0\} + \varepsilon) = \operatorname{cl}(A - K\backslash\{0\}).$$

We have now

$$wSUP_1 A - K \subseteq \operatorname{cl}(A - K\backslash\{0\})$$

and we consider $\alpha \in \operatorname{cl}(A - K\backslash\{0\})\backslash\{+\infty\}$.
If $\alpha \in A - K\backslash\{0\} = wSUP_1 A - K\backslash\{0\}$ then $\alpha \in wSUP_1 A - K$. If $\alpha \notin A - K\backslash\{0\}$ then $\alpha \in wSUP\, A$.
Let $\varepsilon \in K\backslash\{0\}$; there exists $a' \in A - K\backslash\{0\}, a' >_K \alpha - \varepsilon$ because $(\alpha - \varepsilon, \alpha + \varepsilon)$ is a neighborhood for α. We can find $a \in A, a >_K \alpha - \varepsilon$ which implies that $\alpha \in wSUP_1 A \subseteq wSUP_1 A - K$.

Finally we get the equality

$$wSUP_1A - K = \operatorname{cl}(A - K\backslash\{0\})\backslash\{+\infty\} \tag{11}$$

which together with (10) shows that

$$A \subseteq wSUP_1A - K. \tag{12}$$

In a similar way we find that A has property (wP).
Let $\mu \in wSUP\,A\backslash\{+\infty\}, \beta \in wSUP_1A \neq \emptyset$ and $\gamma <_K \mu, \gamma <_K \beta$. Since $\beta \in wSUP_1A$, $\gamma <_K \beta$, we obtain $\gamma \notin wSUP\,A$. The set

$$B = (\mu - \operatorname{cl} K) \cap wSUP\,A \cap (\gamma + \operatorname{cl} Z_+)$$

is nonempty, closed and $\gamma \leq b \leq \mu, \forall b \in B$ so it follows $MIN(B \mid \operatorname{cl} Z_+) \neq \emptyset$.
Let $\alpha \in MIN\,(B \mid \operatorname{cl} Z_+)\,; (\alpha \neq \gamma, \gamma \notin wSUP\,A)$. The set

$$C = (\alpha - \operatorname{cl} Z_+) \cap (\mu - \operatorname{cl} Z_+) \cap (\gamma + \operatorname{cl} Z_+) = (\alpha - \operatorname{cl} Z_+) \cap (\gamma + \operatorname{cl} Z_+)$$

does not contain any different element of η because $\alpha \in MIN(B \mid \operatorname{cl} Z_+)$. For $\varepsilon \in Z_+\backslash\{0\}$, $V = (\alpha - \varepsilon, \alpha + \varepsilon)$ is a convex neighborhood for α and thus $V \setminus \{\alpha\} \cap C \neq \emptyset$. We deduce that $\alpha - \varepsilon \notin wSUP\,A$ (indeed, if not $C \cap wSUP\,A \neq \emptyset$, which is false) and so $\alpha \in wSUP_1A$. In this way we obtain

$$\begin{aligned}
(wSUP\,A)\backslash\{+\infty\} &\subseteq wSUP_1A + \operatorname{cl} Z_+ \\
&\subseteq \bigcap_{\varepsilon \in K\backslash\{0\}} wSUP_1A + K - \varepsilon \\
&\subseteq \operatorname{cl}(wSUP_1A + K) \setminus \{+\infty\} \\
&\subseteq \bigcap_{\varepsilon \in K\backslash\{0\}} (wSUP\,A - \varepsilon) \setminus \{+\infty\} \\
&= \operatorname{cl}(wSUP\,A + K) \setminus \{+\infty\} \\
&= (wSUP\,A) \setminus \{+\infty\}.
\end{aligned}$$

Thus $wSUP\,A = \operatorname{cl}(wSUP_1A + K) \cup \{+\infty\}$. The relation similar to (11)

$$wINF_1A + K = \operatorname{cl}(A + K \setminus \{0\}) \setminus \{-\infty\} \tag{R}$$

will provide

$$\mathrm{cl}\,(wSUP_1A + K)\backslash\{+\infty\} = wSUP_1A \bigcup \mathrm{cl}\,(wSUP_1A + K\backslash\{0\})\backslash\{+\infty\} =$$
$$= wSUP_1A \bigcup (wINF_1 wSUP_1A + K).$$

Let $\alpha \in wINF_1 wSUP_1A \subseteq wSUP\,A$. If there exists $\varepsilon \in K\backslash\{0\}$ such that $\alpha - \varepsilon \in wSUP\,A \subseteq wSUP_1A + K - \varepsilon$ we obtain that $\alpha \in wSUP_1A + K$. But $\alpha \in wINF(wSUP_1A)$ which implies that $\alpha \in wSUP_1A$. Thus

$$\mathrm{cl}\,(wSUP_1A + K) \subseteq (wSUP_1A + K) \cup \{+\infty\}.$$

Finally we get

$$\mathrm{cl}\,(wSUP_1A + K)\backslash\{+\infty\} = wSUP_1A + K$$

and

$$wSUP\,A = (wSUP_1A + K) \cup \{+\infty\}.$$

In a similar way we obtain

$$wINF\,A = (wINF_1A - K) \cup \{-\infty\}.$$

Corollary 4.1. *Under the same hypothesis like in Theorem 4.3, if A has $wSUP\,A \neq \{+\infty\}$, then*

$$(wSUP_1A + \mathrm{Int}\ Z_+) = C[\mathrm{cl}(A - \mathrm{Int}\ Z_+ \cup \{0\})].$$

Proof. From the previous theorem we have that a set A which satisfies $SUP\,A \neq \{+\infty\}$ has the property that

$$wSUP\,A = (wSUP_1A + \mathrm{Int}\ Z_+ \cup \{0\}) \cup \{+\infty\}.$$

Since $wSUP\,A = C(A - \mathrm{Int}\ Z_+ \cup \{0\})$ and $wSUP_1A = Fr(A - \mathrm{Int}\ Z_+ \cup \{0\})$ it follows that

$$\begin{aligned} wSUP_1A + \mathrm{Int}\ Z_+ &= C(A - \mathrm{Int}\ Z_+ \cup \{0\}) \cap C(wSUP_1A) \\ &= C((A - \mathrm{Int}\ Z_+ \cup \{0\}) \cup wSUP_1A) \\ &= C((A - \mathrm{Int}\ Z_+ \cup \{0\}) \cup Fr(A - \mathrm{Int}\ Z_+ \cup \{0\})) \\ &= C(\mathrm{cl}(A - \mathrm{Int}\ Z_+ \cup \{0\}). \end{aligned}$$

Corollary 4.2. *Under the same hypothesis like in Theorem 4.3, a set A with the property that $wINF\, A \neq \{-\infty\}$ satisfies the relation*

$$(wINF_1 A - \text{Int } Z_+) = C[\text{cl}(A + \text{Int } Z_+ \cup \{0\})].$$

Corollary 4.3. *In the same hypothesis like in Theorem 4.3, a set A with the property that $wINF\, A \neq \{-\infty\}$ and $A + \text{Int } Z_+ \cup \{0\}$ is closed has $wMIN\, A \neq \emptyset$ and (wDP) holds.*

Proof. Under the specified hypothesis, the theorem ensures that $wINF_1 A \neq \emptyset$ and since $wINF_1 A = wMIN(\text{cl}(A + \text{Int } Z_+ \cup \{0\}))$, the conclusion follows. □

Corollary 4.4. *In the same hypothesis like in Theorem 4.3, a set A which has the properties that $wSUP\, A \neq \{+\infty\}$ and $A - \text{Int } Z_+ \cup \{0\}$ is closed, has $wMAX\, A \neq \emptyset$ and (wDP') holds.*

Corollary 4.5. *Under the same hypothesis like in Theorem 4.5, if $A \subset Z$ is a nonempty set with the property that $A + \text{Int } Z_+ \cup \{0\}$ is closed then $A + \text{Int } Z_+ \cup \{0\}$ is convex if and only if $INF_1 A + \text{Int } Z_+ \cup \{0\}$ is convex.*

Proof. First we will prove that if $A + \text{Int } Z_+ \cup \{0\}$ is convex, then $A + \text{Int } Z_+$ is convex. Let $a, a' \in A, k, k' \in \text{Int } Z_+, \lambda \in (0,1)$. We have $\lambda(a+k) + (1-\lambda)(a'+k') =$
$= \lambda(a+k) + (1-\lambda)a' + (1-\lambda)k' \in (A + \text{Int } Z_+ \cup \{0\} + \text{Int } Z_+) = A + \text{Int } Z_+$ and thus $A + \text{Int } Z_+$ is convex. From the previous theorem we have that

$$A + \text{Int } Z_+ = INF_1 A + \text{Int } Z_+$$

so $INF_1 A + \text{Int } Z_+$ is convex. We shall prove now that $INF_1 A + \text{Int } Z_+ \cup \{0\}$ is convex. Let $\alpha, \alpha' \in INF_1 A, k, k' \in \text{Int } Z_+ \cup \{0\}, \lambda \in (0,1)$ and

$$\beta = \lambda(\alpha + k) + (1-\lambda)(\alpha' + k').$$

If $k, k' \neq 0$ then

$$\beta \in INF_1 + \text{Int } Z_+ \subset INF_1 + \text{Int } Z_+ \cup \{0\}.$$

If $k \neq 0, k' = 0$ then

$$\beta = \lambda(\alpha + k/2) + (1-\lambda)(\alpha' + \lambda k/2(1-\lambda)) \in$$
$$\in INF_1 A + \text{Int } Z_+ \subset INF_1 A + \text{Int } Z_+ \cup \{0\}.$$

If $k' \neq 0, k = 0$ we proceed in the same manner. Now let $k = 0, k' = 0$. We have

$$\beta + \varepsilon \in A + \text{Int } Z_+ = INF_1 + \text{Int } Z_+ \subset INF_1 + \text{Int } Z_+ \cup \{0\}.$$

Thus

$$\beta \in \bigcap_{\varepsilon > 0} INF_1 A + \text{Int } Z_+ \cup \{0\} - \varepsilon = \text{cl}(INF_1 A + \text{Int } Z_+ \cup \{0\}).$$

From the proof of the previous theorem it follows that

$$INF_1 A + \text{Int } Z_+ \cup \{0\} = \text{cl}(A + \text{Int } Z_+)$$

so $INF_1 A + \text{Int } Z_+ \cup \{0\}$ is a closed set and this implies that $\beta \in INF_1 A + \text{Int } Z_+ \cup \{0\}$.
Conversely, if $INF_1 A + \text{Int } Z_+ \cup \{0\}$ is convex it follows that $INF_1 A + \text{Int } Z_+$ is convex which implies that $A + \text{Int } Z_+$ is convex. In a similar way, it follows that in the hypothesis that $A + \text{Int } Z_+ \cup \{0\}$ is closed, $A + \text{Int } Z_+ \cup \{0\}$ is convex. □

Remark 4.1. The previous corollary remains true only under the supposition that $A \subset Z$, $A \neq \emptyset$, $INF_1 A \neq \{-\infty\}$ and $A, INF_1 A$ are inferior closed and (wP') does hold.

Corollary 4.6. *Under the hypothesis of the Theorem 4.3, if $A \subset Z$ is a nonempty set with the property that $A - \text{Int } Z_+ \cup \{0\}$ is closed then*

$A - \text{Int } Z_+ \cup \{0\}$ is convex if and only if $SUP_1 A - \text{Int } Z_+ \cup \{0\}$ is convex.

Remark 4.2. The previous corollary does hold if we suppose $A \subset Z$, $A \neq \emptyset$, $SUP_1 A \neq \{+\infty\}$ and $A, SUP_1 A$ are superior closed and (wP_1') does hold.

We present in what follows some sufficient conditions for a set to be superior or inferior closed.

In the remaining of this section we suppose that the cone Z_+ is convex, pointed, normal and $Z_+ = \text{Int } Z_+ \cup \{0\}$.

Proposition 4.1. *A nonempty set $A \subset Z$ is inferior closed if and only if $A - Z_+$ is a closed set.*

Proof. By definition, A is inferior closed if $\bigcap\limits_{\varepsilon \in Z_+ \setminus \{0\}} (A + \varepsilon - Z_+) = A - Z_+$ which is equivalent with $\bigcap\limits_{\varepsilon \in Z_+ \setminus \{0\}} (A + (-\varepsilon, \varepsilon) - Z_+) = A - Z_+$. Since Z_+ is a normal cone with nonempty interior, we have that $\bigcap\limits_{V \in \mathcal{V}(0)} (A + V - Z_+) = \text{cl}\,(A - Z_+)$ and we deduce the equality $A - Z_+ = \text{cl}\,(A - Z_+)$. □

Proposition 4.2. *A nonempty set $A \subset Z$, is superior closed if and only if $A + Z_+$ is a closed set.*

Proposition 4.3. *If* cl Z_+ *is a Daniell cone and A is a nonempty set from Z, such that A is a "SUP_1" set (i.e. there exists $B \subset Z$ such that $A = SUP_1 B$) then A is inferior closed.*

Proof. By this hypothesis, from Theorem 4.3 we have

$$B \subseteq SUP_1 B - Z_+.$$

Obviously, the following inclusion does hold

$$SUP_1 B - Z_+ \setminus \{0\} \subseteq B - Z_+ \setminus \{0\}$$

and thus

$$SUP_1 B - Z_+ \setminus \{0\} = B - Z_+ \setminus \{0\}.$$

We obtain that

$$SUP_1 B - Z_+ = A - Z_+ = (B - Z_+ \setminus \{0\}) \cup SUP_1 B.$$

From Corollary 3.2 we have that $SUP_1 B = Fr(B - Z_+ \setminus \{0\})$ and since $Z_+ \setminus \{0\} = \text{Int } Z_+$ we deduce

$$(B - Z_+ \setminus \{0\}) \cup SUP_1 B = cl\,(B - Z_+ \setminus \{0\}).$$

Thus $A - Z_+$ is closed and from Proposition 4.2 we deduce that A is inferior closed. □

Remark 4.3. If cl Z_+ is Daniell and A is a nonempty set such that A is an "INF_1'' set then A is superior closed.

5. Applications of efficient points to the theory of vector duality. In this section we intend to present some applications of the approximative efficient points for the study of a new type of problem which generalizes the problem $MINMAX$, that will be called $INFSUP$.

We will consider $(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$ some locally convex spaces and Z ordered by the convex pointed cone Z_+. Let Z^* denote the topological dual space of Z and we associate in Z^* a dual cone with Z_+ denoted by Z_+^*, that is

$$Z_+^* = \{z^* \in Z^* \mid \ z^*(z) \geq 0, \ \forall z \in Z_+\}.$$

The set of strictly positive functionals in Z_+^* is denoted by

$$Z_+^\# = \{z^* \in Z^* \mid \ z^*(z) > 0, \ \forall z \in Z_+ \setminus \{0\}\}.$$

Let $f : X \times Y \to Z$ be a mapping and denote

$$f(X, y) = \bigcup_{x \in X} f(x, y) \text{ and } f(x, Y) = \bigcup_{y \in Y} f(x, y).$$

A point (x_0, y_0) is called $\mu - \eta$ saddle point of f, where $\mu, \eta \in Z_+$ if

$$f(x_0, y_0) \in {}^{\mu}MAX\, f(x_0, Y) \cap {}^{\eta}MIN\, f(X, y_0).$$

If

$$f(x_0, y_0) \in w\,{}^{\mu}MAX\, f(x_0, Y) \cap w\,{}^{\eta}MIN\, f(X, y_0)$$

then (x_0, y_0) is a weak $\mu - \eta$ saddle point of f.

Proposition 5.1. *Let $\Phi \in Z_+^\#$. If (x_0, y_0) is a $\Phi(\mu) - \Phi(\eta)$ saddle point of $\Phi \circ f$ then (x_0, y_0) is a $\mu - \eta$ saddle point of f. If f is a convex map in each variable and (x_0, y_0) is a $\mu - \eta$ saddle point of f, then there exists $\Phi, \Phi' \in Z_+^*$ such that*

$$\Phi \circ f(x_0, y_0) \in {}^{\Phi(\mu)}MAX\, \Phi \circ f(x_0, Y)$$

and

$$\Phi' \circ f(x_0, y_0) \in {}^{\Phi'(\eta)}MAX\, \Phi' \circ f(X, y_0).$$

Proof. Let $\Phi \in Z_+^{\#}$; this implies that $\Phi(z) > 0, \forall z > 0$. If (x_0, y_0) is a $\Phi(\mu) - \Phi(\eta)$ saddle point, i.e.

$$\Phi \circ f(x_0, y_0) \geq \Phi \circ f(x_0, y) - \Phi(\mu), \forall y \in Y$$

then

$$f(x_0, y_0) \not< f(x_0, y) - \mu, \forall y \in Y$$

and thus $f(x_0, y_0) \in {}^{\mu}MAX\, f(x_0, Y)$. In a similar manner we obtain that $f(x_0, y_0) \in {}^{\eta}MIN\, f(X, y_0)$.
Conversely, if $f(x_0, y_0) \in {}^{\mu}MAX\, f(x_0, Y)$ we will find by the Hahn-Banach separation theorem an element $\Phi \in Z_+^*$ such that

$$\Phi \circ f(x_0, y_0) \in {}^{\Phi(\mu)}MAX\, \Phi \circ f(x_0, Y).$$

In a similar manner we get $\Phi' \in Z_+^*$ such that

$$\Phi' \circ f(x_0, y_0) \in {}^{\Phi'(\eta)}MIN\, \Phi \circ f(X, y_0)$$

and the proposition is proved. □

In [23], Tanaka provides some MINMAX theorems for vectorial functions. In what follows we present some results of the same type as Theorems 5.1, 5.2 from [23] for the problem $INFSUP$. Let us consider the problems:

$$wINF_1 \bigcup_{x \in X} wSUP_1 f(x, Y)$$

and

$$wSUP_1 \bigcup_{y \in Y} wINF_1 f(X, y).$$

In what follows, if not stated otherwise, we consider
(H): Z_+ ordered by a normal closed Daniell cone with nonempty interior.
This conditions ensure by Theorem 4.3 that all the sets $A \neq \emptyset$, with $wSUP\, A \neq \{+\infty\}$, have the properties that

$$wSUP_1 A \neq \emptyset, A \subseteq wSUP_1 A - \text{Int } Z_+ \cup \{0\}$$

and

$$wSUP\ A = (wSUP_1 A + \text{Int } Z_+ \cup \{0\}) \cup \{+\infty\}.$$

If for all $x \in X$ we have that $wSUP_1 f(x, Y) = \{+\infty\}$, then

$$wINF_1(\bigcup_{x \in X} wSUP_1 f(x, Y)) = \{+\infty\}.$$

If there exists $x \in X$ with $wSUP_1 f(x, Y) \neq \{+\infty\}$, let us denote

$$X_0 = \{x \in X, wSUP_1 f(x, Z) \neq \{+\infty\}\}.$$

Thus

$$wINF_1(\bigcup_{x \in X} wSUP_1 f(x, Y)) = wINF_1(\bigcup_{x \in X_0} wSUP_1 f(x, Y)).$$

Since $f(x, Y) \neq \emptyset$ and $wSUP_1 f(x, Y) \neq \{+\infty\}$ for all $x \in X$ we will have

$$\bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \emptyset,\ \{+\infty\},\ \{-\infty\}$$

and

$$f(x, Y) \subseteq wSUP_1 f(x, Y) - \text{Int } Z_+ \cup \{0\}, \forall x \in X_0.$$

If $wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \{-\infty\}$ then

$$\bigcup_{x \in X_0} wSUP_1 f(x, Y) \subseteq wINF_1 \bigcup_{x \in X_0} wSUP_1 f(x, Y) + \text{Int } Z_+ \cup \{0\}.$$

This implies that for all $x \in X_0$,

$$\emptyset \neq wSUP_1 f(x, Y) \subseteq wINF_1 \bigcup_{x \in X_0} wSUP_1 f(x, Y) + \text{Int } Z_+ \cup \{0\}. \tag{13}$$

In a similar way, if $wINF_1 f(X, y) = \{-\infty\}$ for all $y \in Y$, then

$$wSUP_1(\bigcup_{y \in Y} wINF_1 f(X, y)) = \{-\infty\}.$$

If there exists $y \in Z$ such that $wINF_1 f(X, y) \neq \{-\infty\}$, we denote

$$Y_0 = \{y \in Z \mid\ wINF_1 f(X, y) \neq \{-\infty\}\}.$$

If $wSUP \bigcup\limits_{y\in Y_0} wINF_1 f(X,y) \neq \{+\infty\}$ then for all $y \in Y_0$,

$$\emptyset \neq wINF_1 f(X,y) \subseteq wSUP_1 \bigcup_{y\in Y_0} wINF_1 f(X,y) - \text{Int } Z_+ \cup \{0\}. \tag{14}$$

Since $wSUP_1 f(x,Y) \neq \emptyset$, $\{+\infty\}$, $\forall x \in X_0$ it follows that $w^{\mu}MAX\, f(x,Y)$ is nonempty $\forall \mu \in \text{Int } Z_+$, $\forall x \in X_0$ and

$$\begin{aligned} w^{\mu}MAXf(x,Y) &\subseteq wSUPf(x,Y) - \mu \\ &= wSUP_1 f(x,Y) + \text{Int } Z_+ \cup \{0\} - \mu \subseteq \\ &\subseteq wINF_1 \bigcup_{x\in X_0} wSUP_1 f(x,Y) + \text{Int } Z_+ \cup \{0\} - \mu,\ \forall x \in X_0. \end{aligned} \tag{15}$$

Analogously,

$$\begin{aligned} w^{\eta}MINf(X,y) &\subseteq wINFf(X,y) - \eta \\ &= wINF_1 f(X,y) - \text{Int } Z_+ \cup \{0\} - \eta \\ &\subseteq wSUP_1 \bigcup_{x\in X_0} wINF_1 f(X,y) + \text{Int } Z_+ \cup \{0\} - \eta,\ \forall y \in Y_0. \end{aligned} \tag{16}$$

Now, we can present the following theorem:

Theorem 5.1. *If the condition* **(H)** *holds and if* (x_0, y_0) *is a* $\mu-\eta$ *weak saddle point of* f*, with* $\mu, \eta \in \text{Int } Z_+$*, then there exist*

$$\alpha \in wINF_1 \bigcup_{x\in X} wSUP_1 f(x,Y)$$

and

$$\beta \in wSUP_1 \bigcup_{y\in Y} wINF_1 f(X,y)$$

such that

$$\beta + \eta \geq_{\text{Int } Z_+\cup\{0\}} f(x_0,y_0) \geq_{\text{Int } Z_+\cup\{0\}} \alpha - \mu.$$

If (x_0, y_0) *is a weak saddle point of* f*, then there exist*

$$\alpha \in wINF_1 \bigcup_{x\in X} wSUP_1 f(x,Z)$$

and

$$\beta \in wSUP_1 \bigcup_{y \in Y} wINF_1 f(X, y)$$

such that

$$\beta \geq_{intZ_+ \cup \{0\}} f(x_0, y_0) \geq_{\text{Int } Z_+ \cup \{0\}} \alpha.$$

Proof. In the beginning we consider the case

$$wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \{-\infty\},\ wSUP \bigcup_{y \in Y_0} wINF_1 f(X, y) \neq \{+\infty\}.$$

Obviously, if (x_0, y_0) is a $\mu - \eta$ weak saddle point then $x_0 \in X_0, y_0 \in Y_0$ and using relations (15) and (16) we obtain

$$\alpha \in wINF_1 \bigcup_{x \in X} wSUP_1 f(x, Z)$$

and

$$\beta \in wSUP_1 \bigcup_{y \in Y} wINF_1 f(X, y)$$

such that

$$\beta + \eta \geq_{intZ_+ \cup \{0\}} f(x_0, y_0) \geq_{\text{Int } Z_+ \cup \{0\}} \alpha - \mu.$$

If (x_0, y_0) is a weak saddle point, then using the relations (13) and (14) the conclusion follows. Now, if we have

$$wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y) = \{-\infty\}$$

or

$$wSUP \bigcup_{y \in Y_0} wINF_1 f(X, y) = \{+\infty\}$$

the relations do hold obviously for $\alpha = -\infty$, respectively $\beta = +\infty$. □

Remark 5.1. The condition

$$wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \{-\infty\}$$

is equivalent with the fact that the multifunction $G : X_0 \rightrightarrows Z$, $G(x) = f(x, Y)$ has a lower bounded selection.

Indeed, the following relation

$$wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \{-\infty\}$$

does hold if and only if there exists $u \in Z$ such that for all $x \in X$, $\alpha \in wSUP_1 f(x, Y)$, we have $\alpha \not\leq_K u$. This implies that $u \notin wSUP\, f(x, Y)$ because, if we suppose the contrary, from Theorem 4.2 we get an element $\alpha \in wSUP_1 f(x, Y)$ such that $\alpha <_K u$, false. Thus, $u \in f(x, Y) - K \setminus \{0\}$, $\forall x \in X_0$ which implies that G has a lower bounded selection, strictly lower bounded by u with respect to the order induced by K, and also with respect to the order induced by Z_+.

Conversely, if there exists $u \in f(x, Y) - Z_+$, $\forall x \in X_0$, it follows that any $u_1 <_K u$ has the property that

$$u_1 \in f(x, Y) - K \setminus \{0\} = wSUP_1 f(x, Y) - K \setminus \{0\},\ \forall x \in X_0.$$

If there exists $x \in X_0, \alpha \in wSUP_1 f(x, Y)$ such that $u_1 >_K \alpha$, it follows that there exists $\beta \in wSUP_1 f(x, Y)$, $\beta >_K \alpha$, contradiction.

Thus

$$u_1 \in wINF \bigcup_{x \in X_0} wSUP_1 f(x, Y)$$

and

$$INF \bigcup_{x \in X_0} wSUP_1 f(x, Y) \neq \{-\infty\}.$$

In the same manner, the condition

$$wSUP \bigcup_{y \in Y_0} wINF_1 f(X, y) \neq \{+\infty\}$$

is equivalent with the fact that the multifunction $\tilde{G} : Y \rightrightarrows Z$, $\tilde{G}(y) = f(X, y)$ has an upper bounded selection.

In a similar way, we can define the notion of saddle point for a multifunction, notion which was presented by Tanino and Sawaragi in [19] and was extended for vectorial multifunctions in [8]. The study of this notion and other applications of the vectorial efficient points for the study of the perturbed problems will be proposed in a future paper.

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