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TWISTED ENDOMORPHISMS OF RANK TWO STABLE VECTOR BUNDLES ON THE PLANE

BY

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Abstract. Let E a " suitable " rank two stable vector bundle on $\mathbf{P}^2$. Here we compute the largest integer $x > 0$ such that $h^0(\mathbf{P}^2, Hom(E, E(x)) = (x+2)(x+1)/2$, i.e. such that for every $f \in H^0(\mathbf{P}^2, Hom(E, E(x)))$ with $f \neq 0$ there is a degree x plane curve C such that f is obtained composing the identity $E \to E$ with the inclusion $E \to E(C)$ obtained from a homogeneous equation of C.

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Let E be a rank 2 stable vector bundle on $\mathbf{P}^2$. We would like to find an integer $x > 0$ (as large as possible) such that $h^0(\mathbf{P}^2, Hom(E, E(x)) = (x+2)(x+1)/2$, i.e. such that for every $f \in H^0(\mathbf{P}^2, Hom(E, E(x)))$ with $f \neq 0$ there is a degree x plane curve C such that f is obtained composing the identity $E \to E$ with the inclusion $E \to E(C)$ obtained from a homogeneous equation of C. We are interested in bundles which are not general in their moduli space. Set $\mathbf{O} := \mathbf{O}_{\mathbf{P}^2}$. This problem has the same solution for E and for the vector bundle $E(t), t \in \mathbf{Z}$. Hence without loosing generality it is sufficient to consider rank 2 vector bundles on $\mathbf{P}^2$ with 0 or -1 as first Chern class. Let $M(\mathbf{P}^2; 2, 0, c_2)$ (resp. $M(\mathbf{P}^2; 2, -1, c_2)$) be the set of all stable vector bundles E on $\mathbf{P}^2$ with $\text{rank}(E) = 2, c_2(E) = c_2$ and $c_1(E) = 0$ (resp. $c_1(E) = -1$). We have $M(\mathbf{P}^2; 2, 0, c_2) \neq \emptyset$ (resp. $M(\mathbf{P}^2; 2, -1, c_2) \neq \emptyset$) if and only if $c_2 \geq 2$ (resp. $c_2 > 0$) ([2] and [5]). $M(\mathbf{P}^2; 2, 0, c_2)$ (resp. $M(\mathbf{P}^2; 2, -1, c_2)$) is smooth and irreducible of dimension $4c_2 - 3$ (resp. $4c_2 - 4$). For any rank two vector bundle E on $\mathbf{P}^2$, set $\text{lev}(E) := \inf\{t \in \mathbf{Z} : h^0(\mathbf{P}^2, E(t)) \neq 0\}$ and call the integer $\text{lev}(E)$ the level of E.

There is a natural stratification of the moduli spaces $M(\mathbf{P}^2; 2, 0, c_2)$ and $M(\mathbf{P}^2; 2, -1, c_2)$; set $M(\mathbf{P}^2; 2, 0, c_2; t) := \{E \in M(\mathbf{P}^2; 2, 0, c_2) : \mathrm{lev}(E) = t\}$ and $M(\mathbf{P}^2; 2, -1, c_2; t) := \{E \in M(\mathbf{P}^2; 2, -1, c_2) : \mathrm{lev}(E) = t\}$. By the very definition of stability we have $\mathrm{lev}(E) > 0$ for every $E \in M(\mathbf{P}^2; 2, 0, c_2)$ and every $E \in M(\mathbf{P}^2; 2, -1, c_2)$. Set $c_1 = 0$ or -1. By [3, Th. 5.1] we have $M(\mathbf{P}^2; 2, c_1, c_2; t) = \emptyset$ if $t > a(c_1, c_2)$, where $a(c_1, c_2)$ is the minimal integer, z, such that $(c_1+2z+1)(c_1+2z+2)/2 - c_2 - zc_1 - z^2 + 1 > 0$. Furthermore, for a Zariski dense subset U of $M(\mathbf{P}^2; 2, c_1, c_2)$ we have $\mathrm{lev}(E) = a(c_1, c_2)$ for every $E \in U$. If $0 < t < a(c_1, c_2)$ we would like to say that the set $M(\mathbf{P}^2; 2, c_1, c_2; t)$ is parameterized (perhaps not one-to-one and even not finite-to-one, but generically finite-to-one) by a dense open subset of a projective space of dimension $c_2 + t^2 + tc_1 + (2t + c_1 - 2)(2t + c_1 - 1)/2$. To make this assertion rigorous, we will restrict to the following subset $U(c_1, c_2, t)$ of $M(\mathbf{P}^2; 2, c_1, c_2; t)$. Every $E \in M(\mathbf{P}^2; 2, c_1, c_2; t), c_1 = 0$ or -1, fits in an exact sequence

$$0 \to \mathbf{O} \to E(t) \to \mathbf{I}_A(c_1 + 2t) \to 0 \tag{1}$$

with A zero-dimensional locally complete intersection subscheme of $\mathbf{P}^2$ with $\deg(A) = c_2(E(t)) = c_2 + t^2 + tc_1$. We will use only A reduced and with natural cohomology, i.e. with $h^0(\mathbf{P}^2, \mathbf{I}_A(z)) = 0$ if $(z+2)(z+1)/2 \leq card(A) = c_2$ and $h^1(\mathbf{P}^2, \mathbf{I}_A(z)) = 0$ if $(z+2)(z+1)/2 > card(A)$. Since $t < a(c_1, c_2)$ we have $h^0(\mathbf{P}^2, \mathbf{I}_A(t)) = 0$. Thus for any sheaf E fitting in (1) we have $h^0(\mathbf{P}^2, E(t)) = 1$. Hence for such sheaf E both A and the extension (1) are uniquely determined, up to a non-zero multiplicative constant. When $0 \leq t \leq a(c_1, c_2)$ we will always use an extension (1) with $\deg(A) = c_2 + t^2 + tc_1$, A reduced and A general subset of $\mathbf{P}^2$ with $\mathrm{card}(A) = c_2 + t^2 + tc_1$; more precisely, we need that A has natural cohomology. With this assumption, for any fixed A the set of all extensions (1) is a vector space $Ext^1(\mathbf{P}^2; \mathbf{I}_A(2t), \mathbf{O})$ of dimension $deg(A) + (2t + c_1 - 2)(2t + c_1 - 1)/2$ if $t \geq 1$ and $deg(A)$ if $t = 0$ ([1] or [7]). For any such A and a general extension (1) the sheaf E is locally free because the canonical divisor of $\mathbf{P}^2$ has degree -3 and hence the Cayley - Bacharach condition is satisfied ([4, §1]). Varying A in $\mathbf{P}^2$ with that cohomological restriction (i.e. over a certain Zariski open non-empty subset U of the symmetric product $S^{deg(A)}(\mathbf{P}^2)$) the set of all extensions (1) is parametrized by a vector bundle over U. In this way we may say define an irreducible algebraic subset $U(c_1, c_2, t)$ of $M(\mathbf{P}^2; 2, c_1, c_2; t)$. $U(c_1, c_2, t)$ is parametrized (generically one-to-one) by an irreducible variety of dimension $3\, deg(A) + (2t + c_1 - 2)(2t + c_1 - 1)/2 (t \geq 1)$ and it is unirational. Having defined $U(c_1, c_2, t)$ we may study its general element. But we will need more. To study the general bundle of each $U(c_1, c_2, t), 0 < t < a(c_1, c_2)$, we need to consider also the bundles $M(\mathbf{P}^2; 2, 0, c_2; 0)$, i.e. the set of all isomorphic classes of bundles which fits in an exact sequence (1) with $c_1 = 0$, A zero-dimensional locally complete intersection subscheme of $\mathbf{P}^2$ with $deg(A) = c_2 > 0$ and $t = 0$; the local freeness of E implies that (1) does not split and that the so-called Cayley - Bacharach property is satisfied by A ([4, §1]); for us it will be sufficient to say that since the canonical divisor of $\mathbf{P}^2$ has degree -3 this condition is satisfied (for $t = 0$) by any zero-dimensional locally complete intersection subscheme A; the set of all extensions (1) with $t = 0$ is parametrized by a projective space, Π, of dimension $deg(A)$ and the set of all extensions with locally free middle term is the complementary in Π of certain proper linear subspaces, one for each connected component of A; we will need only the case A reduced. Notice that for any exact sequence (1) with $t = 0$ and $A \neq \emptyset$ we have $h^0(\mathbf{P}^2, E) = 1$. Every $E \in M(\mathbf{P}^2; 2, 0, c_2; 0)$ is semistable but not stable. In this note we will prove the following result.

Theorem 1.1. *Fix integers c_1, c_2, t and x with $0 \leq c_1 \leq 1, x \geq 0, 0 < t \leq a(c_1, c_2)$ and $c_2 > (x + 2 - c_1)(x + 1 - c_1)/2$. Then for a general $E \in U(c_1, c_2, t)$ we have $h^0(\mathbf{P}^2, Hom(E, E(x))) = (x + 2)(x + 1)/2$.*

In particular for $t = a(c_1, c_2)$ we obtain the following result.

Corollary 1.2. *Fix integers c_1, c_2 and x with $0 \leq c_1 \leq 1, x \geq 0$ and $c_2 > (x + 2 - c_1)(x + 1 - c_1)/2$. Then for a general $E \in M(\mathbf{P}^2; 2, c_1, c_2)$ we have $h^0(\mathbf{P}^2, Hom(E, E(x))) = (x + 2)(x + 1)/2$.*

To prove Theorem 1.1 and hence Corollary 1.2 we use the following lemma.

Lemma 1.3. *Fix integers c_1, c_2 and t with $c_1 = 0$ or $c_1 = 1, c_2 > 0$ and $0 \leq t < a(c_1, c_2)$ and a general $E \in U(c_1, c_2, t)$. Then there exists an integral affine curve T, $o \in T$ and a flat family of vector bundles $\{E_\lambda\}_{\lambda \in T}$ on T with $E_o \cong E$ and E_λ general element of $U(c_1, c_2; t+1)$ for $\lambda \in (T \backslash \{o\})$.*

Proof. Since $M(\mathbf{P}^2; 2, c_1, c_2; t+1)$ is dense in $M(\mathbf{P}^2; 2, c_1, c_2)$ if $t+1 = a(c_1; c_2)$, we may assume $t + 1 < a(c_1, c_2)$. Fix a line $D \subset \mathbf{P}^2$ and a rank 2 vector bundle F on $\mathbf{P}^2$ with $0 \leq c_1(F) \leq 1$. We assume that $F|D \cong \mathbf{O}_D \oplus \mathbf{O}_D(c_1)$. Since we are in characteristic 0, this condition on $F|D$ is satisfied by every stable vector bundle if we fix F and take as D a general line (Grauert - Mlich restriction theorem (see e.g. [2])). If F fits in an exact sequence (1) with $t = 0$ this condition is satisfied if $D \cap A = \emptyset$. Fix an integer $a \geq c_1$ and a surjection $u : \mathbf{O}_D \oplus \mathbf{O}_D(c_1) \to \mathbf{O}_D(a)$. Since $F|D \cong \mathbf{O}_D \oplus \mathbf{O}_D(c_1)$, the surjection u induces a surjection $\rho : F \to \mathbf{O}_D(a)$. Set $G := Ker(\rho)$. Since D is a Cartier divisor, G is locally free ([8]). Following [8] we will say that G is obtained from F making an elementary transformation supported by D and associated to the surjection u or the surjection ρ. For every integer z we have an exact sequence

$$0 \to \mathbf{O}(z-1) \to \mathbf{O}(z) \to \mathbf{O}_D(z) \to 0 \tag{2}$$

from which one can compute the Chern classes of $\mathbf{O}_D(z)$ seen as a sheaf on $\mathbf{P}^2$. We obtain $rank(\mathbf{O}_D(z)) = 0, c_1(\mathbf{O}_D(z)) = -1$ and $c_2(\mathbf{O}_D(z)) = z$. By its very definition the rank 2 vector bundle G fits in an exact sequence

$$0 \to G \to F \to \mathbf{O}_D(a) \to 0 \tag{3}$$

from which one can compute the Chern classes of G. We obtain $c_1(G) = c_1(F) - 1$ and $c_2(G) = c_2(F) + c_1 - 1 - a$. We have $Ker(u) \cong \mathbf{O}_D(c_1 - a)$. Since the normal bundle of D in $\mathbf{P}_2$ has degree 1, by tensoring (2) for $z = 0$ with $\mathbf{O}_D$ we obtain that $Tor^1_{\mathbf{O}}(\mathbf{O}_D, \mathbf{O}_D) \cong \mathbf{O}_D(-1)$. Thus restricting (3) to D we obtain that $G|D$ fits in an exact sequence

$$0 \to \mathbf{O}_D(a-1) \to G|D \to \mathbf{O}_D(c_1 - a) \to 0 \tag{4}$$

The surjective map in (4) induces a surjection $\tau : G \to \mathbf{O}_D(c_1 - a)$ and we have $Ker(\tau) \cong F(-1)$. Fix an integer $t \geq 0$. If $h^0(\mathbf{P}^2, G(t)) > 0$, then $h^0(\mathbf{P}^2, F(t)) > 0$ because G is a subsheaf of F. If $h^0(\mathbf{P}^2, F(t)) > 0$, then $h^0(\mathbf{P}^2, G(t+1)) > 0$ because $Ker(\tau) \cong F(-1)$. If $h^0(\mathbf{P}^2, F(t)) > 0$, then $H^0(\mathbf{P}_2, G(t))$ is the kernel of the induced map $\alpha : H^0(\mathbf{P}^2, F(t)) \to H^0(D, \mathbf{O}_D(a+t))$. Assume $h^0(\mathbf{P}^2, F(t)) = 1$ and take $\sigma \in H^0(\mathbf{P}_2, F(t))$ with $\sigma \neq 0$. If $a = 0$ and $c_1 = +1$ the surjection u is unique up to a non-zero multiplicative constant and hence G is uniquely determined. In the other cases we may always find a surjection u such that $\alpha(\sigma) \neq 0$. Thus for such choice of u we have $h^0(\mathbf{P}^2, G(t)) = 0$ and $h^0(\mathbf{P}^2, G(t+1)) = 0$. Take any surjection $v : G \to \mathbf{O}_D(c_1 - a)$. For all $\lambda, \mu \in \mathbf{K}$ set $v_{\lambda,\mu} := \lambda\tau + \mu v$. For a general pair (λ, μ) the map $v_{\lambda,\mu} : G \to \mathbf{O}_D(c_1 - a)$ is surjective. Set $G(\lambda, \mu) = Ker(v_{\lambda,\mu})$ for any pair (λ, μ) with $v_{\lambda,\mu}$ surjective. The family $G(\lambda, \mu)$ is a flat family of rank 2 vector bundles on $\mathbf{P}^2$ parameterized by an irreducible variety (a Zariski open non-empty subset of $\mathbf{K}^2$).

Since $G(1,0) \cong F(-1), F$ is the flat limit of the family $\{G(\lambda,\mu)(1)\}$. Since semistability and stability are open conditions, if F is semistable (resp. stable), then a general $G(\lambda,\mu)$ is semistable (resp. stable). For the same reason if $F \in M(\mathbf{P}^2;2,c_1,c_2;t)$, then for a general pair (λ,μ) we have $G(\lambda,\mu) \in M(\mathbf{P}^2;2,c_1,c_2;z)$ for some $z \geq t$. First assume $c_1 = 0$ and take $a = 0$. We fix the integers c_2 and t with $c_2 > 0$ and $0 \leq t < a(0,c_2) - 1$. We assume $F \in M(\mathbf{P}^2;2,0,c_2;t)$ and $h^0(\mathbf{P}^2,F(t)) = 1$. Then $G|D \cong \mathbf{O}_D \oplus \mathbf{O}_D(-1)$ and we have a large family of surjections $v : \mathbf{O}_D \oplus \mathbf{O}_D(-1) \to \mathbf{O}_D$. For any such v the bundle $Ker(v)(1)|D$ is trivial and $h^0(\mathbf{P}^2, Ker(v)(t+1)) \neq 0$. We assume $t+1 < a(0,c_2)$ and that F is general in $U(0,c_2,t)$, i.e. that $F(t)$ (instead of $E(t)$) fits in a general extension (1) with A general. Since $t+1 < a(0,c_2)$, we obtain $h^0(\mathbf{P}^2,F(t+1)) = 3$. For general v we obtain as $Ker(v)(1)$ any rank 2 vector bundle G' fitting in an exact sequence

$$0 \to G' \to G(1) \to \mathbf{O}_D \to 0 \tag{5}$$

From the exact sequence

$$0 \to G \to F \to \mathbf{O}_D \to 0 \tag{6}$$

and the assumption $h^0(\mathbf{P}^2,F(t+1)) = 3$ we obtain $h^0(\mathbf{P}^2,G(t+1)) = 0$ for general v. Hence if $t+1 < a(0,c_2)$ we obtain F as flat limit of a family, $\{F_\alpha\}$ of bundles with level $\leq t+1$ but not with level t, i.e. F is in the closure of $M(\mathbf{P}^2;2,0,c_2;t+1)$. Furthermore, for a general a we have $h^0(\mathbf{P}^2,F_\alpha(t+1)) \leq 3$. Hence for a general α the bundle F_α fits in an exact sequence 1 with $t' := t+1$ instead of t and with a certain $A\alpha$ with $h^0(\mathbf{P}^2,\mathbf{I}_{A_\alpha}(c_1+2t+2)) \leq 2$ and hence $h^0(\mathbf{P}^2,\mathbf{I}_{A_\alpha}(c_1+2t+1)) = 0$. Since the anticanonical divisor of $\mathbf{P}^2$ has degree 3 and $dim(Ext^1(\mathbf{P}^2;\mathbf{O},\mathbf{I}_{A_\alpha}(c_1+2t+2)))$ depends only on the integer $deg(A_\alpha) = c_2 + (t+1)^2 + (t+1)c_1$, we see that we may deform simultaneously A_α to a general subset of $\mathbf{P}^2$ and the extension (1) having as parameter space a vector bundle over a Zariski open subset of a symmetric product of $\mathbf{P}^2$ ([1] or [7]). Since the anticanonical divisor has degree 3, in this deformation the Cayley - Bacharach property is preserved. Hence we see that a general F_α is a flat limit of a family of elements of $U(c_1,c_2,t+1)$, proving the lemma for $c_1 = 0$. If $c_1 = 1$ we take $a = 1$ and work in the same way.

Proposition 1.4. *Fix integers c_1, c_2 and x with $0 \leq c_1 \leq 1, x \geq 0$ and $c_2 > (x+2-c_1)(x+1-c_1)/2$. Then for a general $E \in M(\mathbf{P}^2;2,c_1,c_2;c_1)$ we have $h^0(\mathbf{P}^2,Hom(E,E(x))) = (x+2)(x+1)/2$.*

Proof. Fix an extension (1) with $t = 0$ giving E. Since E is general, we may assume that A has natural cohomology. Since $rank(E) = 2$ we have $E^* \cong E(-c1)$. First assume $c_1 = 0$. Tensor the extension (1) by $E^*(x) \cong E(x)$. We obtain an exact sequence

$$0 \to E(x) \to Hom(E, E(x)) \to \mathbf{I}_A \otimes E(x) \to 0 \tag{7}$$

Hence by (7) to prove 1.4 it is sufficient to prove that $h^0(\mathbf{P}^2, E(x)) = (x+2)(x+1)/2$ and $h^0(\mathbf{P}^2, \mathbf{I}_A \otimes E(x)) = 0$. Twisting (1) with $\mathbf{O}(x)$ and using that A has natural cohomology, we obtain $h^0(\mathbf{P}^2, E(x)) = (x+2)(x+1)/2$. Now we will check the vanishing of $h^0(\mathbf{P}^2, \mathbf{I}_A \otimes E(x))$. Fix $\sigma \in H^0(\mathbf{P}^2, \mathbf{I}_A \otimes E(x))$ and call σ' its image in $H^0(\mathbf{P}^2, \mathbf{I}_A(x + c_1))$ using the surjection obtained twisting (1) for $t = 0$ with $\mathbf{O}(x)$. Since $card(A) > h^0(\mathbf{P}^2, \mathbf{O}(x))$ and A has natural cohomology, we obtain $\sigma' = 0$. Thus σ comes from $\sigma" \in H^0(\mathbf{P}^2, \mathbf{O}(x))$. Since $\sigma \in H^0(\mathbf{P}^2, \mathbf{I}_A \otimes E(x)), \sigma"$ vanishes on A. Since $h^0(\mathbf{P}^2, \mathbf{I}_A(x)) = 0$, we obtain $\sigma" = 0$ and hence $\sigma = 0$, as wanted. Now assume $c_1 = 1$ and take a homomorphism $f : E \to E(x)$. Thus f induces $g : E \to \mathbf{I}_A(x + 1)$. By the assumption on x and the generality of A we obtain $g(\mathbf{O}) = 0$. Thus g induces $f" : \mathbf{I}_A(1) \to \mathbf{I}_A(x + 1)$ and we conclude.

Remark 1.5. If $x = 0$ the statement of Proposition 1.4 means that E is simple. Assume $x > 0$ and $0 < (x+2)(x+1)/2 < c_2$. Proposition 1.4 means that for every $f \in H^0(\mathbf{P}^2, Hom(E, E(x)))$ with $f \neq 0$ there is a degree x plane curve C such that f is obtained composing the identity $E \to E$ with an inclusion $E \to E(C)$ obtained from a homogeneous equation of C.

Proof of 1.1. From 1.3, 1.4 and semicontinuity we obtain Theorem 1.1.

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