ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIX, s.I a, Matematică, 2003, f.1.

SOME POISSON DYNAMICS AND THE SPLITTING GAUSS-LEGENDRE INTEGRATORS

BY

MIRCEA CRAIOVEANU, IONEL MOŞ and MIRCEA PUTA

Dedicated to Prof. Dr. Radu MIRON on the occasion of his 75th birthday

Abstract. The splitting Gauss-Legendre integrators for two optimal control problems and for the Maxwell-Bloch equations with a quadratic control about Ox_1 axis are discussed and some of their geometrical and dynamical properties are pointed out.

1. Introduction. Let $(\mathbb{R}^n, \{\cdot,\cdot\}, H)$ be a Hamilton-Poisson mechanical system with the dynamics described by the system

$$\dot{x} = \Pi(x) \cdot \nabla H(x),\ x \in \mathbb{R}^n\,,$$

where Π is the matrix which generates the Poisson structure $\{\cdot,\cdot\}$ and H the Hamiltonian or the energy of the system. Among the methods used in the geometrical integration of the mechanical systems we can often find the splitting and composition methods. The splitting methods exploit the natural decomposition of the problem, and in particular of the Hamiltonian and they have been used with success in the study of many dynamical systems. The main idea behind the splitting Gauss-Legendre method is to decompose the Hamiltonian H as follows:

$$H = H_1 + H_2 + \cdots + H_m\,,$$

and to determine the Gauss-Legendre integrators $\phi_{H_1}, \phi_{H_2}, \ldots, \phi_{H_m}$ associated to the dynamics generated respectively by $H_1, H_2, \ldots, H_m$. The integrator

$$\phi = \phi_{H_1} \circ \phi_{H_2} \circ \cdots \circ \phi_{H_m}$$

is called *the splitting Gauss-Legendre integrator*.

The splitting Gauss-Legendre integrator has been introduced in [4] for the free rigid body and the heavy top. For the Maxwell-Bloch equations and the 3-dimensional Toda lattice it has been discussed in [1]. The goal of our paper is to determine the splitting Gauss-Legendre integrators for the kinematic car, the ball-plate problem and the Maxwell-Bloch equations with a quadratic control about Ox_1 axis and to point out some of their properties.

2. The kinematic car and the splitting Gauss-Legendre integrator. The dynamics of the kinematic car can be written as a left invariant control system on a 4-dimensional Lie group (see [3], [5]). More exactly, let G_4 be the Lie group given by

$$G_4 = \left\{ X = \begin{bmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & \frac{x_1^2}{2} \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; x_1, x_2, x_3, x_4 \in \mathbb{R} \right\} .$$

A basis for its corresponding Lie algebra $\mathcal{G}_4$ is given by $\{A_1, A_2, A_3, A_4\}$, where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_3 = [A_2, A_1], \; A_4 = [A_3, A_1] .$$

Then the dynamics of the kinematic car can be written in the form

$$\dot{X} = X(A_1\nu_1 + A_2\nu_2) , \tag{2.1}$$

where $X \in G_4$.

Theorem 2.1. *([3],[5]) Let $\mathcal{T}$ be the cost function given by*

$$\mathcal{T}(\nu_1, \nu_2) = \frac{1}{2}\int_0^{t_f} [c_1\nu_1^2(t) + c_2\nu_2^2(t)]dt , \;\; c_1 > 0, c_2 > 0 .$$

Then the controls which minimize $\mathcal{T}$ and steer the system (2.1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by

$$\nu_1 = \frac{1}{c_1} P_1 \, , \, \nu_2 = \frac{1}{c_2} P_2 \, ,$$

where P_i's are solutions of the following system of differential equations

$$\begin{cases} \dot{P}_1 = \frac{1}{c_2} P_2 P_3 \, , \\ \dot{P}_2 = -\frac{1}{c_1} P_1 P_3 \, , \\ \dot{P}_3 = -\frac{1}{c_1} P_1 P_4 \, , \\ \dot{P}_4 = 0 \, . \end{cases} \tag{2.2}$$

Let us notice that the system (2.2) can be put in the equivalent form

$$\begin{cases} \dot{P}_1 = \frac{1}{c_2} P_2 P_3 \, , \\ \dot{P}_2 = -\frac{1}{c_1} P_1 P_3 \, , \\ \dot{P}_3 = -\frac{k}{c_1} P_1 \, , \end{cases} \tag{2.3}$$

where

$$P_4 = k = constant \, .$$

Now one has

Theorem 2.2. *([5]) The dynamics (2.3) has the Hamilton-Poisson realization*

$$(\mathbb{R}^3, \Pi, H) \, ,$$

where

$$\Pi = \begin{bmatrix} 0 & P_3 & k \\ -P_3 & 0 & 0 \\ -k & 0 & 0 \end{bmatrix}$$

and the Hamiltonian H is given by

$$H(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2 + \frac{1}{2c_2} P_2^2 \,.$$

□

Remark 2.1. The function $C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$C(P_1, P_2, P_3) = -kP_2 + \frac{1}{2} P_3^2 \tag{2.4}$$

is a Casimir of our configuration. □

Let us note that the Hamiltonian H splits into two parts:

$$H = H_1 + H_2 \,,$$

where

$$H_1(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2, \;\; H_2(P_1, P_2, P_3) = \frac{1}{2c_2} P_2^2 \,.$$

Then the Gauss-Legendre integrators ϕ_{H_1}, ϕ_{H_2} associated to the dynamics generated by H_1 and H_2 are given respectively by

(i) $\phi_{H_1}(P_1^n, P_2^n, P_3^n) = (P_1^{n+1}, P_2^{n+1}, P_3^{n+1})$, where

$$\begin{cases} P_1^{n+1} = P_1^n \\ P_2^{n+1} = P_2^n - \dfrac{ah}{c_1} P_3^n + \dfrac{a^2 h^2 k}{2c_1^2} \\ P_3^{n+1} = P_3^n - \dfrac{ahk}{c_1} \,, \end{cases} \tag{2.5}$$

$$a = P_1(0)$$

and h is the size of the integrator;

(ii) $\phi_{H_2}(P_1^n, P_2^n, P_3^n) = (P_1^{n+1}, P_2^{n+1}, P_3^{n+1})$, where

$$\begin{cases} P_1^{n+1} = P_1^n + \dfrac{hbc}{c_2} \\ P_2^{n+1} = P_2^n \\ P_3^{n+1} = P_3^n \,, \end{cases} \tag{2.6}$$

$$b = P_2(0),$$

$$c = P_3(0)$$

and h is the size of the integrator.

Now one can prove

Theorem 2.3. (i) *The integrators ϕ_{H_1}, ϕ_{H_2} are Poisson integrators.* (ii) *The integrators ϕ_{H_1}, ϕ_{H_2} preserve the Casimir C of our configuration, given by*

$$C(P_1, P_2, P_3) = kP_2 + \frac{1}{2}P_3^2\,.$$

Proof. (i) Let $\phi_{H_1}(P_1^n, P_2^n, P_3^n) = (P_1^{n+1}, P_2^{n+1}, P_3^{n+1})$. Then one has

$$\begin{bmatrix} \frac{\partial P_1^{n+1}}{\partial P_1^n} & \frac{\partial P_1^{n+1}}{\partial P_2^n} & \frac{\partial P_1^{n+1}}{\partial P_3^n} \\ \frac{\partial P_2^{n+1}}{\partial P_1^n} & \frac{\partial P_2^{n+1}}{\partial P_2^n} & \frac{\partial P_2^{n+1}}{\partial P_3^n} \\ \frac{\partial P_3^{n+1}}{\partial P_1^n} & \frac{\partial P_3^{n+1}}{\partial P_2^n} & \frac{\partial P_3^{n+1}}{\partial P_3^n} \end{bmatrix} \begin{bmatrix} 0 & P_3^n & k \\ -P_3^n & 0 & 0 \\ -k & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial P_1^{n+1}}{\partial P_1^n} & \frac{\partial P_1^{n+1}}{\partial P_2^n} & \frac{\partial P_1^{n+1}}{\partial P_3^n} \\ \frac{\partial P_2^{n+1}}{\partial P_1^n} & \frac{\partial P_2^{n+1}}{\partial P_2^n} & \frac{\partial P_2^{n+1}}{\partial P_3^n} \\ \frac{\partial P_3^{n+1}}{\partial P_1^n} & \frac{\partial P_3^{n+1}}{\partial P_2^n} & \frac{\partial P_3^{n+1}}{\partial P_3^n} \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{ah}{c_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & P_3^n & k \\ -P_3^n & 0 & 0 \\ -k & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{ah}{c_1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & P_3^n - \frac{ahk}{c_1} & k \\ -P_3^n + \frac{ahk}{c_1} & 0 & 0 \\ -k & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & P_3^{n+1} & k \\ -P_3^{n+1} & 0 & 0 \\ -k & 0 & 0 \end{bmatrix},$$

as required.

In a similar manner one can prove that ϕ_{H_2} is a Poisson integrator.

(ii) Let $\phi_{H_1}(P_1^n, P_2^n, P_3^n) = (P_1^{n+1}, P_2^{n+1}, P_3^{n+1})$. Then, we have successively:

$$\begin{aligned} C(P_1^{n+1}, P_2^{n+1}, P_3^{n+1}) &= -kP_2^{n+1} + \frac{1}{2}(P_3^{n+1})^2 \\ &= -k(P_2^n - \frac{ah}{c_1}P_3^n + \frac{a^2h^2k}{2c_1^2}) + \frac{1}{2}(P_3^n - \frac{ahk}{c_1})^2 \\ &= -kP_2^n + \frac{1}{2}(P_3^n)^2 \\ &= C(P_1^n, P_2^n, P_3^n)\,. \end{aligned}$$

Also, one has

$$C(\phi_{H_2}(P_1^n, P_2^n, P_3^n)) = C(P_1^n, P_2^n, P_3^n)\,,$$

so that one can conclude that the integrators (2.5) and (2.6) preserve the Casimir C.

□

As a consequence one obtains

Corollary 2.1. *The splitting Gauss-Legendre integrator*

$$\phi = \phi_{H_1} \circ \phi_{H_2}$$

has the following properties:

(i) *It is a Poisson integrator;*

(ii) *It preserves the Casimir C.*

□

Remark 2.2. It is easy to see that the integrators ϕ_{H_1}, ϕ_{H_2} and ϕ do not preserve the energy of our dynamics. □

3. The ball-plate problem and the splitting Gauss-Legendre integrator. The ball-plate problem can be formulated as an optimal control problem on a Lie group (see [2],[6]). More exactly, let $G = \mathbb{R}^2 \times SO(3)$ be the Lie group with the semidirect product group structure. Its Lie algebra can be identified with $\mathbb{R}^5$ and let $\{A_1, A_2, A_3, A_4, A_5\}$, where

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

be its canonical basis.

Then the dynamics of the ball-plate problem can be described by the following left invariant control system on $\mathbb{R}^2 \times SO(3)$:

$$\dot{X} = X[u_1(A_4 - A_2) + u_2(A_5 + A_1)]\,. \tag{3.1}$$

Theorem 3.1. ([2],[6]) *Let $\mathcal{T}$ be the cost function given by*

$$\mathcal{T}(u_1, u_2) = \frac{1}{2}\int_0^{t_f} (u_1^2(t) + u_2^2(t))dt\,.$$

Then the controls which minimize $\mathcal{T}$ and steer the system (3.1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by

$$\begin{cases} u_1 = P_4 - P_2 \\ u_2 = P_5 + P_1\,, \end{cases}$$

where P_i's are solutions of the differential system

$$\begin{cases} \dot{P}_1 = -P_3(P_2 - P_4) \\ \dot{P}_2 = P_3(P_1 + P_5) \\ \dot{P}_3 = -P_2P_5 - P_1P_4 \\ \dot{P}_4 = 0 \\ \dot{P}_5 = 0\,. \end{cases} \tag{3.2}$$

□

Following [6] (supposing $P_4 = 1, P_5 = 0$) the differential system (3.2) takes the form

$$\begin{cases} \dot{P}_1 = P_3(1 - P_2) \\ \dot{P}_2 = P_1P_3 \\ \dot{P}_3 = -P_1\,. \end{cases} \tag{3.3}$$

Then one gets

Theorem 3.2. *([6]) The differential system (3.3) has the Hamilton-Poisson realization*

$$(\mathbb{R}^3, \Pi_{\_}, H)\,,$$

where Π_- *is the matrix of the minus-Lie-Poisson structure on* $\mathbb{R}^3$*, i.e.*

$$\Pi_- = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix},$$

and the Hamiltonian H *is given by*

$$H(P_1, P_2, P_3) = \frac{1}{2}(1 - P_2)^2 + \frac{1}{2}P_1^2 .$$

Remark 3.1. It is not hard to see that the function $C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$C(P_1, P_2, P_3) = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2)$$

is a Casimir of our configuration. □

The Hamiltonian H splits as follows:

$$H = H_1 + H_2 ,$$

where

$$H_1(P_1, P_2, P_3) = \frac{1}{2}P_1^2, H_2(P_1, P_2, P_3) = \frac{1}{2}(1 - P_2)^2 .$$

Then the Gauss-Legendre integrators ϕ_{H_1} and ϕ_{H_2} corresponding to the dynamics generated by H_1 and H_2 are given respectively by

(i) $\phi_{H_1}(P_1^k, P_2^k, P_3^k) = (P_1^{k+1}, P_2^{k+1}, P_3^{k+1})$, where

$$\begin{cases} P_1^{k+1} = P_1^k , \\ P_2^{k+1} = \dfrac{4 - h^2\alpha^2}{4 + h^2\alpha^2}P_2^k + \dfrac{4h\alpha}{4 + h^2\alpha^2}P_3^k , \\ P_3^{k+1} = -\dfrac{4h\alpha}{4 + h^2\alpha^2}P_2^k + \dfrac{4 - h^2\alpha^2}{4 + h^2\alpha^2}P_3^k , \end{cases}$$

$$\alpha = P_1(0)$$

and h is the size step of the integrator;

(ii) $\phi_{H_2}(P_1^k, P_2^k, P_3^k) = (P_1^{k+1}, P_2^{k+1}, P_3^{k+1})$, where

$$\begin{cases} P_1^{k+1} = \dfrac{4-h^2\beta^2}{4+h^2\beta^2}P_1^k - \dfrac{4h\beta}{4+h^2\beta^2}P_3^k\,, \\ \\ P_2^{k+1} = P_2^k\,, \\ \\ P_3^{k+1} = \dfrac{4h\beta}{4+h^2\beta^2}P_1^k + \dfrac{4-h^2\beta^2}{4+h^2\beta^2}P_3^k\,, \end{cases}$$

$$\beta = P_2(0) - 1$$

and h is the size step of the integrator.

Now one can prove

Theorem 3.3. (i) ϕ_{H_1}, ϕ_{H_2} *are Poisson integrators.*
(ii) *The restrictions of the integrators* ϕ_{H_1}, ϕ_{H_2} *to the coadjoint orbits* $(\mathcal{O}_r, \omega_r)$, *where*

$$\mathcal{O}_r = \{(P_1, P_2, P_3) \in \mathbb{R}^3 | P_1^2 + P_2^2 + P_3^2 = r^2\}\,,$$

and

$$\omega_r = -\frac{1}{r}(P_2 dP_1 \wedge dP_3 - P_3 dP_1 \wedge dP_2 - P_1 dP_2 \wedge dP_3)\,,$$

give rise to symplectic integrators.

Proof. (i) Let $\phi_{H_1}(P_1^k, P_2^k, P_3^k) = (P_1^{k+1}, P_2^{k+1}, P_3^{k+1})$. One gets successively

$$\begin{bmatrix} \dfrac{\partial P_1^{k+1}}{\partial P_1^k} & \dfrac{\partial P_1^{k+1}}{\partial P_2^k} & \dfrac{\partial P_1^{k+1}}{\partial P_3^k} \\ \dfrac{\partial P_2^{k+1}}{\partial P_1^k} & \dfrac{\partial P_2^{k+1}}{\partial P_2^k} & \dfrac{\partial P_2^{k+1}}{\partial P_3^k} \\ \dfrac{\partial P_3^{k+1}}{\partial P_1^k} & \dfrac{\partial P_3^{k+1}}{\partial P_2^k} & \dfrac{\partial P_3^{k+1}}{\partial P_3^k} \end{bmatrix} \begin{bmatrix} 0 & -P_3^k & P_2^k \\ P_3^k & 0 & -P_1^k \\ -P_2^k & P_1^k & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial P_1^{k+1}}{\partial P_1^k} & \dfrac{\partial P_1^{k+1}}{\partial P_2^k} & \dfrac{\partial P_1^{k+1}}{\partial P_3^k} \\ \dfrac{\partial P_2^{k+1}}{\partial P_1^k} & \dfrac{\partial P_2^{k+1}}{\partial P_2^k} & \dfrac{\partial P_2^{k+1}}{\partial P_3^k} \\ \dfrac{\partial P_3^{k+1}}{\partial P_1^k} & \dfrac{\partial P_3^{k+1}}{\partial P_2^k} & \dfrac{\partial P_3^{k+1}}{\partial P_3^k} \end{bmatrix}^t =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{4-h^2\alpha^2}{4+h^2\alpha^2} & \dfrac{4h\alpha}{4+h^2\alpha^2} \\ 0 & -\dfrac{4h\alpha}{4+h^2\alpha^2} & \dfrac{4-h^2\alpha^2}{4+h^2\alpha^2} \end{bmatrix} \begin{bmatrix} 0 & -P_3^k & P_2^k \\ P_3^k & 0 & -P_1^k \\ -P_2^k & P_1^k & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{4-h^2\alpha^2}{4+h^2\alpha^2} & \dfrac{4h\alpha}{4+h^2\alpha^2} \\ 0 & -\dfrac{4h\alpha}{4+h^2\alpha^2} & \dfrac{4-h^2\alpha^2}{4+h^2\alpha^2} \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & -P_3^{k+1} & P_2^{k+1} \\ P_3^{k+1} & 0 & -P_1^{k+1} \\ -P_2^{k+1} & P_1^{k+1} & 0 \end{bmatrix},$$

so that ϕ_{H_1} is a Poisson integrator.

In a similar manner one can prove that ϕ_{H_2} is a Poisson integrator.

(ii) Indeed, if $\phi_{H_i}(P_1^k, P_2^k, P_3^k) = (P_1^{k+1}, P_2^{k+1}, P_3^{k+1})$, (i=1,2), then one has

$$(P_1^{k+1})^2 + (P_2^{k+1})^2 + (P_3^{k+1})^2 = (P_1^k)^2 + (P_2^k)^2 + (P_3^k)^2$$

and

$$\frac{1}{r}(P_2^{k+1}dP_1^{k+1} \wedge dP_3^{k+1} - P_3^{k+1}dP_1^{k+1} \wedge dP_2^{k+1} - P_1^{k+1}dP_2^{k+1} \wedge dP_3^{k+1})$$

$$= \frac{1}{r}(P_2^k dP_1^k \wedge dP_3^k - P_3^k dP_1^k \wedge dP_2^k - P_1^k dP_2^k \wedge dP_3^k),$$

as required.

□

As a consequence one gets

Corollary 3.1. *The splitting Gauss-Legendre integrator*

$$\phi = \phi_{h_1} \circ \phi_{H_2}$$

has the following properties:

(i) *It is a Poisson integrator.*

(ii) *Its restriction to each coadjoint orbit* $(\mathcal{O}_r, \omega_r)$ *gives rise to a symplectic integrator.*

□

Remark 3.2. It is not difficult to see that the integrators ϕ_{H_1}, ϕ_{H_2} and ϕ do not preserve the energy of our dynamics. □

4. The splitting Gauss-Legendre integrator for the Maxwell-Bloch equations with a quadratic control about Ox_1 axis. The Maxwell-Bloch equations with a quadratic control about Ox_1 axis can be written in the form

$$\begin{cases} \dot{x}_1 = x_2 - kx_2x_3\,, \\ \dot{x}_2 = x_1x_3\,, \\ \dot{x}_3 = -x_1x_2\,, \end{cases} \tag{4.1}$$

where $k \in \mathbb{R}$ is the feedback gain parameter (see [8]). Then one has

Theorem 4.1. *([8]) The differential system (4.1) has the Hamilton-Poisson realization*

$$(\mathbb{R}^3, \Pi_{MB}, H)\,,$$

where

$$\Pi_{MB} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix},$$

and the Hamiltonian H is given by

$$H(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + \frac{k}{2}x_2^2 + x_3\,.$$

□

Let us notice that the Hamiltonian H splits as follows

$$H = H_1 + H_2 + H_3\,,$$

where

$$\begin{aligned} H_1(x_1, x_2, x_3) &= \frac{1}{2}x_1^2\,, \\ H_2(x_1, x_2, x_3) &= \frac{k}{2}x_2^2\,, \\ H_3(x_1, x_2, x_3) &= x_3\,. \end{aligned}$$

Then the Gauss-Legendre integrators $\phi_{H_1}, \phi_{H_2}, \phi_{H_3}$ corresponding to the dynamics generated by H_1, H_2 and H_3 are given respectively by

(i) $\phi_{H_1}(x_1^n, x_2^n, x_3^n) = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1})$, where

$$\begin{cases} x_1^{n+1} = x_1^n\,, \\ x_2^{n+1} = \dfrac{4-h^2a^2}{4+h^2a^2}x_2^n + \dfrac{4ha}{4+h^2a^2}x_3^n\,, \\ x_3^{n+1} = -\dfrac{4ha}{4+h^2a^2}x_2^n + \dfrac{4-h^2a^2}{4+h^2a^2}x_3^n\,, \end{cases}$$

$$a = x_1(0) = constant$$

and h is the size step of the integrator;

(ii) $\phi_{H_2}(x_1^n, x_2^n, x_3^n) = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1})$, where

$$\begin{cases} x_1^{n+1} = x_1^n - hkbc\,, \\ x_2^{n+1} = x_2^n\,, \\ x_3^{n+1} = x_3^n\,, \end{cases}$$

$$b = x_2(0) = constant$$

$$c = x_3(0) = constant$$

and h is the size step of the integrator;

(iii) $\phi_{H_3}(x_1^n, x_2^n, x_3^n) = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1})$, where

$$\begin{cases} x_1^{n+1} = x_1^n - hb\,, \\ x_2^{n+1} = x_2^n\,, \\ x_3^{n+1} = x_3^n\,, \end{cases}$$

$$b = x_2(0) = constant$$

and h is the size step of the integrator.

Now one can prove

Theorem 4.2. *(i) $\phi_{H_1}, \phi_{H_2}, \phi_{H_3}$ are Poisson integrators.*
(ii) The restrictions of the integrators $\phi_{H_1}, \phi_{H_2}, \phi_{H_3}$ to the coadjoint orbits $(\mathcal{O}_r, \omega_r)$, where

$$\mathcal{O}_r = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2^2 + x_3^2 = r^2\}$$

and

$$\omega_r = \frac{1}{r}(x_3 dx_1 \wedge dx_2 + x_2 dx_1 \wedge dx_3)\,,$$

give rise to symplectic integrators. □

As a consequence one obtains **Corollary 4.1.** *The splitting Gauss-Legendre integrator*

$$\phi = \phi_{H_1} \circ \phi_{H_2} \circ \phi_{H_3}$$

has the following properties:

(i) *It is a Poisson integrator;*

(ii) *Its restriction to each coadjoint orbit* $(\mathcal{O}_r, \omega_r)$ *gives rise to a symplectic integrator.*

□

Remark 4.1. It is easy to see that the integrators $\phi_{H_1}, \phi_{H_2}, \phi_{H_3}$ and ϕ are not energy preserving of our dynamics. □

Remark 4.2. In the particular case $k = 0$ we rediscover some results from [1]. □

REFERENCES

1. Craioveanu, M., Mos, I., Puta, M. and Voitecovici, A. – *The splitting midpoint integrators for the Maxwell-Bloch equations and Toda lattice equations*, The Proceedings of the fifth International Workshop on Differential Geometry and its Applications, Timişoara, September 18-22, 2001.

2. Jurdjevic, V. – *The geometry of the ball-plate problem*, Arch. Rat. Mech. Anal. 124(1993), 305-328.

3. Leonard, N. – *Averaging and motion control systems on Lie groups*, Ph. D. Thesis, University of Maryland, College Park, MD, 1994.

4. Li, S., Qin, M. – *Lie-Poisson integration for rigid body dynamics*, Computers Math. Appl. vol. 20, no. 9(1995), 105-118.

5. Puta, M. – *Geometrical aspects in the kinematic car's dynamics*, in the Mathematical Theory of Networks and Systems, Proc. of the MTNS 98, Symposium held in Padova, Italy, July 1998, eds. A.Beghi, L.Finesso, G.Picci, 1063-1066, Il Poligrafo, Padova.

6. Puta, M., Butur, M. and Moş, I. – *Some remarks on the ball-plate problem*, An. Univ. Timişoara (to appear).

7. Puta, M., Lăzureanu, C. and Moş, I. – *The complete integrability and geometric prequantization for two optimal control problems*, An Univ. Timişoara (to appear).

8. Puta, M., Butur, M., Goldenthal, G., Moş, I. and Rujescu, C. – *Maxwell-Bloch equations with a quadratic control about Ox_1 axis*, in the volume "Geometry, Integrability and Quantization" II, eds. I.Mladenov and L.Naber, June 7-15, 2000, Varna, Bulgaria, 280-286, Coral Press, Sofia.

Received: 15.XI.2002

Seminarul de Geometrie şi Topologie,
The West University of Timişoara,
B-dul V.Pârvan 4,
1900-Timişoara,
ROMÂNIA