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AN EXISTENCE RESULT FOR PERIODIC SOLUTIONS VIA MIRANDA'S THEOREM

BY

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Abstract. In this paper one proves an existence theorem for the bilocal boundary value problem $\dot{x} = f(t,x)$, $x(0) = x(T)$, by using a known theorem of Miranda, theorem which gives rise suficient conditions for an operator F defined in a finite dimensional space to admit al least one zero.

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1. Introduction. Let $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function. As is well known, the bilocal boundary value problem

$$\dot{x} = f(t,x), \tag{1}$$

$$x(0) = x(T) \tag{2}$$

is closely related to the existence of periodic solutions. Indeed, if f can be extended on $\mathbb{R} \times \mathbb{R}^N$ as a $T-$periodic and continuous function, then every $T-$periodic extension of a solution for the problem $(1), (2)$ will represent a periodic solution. Therefore, without risk of confusion, one uses to call each solution of problem $(1), (2)$ a periodic solution.

The problem $(1), (2)$ has been intensely studied through different methods. Last years, the topological methods has been especially remarked in the study of problem $(1), (2)$; one of the starting points has been Krasnoselski's book ([3]), followed by other works due to this author and his disciples. A substantial contribution in this field has been constituted the works of

MAWHIN ([4], [5]), followed by many others. In [4] and [5] an exhaustive bibliography can be found.

In the case when the equation (1) admits an unique solution $x(t;c)$ defined on $[0,T]$ satisfying the initial condition

$$x(0)=c, \tag{3}$$

the problem of the existence of solutions for $(1),(2)$ can be considered by using the so called **Poincaré operator**, which is defined in $\mathbb{R}^N$ by $c \to x(T;c)$. Obviously, $x(T;c)$ will be a solution for $(1),(2)$ if and only if c will be a fixed point of the Poincaré operator. The Poincaré operator has been used by many authors; in particular, in [3], [4], [6] the Brouwer topological degree is used to assure the existence of a fixed point of this operator.

In this note we use another operator defined in $\mathbb{R}^N$, i.e.

$$F(c)=\int_0^T f(s,x(s;c))\,ds, \tag{4}$$

which is well defined when the problem $(1),(3)$ has an unique solution $x(t;c)$ defined on $[0,T]$. In this case, since the mapping $c \to x(\cdot;c)$ is continuous from $\mathbb{R}^N$ into $C\left([0,T],\mathbb{R}^N\right)$ will result that the mapping $c \to F(c)$ is continuous in $\mathbb{R}^N$.

Obviously, $x(t;c)$ is a solution for the problem $(1),(2)$ if and only if

$$F(c)=0. \tag{5}$$

Miranda's theorem gives a sufficient condition in which the equation (5) admits solutions.

Let $l>0$ a fixed number. Set

$$\begin{aligned}\Pi_l &:= \left\{x=(x_i)_{i\in\overline{1,N}}\in\mathbb{R}^N,\ |x_i|\le l,\ i\in\overline{1,N}\right\},\\ S_i^+ &:= \{x\in\Pi_l,\ x_i=l\},\ S_i^-:=\{x\in\Pi_l,\ x_i=-l\}.\end{aligned}$$

Let $F=(F_i)_{i\in\overline{1,N}}$, $F:\Pi_l\to\mathbb{R}^N$ be a continuous application. Miranda's theorem asserts that if the following conditions are fulfilled:

$$F_i(c)\le 0,\ (\forall)\ c\in S_i^+;\ F_i(c)\ge 0,\ (\forall)\ c\in S_i^-,\ (\forall)\ i\in\overline{1,N}, \tag{6}$$

then the equation (5) admits solutions in Π_l. This result is established immediately by using Brouwer's fixed point theorem.

In [1], [2] Miranda's theorem is used for proving the existence of solutions for the problem (1) , (2), but the admitted hypotheses are more restrictive. In this paper we shall suppose two hypotheses about function f: one of sublinearity type and the second which assures the fulfilling of conditions (6) .

The conditions to achieve this last request will be of *Landesman-Lazer type*. In [4], chapter VI, these conditions are used in the case of a scalar equation (1) . Last years, conditions of Landesman-Lazer type has been successfully used in the problem of the existence of bounded solutions for the differential equation of second order. In this direction the reader can find in [5], [7] a rich bibliography.

Finally we enumerate some of the used notations.

In $\mathbb{R}^N$ consider the norm

$$|x| := \max_{i\in\overline{1,N}} \{|x_i|\}\,,\ x = (x_i)_{i\in\overline{1,N}} \in \mathbb{R}^N.$$

Denote by f_i, F_i, $i \in \overline{1,N}$ the components of the continuous function $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$, respectively of the function F given by (4) .

Set in addition

$$\begin{aligned} \overline{B_\rho} &: = \left\{x \in \mathbb{R}^N,\ |x| \le \rho\right\}, \\ \overline{\Sigma_\rho} &: = \left\{x \in C\left([0,T], \mathbb{R}^N\right),\ |x(t)| \le \rho,\ t \in [0,T]\right\}, \\ M_\rho &: = \sup\left\{|f(t,x)|,\ t \in [0,T],\ x \in \overline{B_\rho}\right\}. \end{aligned}$$

2. Main result. We shall state and prove the following theorem.

Theorem 1. *Suppose that the following hypotheses are fulfilled:*
i) the inequality

$$\limsup_{\rho\to\infty} \frac{M_\rho}{\rho} < \frac{1}{2T} \tag{7}$$

holds true;
ii) there exists a number $r > 0$ such that

$$\begin{aligned} &f_i(t,x) < \delta_-(t),\ t \in [0,T],\ x_i \ge r, \\ &f_i(t,x) < \delta_+(t),\ t \in [0,T],\ x_i \le -r, \end{aligned} \tag{8}$$

where δ_-, $\delta_+ : [0,T] \to \mathbb{R}$ *are continuous functions satisfying the conditions*

$$\begin{aligned} &\int_0^T \delta_- (t)\, dt < -\delta < 0, \\ &\int_0^T \delta_+ (t)\, dt > \delta > 0, \end{aligned} \tag{9}$$

where δ *is a given positive number.*

Then the problem $(1), (2)$ *admits solutions.*

Proof. Let

$$L := \limsup_{\rho \to \infty} \frac{M_\rho}{\rho}. \tag{10}$$

By hypothesis i) we have

$$L < \frac{1}{2T}. \tag{11}$$

For each ρ satisfying

$$\rho > \max \left\{ r, \frac{r}{1-2TL} \right\} \tag{12}$$

we have

$$L < \frac{1}{2T} \left(1 - \frac{r}{\rho} \right) < \frac{1}{2T} \tag{13}$$

and therefore for each ρ big enough we will have

$$r + 2TM_\rho < \rho. \tag{14}$$

Let us consider a ρ such that (14) is satisfied.

Firstly suppose that f satisfies a Lipschitz condition

$$|f(t,x) - f(t,y)| \leq L|x-y|, \ t \in [0,T], \ x, y \in B_\rho. \tag{15}$$

Hence the problem $(1), (3)$ has an unique solution $x(t;c)$. But $x(t;c)$ is the unique fixed point of the operator $x \to Hx$ defined by the equality

$$(Hx)(t) = c + \int_0^t f(s, x(s))\, ds. \tag{16}$$

But, for $x \in \overline{\Sigma_\rho}$ and each c satisfying the condition

$$c \in \overline{B_{r+M_\rho T}} \tag{17}$$

one has

$$|(Hx)(t)| \leq |c| + \int_0^t |f(s, x(s))|\, ds \leq r + M_\rho T + M_\rho T = r + 2M_\rho T < \rho.$$

Therefore,

$$H\overline{\Sigma_\rho} \subset \overline{\Sigma_\rho}$$

and so for every c satisfying (17) one has

$$|x(t;c)| \leq \rho,\ t \in [0,T]. \tag{18}$$

Consequently, the mapping $c \to x(\cdot;c)$ is defined from $\overline{B_{r+M_\rho T}}$ into $C\left([0,T], \mathbb{R}^N\right)$; a classical result in the qualitative theory of differential equations assures us that this mapping is continuous too.

Let l be a fixed number such that

$$r + M_\rho T < l < r + 2M_\rho T.$$

Consider the operator $F : \Pi_l \to \mathbb{R}^N$ given by (4); from above it results that F is a continuous operator.

Let $c \in S_i^+$; then

$$x_i(t;c) = l + \int_0^t f_i(s, x(s;c))\, ds > r + M_\rho T - M_\rho T = r, \tag{19}$$

since

$$(|x_i(t;c)| < \rho) \Longrightarrow (|f_i(t, x(t;c))| \leq M_\rho).$$

By (8) and (19) it results

$$F_i(c) := \int_0^T f_i(s, x(s,c))\, ds < \int_0^T \delta_i(t)\, dt < -\delta T < 0. \tag{20}$$

Similarly one proves that

$$(c \in S_i^-) \Longrightarrow (F_i(c) > \delta T > 0). \tag{21}$$

Therefore, the equation (15) admits solutions; if c is such a solution, then since

$$x(T;c) - x(0;c) = \int_0^T \dot{x}(s;c)\,ds = \int_0^T f(s, x(s,c))\,ds = F(c),$$

it follows that $x(t;c)$ is a solution of the problem $(1), (2)$.

The passing to the case when f is continuous is classical. Indeed, from Weierstrass's theorem there exists a sequence of polynomials $f^n : [0,T] \times \overline{B_\rho} \to \mathbb{R}^N$ which approximates uniformly the function f in the sense that

$$(22) \qquad \begin{array}{c} (\forall)\,\epsilon > 0,\ (\exists)\,n_0 = n_0(\epsilon),\ (\forall)\,n \geq n_0,\ (\forall)\,(t,x) \in [0,T] \times \overline{B_\rho}, \\ |f^n(t,x) - f(t,x)| < \epsilon, \end{array}$$

where ρ is a fixed number satisfying (13).

One can extend f^n to $\mathbb{R} \times \mathbb{R}^N$, by using Tietze-Dugundji theorem (see [9], p. 50); denote with $\widetilde{f^n}$ this extension; one remarks in addition that

$$\left|\widetilde{f}(t,x)\right| \leq M_\rho,\ (\forall)\,x \in \mathbb{R}^N,\ (\forall)\,t \in [0,T].$$

For a ρ satisfying (20) and for c satisfying (17) one has for the solution $x^n(t;c)$ of the problem

$$(23) \qquad \dot{x} = \widetilde{f^n}(t,x),\ x(0) = c$$

the estimation

$$|x^n(t;c)| \leq r + M_\rho T + (M_\rho + \epsilon)\,T = r + 2M_\rho T + \epsilon T,\ n \geq n_0.$$

From (14) it results that for ϵ small enough,

$$(24) \qquad |x^n(t;c)| < \rho,\ n \geq n_0$$

and therefore $x^n(t;c)$ is the solution of the problem

$$(25) \qquad \dot{x} = f^n(t;c),\ x(0) = c.$$

From the first part of theorem it results that for every $n \in \mathbb{N}^*$ there exists $c^n \in \Pi_l$ such that the problem

$$(26) \qquad \dot{x} = \widetilde{f^n}(t,x),\ x(0) = x(T)$$

admits a solution x^n given by

$$x^n(t) = x(t; c^n). \tag{27}$$

Moreover, since

$$|x^n(t)| \leq \rho, \quad (\forall)\, t \in [0,T], \quad (\forall)\, n \in \mathbb{N}^*, \tag{28}$$

it will follows that $x^n(t)$ is a solution for $(1), (2)$.

By (28) and (29) it results that the sequence $(x^n)_{n\in\mathbb{N}}$ is bounded in $C([0,T], \mathbb{R}^N)$; by other part, from

$$\dot{x}^n(t) = f(t, x^n(t)),$$

it follows that

$$|\dot{x}^n(t)| \leq \rho + M_\rho.$$

By (28), (29) and Ascoli-Arzelà theorem it results that the sequence $(x^n)_{n\in\mathbb{N}}$ is relatively compact in $C\left([0,T], \mathbb{R}^N\right)$. Without loss of generality one can suppose that $c^n \to c$ in $\mathbb{R}$ and $x^n \to x$ in $C\left([0,T], \mathbb{R}^N\right)$; then, since

$$x^n(t) = c^n + \int_0^t f^n(s, x^n(s))\, ds,$$

we have

$$x^n(t) = c + \int_0^t f(s, x(s))\, ds$$

and hence $x(t)$ is a solution for (1).

By other part, by

$$x^n(0) = x^n(T)$$

it results that x fulfills (2), which ends the proof. □

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