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AN INTEGRAL FOR MULTIFUNCTIONS WITH RESPECT TO A MULTIMEASURE

BY

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Abstract. In this paper we consider an integral for multifunctions with respect to a multimeasure. In [6] we built such an integral in the case of a real Banach algebra, using the way of defining sequences, analogously to that done in DUNFORD and SCHWARTZ [8]. In the same manner, we give here a set - valued integral, but in the case of a locally convex algebra. We also get convergence theorems of Vitaly and Lebesgue type.

1. Terminology and notations. Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex vector space and let Q a filtering family of seminorms which defines the topology of X.

We consider $(x, y) \mapsto xy$ having the following properties:

$$\left.\begin{array}{l}\text{(i) } x(yz) = (xy)z,\\ \text{(ii) } xy = yx,\\ \text{(iii) } x(y+z) = xy + xz,\\ \text{(iv) } (\alpha x)(\beta y) = (\alpha\beta)(xy),\end{array}\right\} \forall\, x, y, z \in X, \alpha, \beta \in \mathbb{R};$$

$$\text{(v) } p(xy) \leq p(x)p(y),\ \forall x, y \in X, p \in Q.$$

Examples of such spaces X are:

1) $X = \{f : T \to \mathbb{R} | f \text{ is bounded } \}$ where T is a topological space. Let $\mathcal{K} = \{K \subseteq T | K \text{ is compact } \}$ and $Q = \{p_k | K \in \mathcal{K}\}$ where $p_K(f) = \sup_{t \in K} |f(t)|, \forall f \in X$.

2) $X = \{f : T \to \mathbb{R}\}$ where T is a nonempty set and $Q = \{p_t | t \in T\}, p_t(f) = |f(t)|,\ \forall f \in X$.

$\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ is the family of all nonempty compact convex subsets of X. If $A, B \in \mathcal{P}_{kc}, \alpha \in \mathbb{R}$,

$$A + B = \{x + y | x \in A, y \in B\}$$
$$\alpha A = \{\alpha x | x \in A\}$$
$$A \cdot B = \{xy | x \in A, y \in B\}.$$

For every $p \in Q, A, B \in \mathcal{P}_{kc}$, let $e_p(A, B) = \sup_{x \in A} \inf_{y \in B} p(x-y)$ and $h_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}$ - the Hausdorff semimetric defined by p on $\mathcal{P}_{kc}$. We define $\|A\|_p = h_p(A, 0),\ \forall A \in \mathcal{P}_{kc}$, where $0 = \{0\}$. Then $\{h_p\}_{p \in Q}$ is a filtering family of semimetrics on $\mathcal{P}_{kc}$ which defines a Hausdorff topology on $\mathcal{P}_{kc}$.

Lemma 1.1. *For every $A, B, C \in \mathcal{P}_{kc}, h_p(A \cdot B, A \cdot C) \leq \|A\|_p \cdot h_p(B, C)$, $\forall p \in Q$.*

Let $Y \subseteq \mathcal{P}_{kc}$ satisfying the conditions:

(y_1) Y is complet with respect to $\{h_p\}_{p \in Q}$;

(y_2) $0 \in Y$;

(y_3) $\forall A, B \in Y \Rightarrow A + B, A \cdot B \in Y$;

(y_4) $\forall A, B, C \in Y \Rightarrow A \cdot (B + C) = A \cdot B + A \cdot C$.

It is easy to see that the following classes:

1. $Y = \{\{x\} | x \in X\}$ for X like above and
2. $Y = \{A \in \mathcal{P}_{kc} | A \subset \mathbb{R}_+\}$ for $X = \mathbb{R}$

are such subsets Y of $\mathcal{P}_{kc}$.

Definition 1.2. Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$. φ is said to be a multimeasure if

(i) $\varphi(\phi) = 0$;

(ii) $\varphi(A \cup B) = \varphi(A) + \varphi(B),\ \forall A, B \in \mathcal{A}, A \cap B = \phi$.

Definition 1.3. Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$. For every $p \in Q$, the p - variation of φ is the non - negative (possibly infinite) set function $v_p(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$v_p(\varphi, A) = \sup \left\{ \sum_{i=1}^{n} \|\varphi(E_i)\|_p; \right.$$
$$\left. (E_i)_{i=1}^{n} \subset \mathcal{A}, E_i \cap E_j = \emptyset, i \neq j, \bigcup_{i=1}^{n} E_i = A \right\}, \ \forall A \in \mathcal{A}.$$

If φ is a multimeasure, then $v_p(\varphi, \cdot)$ is finitely additive.

Throughout this paper, $\varphi : \mathcal{A} \to Y$ will be a multimeasure and denote $v_p(\varphi, \cdot)$ with $\mathcal{V}_p$. We shall assume there is $\alpha > 0$ such that $\mathcal{V}_p(S) \leq \alpha, \ \forall p \in Q$.

2. Simple multifunctions.

Definition 2.1. A multifunction $F : S \to Y$ is said to be a simple multifunction if $F = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$, where $C_i \in Y, A_i \in \mathcal{A}, i = 1, \cdots, n, A_i \cap A_j = \phi (i \neq j), \bigcup_{i=1}^{n} A_i = S$ and 1_{A_i} is the characteristic function of A_i.

Definition 2.2. If $F : S \to Y$ is a simple multifunction $F = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$, then the integral of F over E is defined to be

$$\int_E F d\varphi = \sum_{i=1}^{n} C_i \cdot \varphi(A_i \cap E) \in Y, E \in \mathcal{A}$$

Since (y_4) the integral of F is independent of the representation chosen for F.

Using a classical reasoning (see [8]) the following statements are obtained.

Theorem 2.3. *Let $F : S \to Y$ a simple multifunction and $\Gamma(E) = \int_E f d\varphi, \ \forall E \in \mathcal{A}$. Then Γ is a multimeasure*

Theorem 2.4. *Let $F, G : S \to Y$ be simple multifunctions. We have:*

$$a) \ h_p(\int_E F d\varphi, \int_E G d\varphi) \leq \int_E h_p(F, G) d\mathcal{V}_p, \forall E \in \mathcal{A}, p \in Q,$$

where $h_p(F,G)$ *is the function defined as* $(h_p(F,G))(s) = h_p(F(s),G(s))$, $\forall s \in S$.

b) If $\Gamma(E) = \int\limits_E F d\varphi$, $\forall E \in \mathcal{A}$, *then* $v_p(\Gamma,E) = \int\limits_E \|F\|_p d\mathcal{V}_p$, $\forall E \in \mathcal{A}, p \in Q$, *where* $\|F\|_p$ *is the function defined as* $(\|F\|_p)(s) = \|F(s)\|_p$, $\forall s \in S$.

Corollary 2.5. *If* $F : S \to Y$ *is a simple multifunction, then*

$$\| \int\limits_E F d\varphi \|_p \leq \int\limits_E \|F\|_p d\mathcal{V}_p, \ \forall E \in \mathcal{A}, p \in Q.$$

3. φ - Integrable multifunctions.

Definition 3.1. A multifunction $F : S \to Y$ is said to be φ - totally measurable if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to Y$ satisfying the following conditions for every $p \in Q$:

(i) $h_p(F_n,F)$ is $\mathcal{V}_p$ - measurable (in the sense of [8]), $\forall n \in \mathbb{N}$;

(ii) $h_p(F_n,F)$ converges to 0 in $\mathcal{V}_p$- measure (in the sense of (8)) (denoted $h_p(F_n,F) \xrightarrow{\mathcal{V}_p} 0$).

Proposition 3.2. *Let* $F,G : S \to Y$ *be multifunctions such that* F *is* φ *- totally measurable and* G *is simple. Then* $h_p(F,G)$ *is* $\mathcal{V}_p$ *- measurable* $\forall p \in Q$.

Proof. Suppose $(F_n)_n$ is a sequence of simple multifunctions satisfying (i) and (ii) of Definition 3.1. Then we can easily show that $h_p(F_n,G) \xrightarrow{\mathcal{V}_p} h_p(F,G)$. Since $h_p(F_n,G)$ is a simple function for every $n \in \mathbb{N}$ it follows that $h_p(F,G)$ is $\mathcal{V}_p$ - measurable, $\forall p \in Q$. ■

Definition 3.3. Let $F : S \to Y$ be a φ - totally measurable multifunction. F is said to be φ - integrable (over S) if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to Y$ satisfying the following conditions for every $p \in Q$:

(i) $h_p(F_n,F) \xrightarrow{\mathcal{V}_p} 0$;

(ii) $\lim\limits_{n,m\to\infty} \int\limits_S h_p(F_n,F_m) d\mathcal{V}_p = 0$;

Remark 3.4. From Proposition 3.2. it results that $h_p(F_n, F)$ is $\mathcal{V}_p$ - measurable, $\forall n \in \mathbb{N}, p \in Q$.

Definition 3.5. Let $F : S \to Y$ be a φ - integrable multifunction. The sequence $(F_n)_n$ in definition 3.3. is said to be a defining sequence for F. For every $E \in \mathcal{A}, p \in Q$ we have:

$$h_p(\int_E F_n d\varphi, \int_E F_m d\varphi) \leq \int_E h_p(F_n, F_m) d\mathcal{V}_p \leq \int_S h_p(F_n, F_m) d\mathcal{V}_p.$$

Since $\lim_{n,m\to\infty} \int_S h_p(F_n, F_m) d\mathcal{V}_p = 0$, the sequence $\left\{\int_E F_n d\varphi\right\}_n$ is Cauchy in Y. The completeness of Y implies that $\lim_{n\to\infty} \int_E F_n d\varphi$ exists in Y and this limit is uniformly with respect to $E \in \mathcal{A}$.

Define $\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi$ to be the integral of F over E, $\forall E \in \mathcal{A}$.

Theorem 3.6. *If $F : S \to Y$ is a φ - integrable multifunction and $\Gamma(E) = \int_E F d\varphi, \forall E \in \mathcal{A}$, then Γ is a multimeasure.*

Proof. It results straightforward from Definition 3.3, 3.5 and Theorem 2.3. In fact, let $(F_n)_n$be a defining sequence for F and $E_1, E_2 \in \mathcal{A}, E_1 \cap E_2 = \emptyset$. Then we have

$$\Gamma(E_1 \cup E_2) = \int_{E_1 \cup E_2} F d\varphi = \lim_{n\to\infty} \int_{E_1 \cup E_2} F_n d\varphi = \lim_{n\to\infty} \int_{E_1} F_n d\varphi + \lim_{n\to\infty} \int_{E_2} F_n d\varphi =$$
$$= \int_{E_1} F d\varphi + \int_{E_2} F d\varphi = \Gamma(E_1) + \Gamma(E_2).$$

Proposition 3.7. *Let $F : S \to Y$ be a φ - integrable multifunction with defining sequence $(F_n)_n$. Then for each $p \in Q$ we have:*

a) The function $\|F\|_p$ is $\mathcal{V}_p$ - integrable and $\int_E \|F\|_p \, d\mathcal{V}_p =$

$= \lim_{n\to\infty} \int_E \|F_n\|_p \, d\mathcal{V}_p$, $\forall E \in \mathcal{A}$;

b) $\lim_{n\to\infty}\int\limits_S h_p(F_n,F)d\mathcal{V}_p=0.$

Proof. It follows once again from Definitions 3.3. and 3.5. ■

Remarks 3.8. 1) If $X=\mathbb{R},Y=\{\{x\}|x\in\mathbb{R}\},F=\{f\}(f$ is a function), $\varphi=\{\mu\}(\mu$ is finitely additive) and F is φ - integrable, then $\int\limits_E Fd\varphi=\{\int\limits_E fd\mu\},E\in\mathcal{A}$, where $\int\limits_E fd\mu$ is the Dunford integral [8].

2) If $X=\mathbb{R},Y=\{\{x\}|x\in\mathbb{R}\},F=\{f\}$ (f is a function) and F is φ - integrable, then f is Brooks - integrable with respect to φ and $[B]\int\limits_E fd\varphi=\int\limits_E Fd\varphi,E\in\mathcal{A}$, where $[B]\int\limits_E fd\varphi$ is the Brooks integral [2].

3) If $X=\mathbb{R}$ and $\varphi=\{\mu\}$ (μ is finetely additive), then we get the integral defined by Martellotti - Sambucini ([10]) for F with respect to μ.

We can get the following results.

Theorem 3.9. *Let $F,G:S\to Y$ be φ - integrable multifunctions. Then for every $p\in Q$,*

$$h_p\left(\int\limits_E Fd\varphi,\int\limits_E Gd\varphi\right)\leq\int\limits_E h_p(F,G)d\mathcal{V}_p,\ \forall E\in\mathcal{A}.$$

Theorem 3.10. *If $F:S\to Y$ is a φ - integrable multifunction, then for every $p\in Q$,*

$$\|\int\limits_E Fd\varphi\|_p\leq\int\limits_E\|F\|_p d\mathcal{V}_p,\ \forall E\in\mathcal{A}.$$

If we set $\Gamma(E)=\int\limits_E Fd\varphi,\ \forall E\in\mathcal{A}$, then $v_p(\Gamma,E)=\int\limits_E\|F\|_p d\mathcal{V}_p,\ \forall E\in\mathcal{A}$.

Theorem 3.11. *If $F:S\to Y$ is a φ - integrable multifunction and $\Gamma(E)=\int\limits_E Fd\varphi,\ \forall E\in\mathcal{A}$, then $\Gamma\ll\mathcal{V}_p,\ \forall p\in Q$ (i.e. $\forall p\in Q,\forall\varepsilon>0,\exists\delta(\varepsilon,p)>0$ such that $v_p(\Gamma,E)<\varepsilon$ for all $E\in\mathcal{A}$ with $\mathcal{V}_p(E)<\delta$).*

The definition of integral (3.5) is consistent:

Theorem 3.12. *Let $(F_n)_n, (G_n)_n$ be two defining sequences for a φ - integrable multifunction $F : S \to Y$. Then:*

$$\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi = \lim_{n\to\infty} \int_E G_n d\varphi, \forall E \in \mathcal{A}.$$

Proof. From Definitions 3.3 and 3.5 it follows that $h_p(F_n, G_n) \xrightarrow{\mathcal{V}_p} 0$. Let $\varepsilon > 0$ and $A_n = \{s \in S; h_p(F_n(s), G_n(s)) > \varepsilon\}$. Since a) - Theorem 2.4, for all sufficiently large $n > n_0$, we have:

$$h_p(\int_E F_n d\varphi, \int_E G_n d\varphi) \leq \int_E h_p(F_n, G_n) d\mathcal{V}_p =$$

$$= \int_{E\cap A_n} h_p(F_n, G_n) d\mathcal{V}_p + \int_{E\setminus A_n} h_p(F_n, G_n) d\mathcal{V}_p \leq \tag{1}$$

$$\leq \int_{E\cap A_n} h_p(F_n, G_n) d\mathcal{V}_p + \varepsilon\alpha.$$

Using the Vitali - Hahn - Saks Theorem, by a standard proof, it results that $\lim_{n\to\infty} \int_{E\cap A_n} h_p(F_n, G_n) d\mathcal{V}_p = 0$. So, from (1), we obtain $\lim_{n\to\infty} h_p\left(\int_E F_n d\varphi, \int_E G_n d\varphi\right) = 0, \forall p \in Q$ and the theorem is proved. ■

Definition 3.13. A finite or countable family of pairwise disjoint sets $(E_i)_i \subset \mathcal{A}$ will be called a Q - uniformly exhaustion of S is for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\mathcal{V}_p(S \setminus \bigcup_{i=1}^{n_0} E_i) < \varepsilon,\ \forall p \in Q$.

Proposition 3.14. *Let $\{E_i\}_{i\in\mathbb{N}^*} \subset \mathcal{A}$ be a Q - uniformly exhaustion of S such that $S = \bigcup_{i=1}^{\infty} E_i$. Let $(C_n)_{n\in\mathbb{N}^*} \subset Y$ such that $\forall p \in Q, \exists r_p > 0$ such that $\|C_n\|_p \leq r_p, \forall n \in \mathbb{N}^*$. We define $F = \sum_{i=1}^{\infty} C_i \cdot 1_{E_i}$ (in the sense that for every $s \in S$ there exists a unique $i = i(s) \in \mathbb{N}^*$ such that $s \in E_{i(s)}$ and so $F(s) = C_{i(s)}$).*

Then F is φ integrable.

Proof. It easily results that $F_n = \sum_{i=1}^{n} C_i \cdot 1_{E_i}, n \in \mathbb{N}^*$, is a defining sequence for F. ■

Definition 3.15. A multifunction $F : S \to Y$ is said to be strong φ - integrable if there is a sequence $(F_n)_n$ of simple multifunctions such that:

$$\left.\begin{array}{l}(i)\ h_p(F_n, F) \xrightarrow{\mathcal{V}_p} 0; \\ (ii)\ \lim\limits_{n,m\to\infty} \int_S h_p(F_n, F_m) d\mathcal{V}_p = 0;\end{array}\right\} \text{uniformly with respect to } p \in Q.$$

Remarks 3.16.

a) From (ii) - Definition 3.15. it follows that $h_p(F_n, F)$ is $\mathcal{V}_p$- measurable, $\forall n \in \mathbb{N}, p \in Q$.

b) Let $F = \sum_{i=1}^{\infty} C_i \cdot 1_{E_i}$, where $\{E_i\}_{i\in\mathbb{N}^*}$ is a Q - uniformly exhaustion for S and $(C_n)_{n\in\mathbb{N}^*} \subset Y$ has the property: $\exists r > 0$ such that $\|C_n\|_p \leq r,\ \forall n \in \mathbb{N}^*, p \in Q$. Then F is strong φ - integrable.

Theorem 3.17(Vitali). *Let $F : S \to Y$ be a multifunction, $(F_n)_n$ be a sequence of strong φ - integrable multifunctions $F_n : S \to Y$ and $\Gamma_n(E) = \int_E F_n d\varphi,\ \forall E \in \mathcal{A}, n \in \mathbb{N}$ such that, for every $p \in Q$ we have:*

(i) $h_p(F_n, F) \xrightarrow{\mathcal{V}_p} 0$;

(ii) $\Gamma_n \ll \mathcal{V}_p$ *uniformly in* $n \in \mathbb{N}$.

Then F is φ - integrable and $\int_E F d\varphi = \lim\limits_{n\to\infty} \int_E F_n d\varphi,\ \forall E \in \mathcal{A}$.

Proof. Since (i) we have $\lim\limits_{n,m\to\infty} \mathcal{V}_p(\{s \in S; h_p(F_n(s), F_m(s)) > \varepsilon\}) = 0,\ \forall \varepsilon > 0$.

From (ii) it follows that the sequence $(\int\limits_E F_n d\varphi)_n$ is Cauchy in Y and consequently it is convergent, $\forall E \in \mathcal{A}$. Since F_n is strong φ - integrable for all n, there is a defining sequence $(G_k^n)_{k\in\mathbb{N}}$ for F_n such that:

$$\left.\begin{array}{ll} (2) & h_p(G_k^n, F_n) \xrightarrow{\mathcal{V}_p} 0 \\ (3) & \lim\limits_{k,l\to\infty} \int\limits_S h_p(G_k^n, G_l^n) d\mathcal{V}_p = 0 \end{array}\right\} \text{uniformly in } p \in Q.$$

Let $H_{n,k}^p = \left\{s \in S; h_p(G_k^n(s), F_n(s)) > \frac{1}{2^n}\right\}$. Then uniformly in $p \in Q$, for all $n \in \mathbb{N}$ there is $\overline{k}(n) \in \mathbb{N}$ such that $\mathcal{V}_p(H_{n,k}^p) < \frac{1}{2^n}$ and $\int\limits_S h_p(G_k^n, F_n) d\mathcal{V}_p \leq \dfrac{1}{2^n}, \forall k \geq \overline{k}$.

Let $B_n^p = H_{n,\overline{k}(n)}^p$. Then $\lim\limits_{n\to\infty} \mathcal{V}_p(B_n^p) = 0$ uniformly in $p \in Q$ and if we put $G_n = G_{\overline{k}(n)}^n$, then $h_p(G_n(s), F(s)) \leq \frac{1}{2^n},\ \forall s \in cB_n^p$. Since $|h_p(G_m, G_n) - h_p(G_m, F)| \leq h_p(G_n, F) \leq h_p(G_{\overline{k}(n)}^n, F_n) + h_p(F_n, F), h_p(G_n, F)$ is $\mathcal{V}_p$ - measurable and $h_p(G_n, F) \xrightarrow{\mathcal{V}_p} 0$.

Since for all sufficiently large n, k, $\int\limits_S h_p(G_n, F_n) d\mathcal{V}_p \leq \frac{1}{2^n}$, $\int\limits_S h_p(G_k, F_k) d\mathcal{V}_p \leq \frac{1}{2^k}$ and from (ii), $\int\limits_S h_p(F_n, F_k) d\mathcal{V}_p < \varepsilon$, we have:

$$\int\limits_S h_p(G_n, G_k) d\mathcal{V}_p \leq \int\limits_S h_p(G_n, F_n) d\mathcal{V}_p + \int\limits_S h_p(F_n, F_k) d\mathcal{V}_p + \\ + \int\limits_S h_p(F_k, G_k) d\mathcal{V}_p < \frac{1}{2^n} + \varepsilon + \frac{1}{2^k}$$

so $\lim\limits_{n,k\to\infty} \int\limits_S h_p(G_n, G_k) d\mathcal{V}_p = 0$. Thus F is φ - integrable and

$$(4) \qquad \int\limits_E F d\varphi = \lim_{n\to\infty} \int\limits_E G_n d\varphi,\ \forall E \in \mathcal{A}.$$

From theorem 3.9. it follows $h_p(\int\limits_E G_n d\varphi, \int\limits_E F_n d\varphi) \leq \int\limits_E h_p(G_n, F_n) d\mathcal{V}_p \leq \frac{1}{2^n}$, therefore $\lim\limits_{n\to\infty} h_p\left(\int\limits_E G_n d\varphi, \int\limits_E F_n d\varphi\right) = 0,\ \forall p \in Q$ which is equivalently

to

$$\lim_{n\to\infty}\int_E F_n d\varphi = \lim_{n\to\infty}\int_E G_n d\varphi. \tag{5}$$

From (4) and (5) it results $\int_E F d\varphi = \lim_{n\to\infty}\int_E F_n d\varphi,\ \forall E \in \mathcal{A}$. This completes the proof. ■

As a consequence of Vitali Theorem we have a Lebesgue type theorem.

Theorem 3.18(Lebesgue). *Let $(F_n)_n$ be a sequence of strong φ - integrable multifunctions $F_n : S \to Y$. Let $F : S \to Y$ be a multifunction such that $h_p(F_n, F) \xrightarrow{\mathcal{V}_p} 0,\ \forall p \in Q$. Suppose there exists $g \in L^1(S, \mathcal{A}, \mathcal{V}_p)$ such that $\|F_n(s)\|_p \leq |g(s)|,\ \forall s \in S, n \in \mathbb{N}, p \in Q$. Then F is φ - integrable and $\int_E F d\varphi = \lim_{n\to\infty}\int_E F_n d\varphi,\ \forall E \in \mathcal{A}$.*

Proposition 3.19. *Let $(F_n)_n$ be a sequence of strong φ - integrable multifunctions that converges to F uniformly with respect to $s \in S$ and $p \in Q$. Then F is strong φ - integrable and $\int_E F d\varphi = \lim_{n\to\infty}\int_E F_n d\varphi,\ \forall E \in \mathcal{A}$.*

Proof. We act like in Theorem 3.17(Vitali). Since F_n is strong φ - integrable for all $n \in \mathbb{N}$, there is $(G_k^n)_{k\in\mathbb{N}}$ a defining sequence of simple multifunctions for F_n. Then for every $\varepsilon > 0, n \in \mathbb{N}$, there exists $k(n, \varepsilon) \in \mathbb{N}$ such that, for $k > k(n, \varepsilon)$

$$\{s \in S; h_p(G_k^n(s), F_n(s)) > \varepsilon\} = \emptyset$$

and for n suitable large

$$\{s \in S; h_p(F(s), F_n(s)) > \varepsilon\} = \emptyset \text{ uniformly in } p \in Q.$$

If we take $G_n = G^n_{k(n,\frac{1}{2^n})}$, then $(G_n)_n$ is a defining sequence for F uniformly in $p \in Q$. ■

Examples 3.20.

1) Let $X = \mathbb{R}, Y = \{A \in \mathcal{P}_{kc} | A \subset \mathbb{R}_+\}, \varphi(E) = [\mu_1(E), \mu_2(E)]$ for any $E \in \mathcal{A}$, where μ_1, μ_2 are positive bounded finitely additive and $\mu_1 \leq \mu_2$. Let F be the interval $[f, g]$, where f, g are real functions such that $f \leq g$. Suppose there exists $(f_n)_n, (g_n)_n$ defining sequences (with respect to μ_2) for

f, g respectively, such that $0 \leq f_n \leq g_n, \forall n \in \mathbb{N}$. Then F is φ - integrable and $\int_E F d\varphi = \left[\int_E f d\mu_1, \int_E g d\mu_2\right]$, $E \in \mathcal{A}$.

2) Let $X = \{f : S \to \mathbb{R}\}, Q = \{p_x | x \in S\}$ where $p_x(f) = |f(x)|$, $\forall f \in X$.
Let $Y = \{\{f\} | f \in X\}$ and $\varphi : \mathcal{A} \to Y, \varphi(E) = \{1_E\}$, $\forall E \in \mathcal{A}$.
Let $F : S \to Y$ be a multifunction $F(s) = \{g_s\}, s \in S$, and suppose there is a partition of $S, \{A_i\}_{i=1}^l \subset \mathcal{A}$, and for every $i \in \{1, 2, \cdots, l\}$, there exist $(f_i^n)_n : S \to \mathbb{R}$ satisfying the condition:

$$\sum_{i=1}^{l} f_i^n \cdot 1_{A_i} \text{ is pointwise convergent to } u, \text{ where}$$

$$u : S \to \mathbb{R} \text{ is defined by } u(x) = g_x(x), \forall x \in S.$$

Then F is φ - integrable and $\int_S F d\varphi = \{u\}$.

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