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COMMON FIXED POINT THEOREMS FOR COINCIDENTALLY COMMUTING PAIRS OF NONSELF MAPPINGS IN METRICALLY CONVEX METRIC SPACES

BY

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Abstract. In this paper some common fixed point theorems for coincidentally commuting pairs of nonself single-valued and set-valued maps are proved on closed subsets of a metrically convex metric space under certain weak commutativity condition and the weaker boundary condition than that used in Assad and Kirk [1], Itoh [7], Rhoades [9] and Huang and Cho [6].

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1. Introduction. The fixed point theorems for set - valued contraction mappings are first proved by Nadler [8] in metric spaces. Later, Assad and Kirk [1] first proved some fixed point theorems for contraction nonself mappings of a closed subset of a Banach space. Further Itoh [7] and Rhoades [9] generalized these results to a wider classes of contraction nonself set - mappings on closed subsets of a metrically convex metric space. In the case of fixed point theorems for nonself mappings of a convex space, one needs certain additional boundary condition. See the references [1], [2], [9] and Huang and Cho [6]. In a recent paper [3], the present author proved some fixed point theorems for sequences of nonself set-maps of closed subset of a Banach space, under a slightly stranger condition than that in Huang and Cho [6] but under a weaker boundary condition than that of Assad and Kirk [1], Itoh [7], Rhoades [9] and Huang and Cho [6]. Most recently, some common fixed point theorems have been proved

in DHAGE [4] for pairs of coincidentally commuting single - valued and set - valued selfmappings of a metric space satisfying generalized contraction condition.

In the present paper we combine the two approaches given in [3] and [4] and prove some common fixed point theorems for coincidentally commuting pairs of a nonself point - map and a nonself set - map on a closed subsets of a metrically convex metric space under a weaker boundary condition of the type given in DHAGE [3].

2. Preliminaries. In the sequel, throughout this paper, let X denote a metric space with metric d and let $CB(X)$ denote the class of all non-empty closed and bounded subsets of X.

Definition 2.1. A metric space X is said to be metrically convex if for any $x, y \in X$ with $x \neq y$, there is a point $z \in X$ such that

$$d(x, y) = d(x, z) + d(z, y)$$

We need the following lemma in the sequel.

Lemma 2.1.[2] *If K is a non-empty, closed subset of a metrically convex metric space X, then for any $x \in X$ and $y \in X \backslash K$, there exists a point $z \in \partial K$ such that*

$$d(x, y) = d(x, z) + d(z, y).$$

where ∂K is a boundary of K.

Denote by

$$D(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$$

$$\delta(A, B) = \sup\{d(a, b) | a \in A, b \in B\}$$

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}$$

for any $A, B \in CB(X)$.

The function H is called the Hausdorff metric on $CB(X)$.

It is clear that

$$D(A, B) \leq H(A, B) \leq \delta(A, B).$$

From Fisher [6], we have

(i) $\delta(A,B) \geq 0$ and equality holds iff $A = B = a$ singleton set,

(ii) $\delta(A,B) = \delta(B,A)$

(iii) $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$ for $A, B, C \in CB(X)$.

The following lemmas are useful in the sequel.

Lemma 2.1.[5] *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of sets in $CB(X)$ converging respectively to the sets A and B in $CB(X)$ in the Hausdorff metric H. Then $\lim_n \delta(A_n, B_n) = \delta(A,B)$, and $\lim_n D(A_n, B_n) = D(A,B)$.*

In [3] and [6] the present author proved some fixed point theorems for the set - maps $F : K \subset X \to CB(X)$ satisfying the contraction condition

$$\begin{aligned} \delta(Fx, Fy) &\leq \alpha d(x,y) + \\ &+\beta \max\{1/2[D(x,Fx) + D(y,Fy)], 1/2[D(x,Fy) + D(y,Fx)]\} \end{aligned} \tag{2.1}$$

for all $x, y \in K$, where $\alpha \geq 0, \beta \geq 0$ satisfy

$$2\alpha + \alpha\beta + \beta < 1. \tag{2.2}$$

In this paper we extend the fixed point results in Dhage [3] to a pair of point and set maps on closed subsets of a metrically convex metric space satisfying the contraction condition of the type (2.1) - (2.2) and under the weaker commutativity conditions.

3. Limit coincidentally commuting maps. Let $F : X \to CB(X)$ and $g : X \to X$ be two mappings. Then the pair $\{F, g\}$ of maps is called *limit commuting* if there exists a sequence $\{x_n\}$ in X such that $\lim_n \delta(gFx_n, Fx_n) = 0$, and *commuting* if $\delta(gFx, Fgx) = 0$ for all $x \in X$. Similarly the pair $\{F, g\}$ is called *limit coincident* if there exists a sequence $\{x_n\}$ in X such that $\lim_n \delta(Fx_n, gx_n) = 0$ and *coincident* if there exists a point $x \in X$ such that $\delta(Fx, gx) = 0$. Finally the pair $\{F, g\}$ of maps is called *limit coincidentally commuting* if $\{F, g\}$ is limit commuting, whenever it is limit coincident on X. It is known that limit coincidentally commutativity condition for pairs of maps on metric spaces. See Dhage [4].

Theorem 3.1. *Let K be a non - empty closed and bounded subsets of a metrically convex metric space X and let $F : K \to CB(X)$ and $g : K \to X$*

be two mappings satisfying

$$\delta(Fx, Fy) \leq \alpha d(gx, gy) + \beta \max\{1/2[D(gx, Fx) + D(gy, Fy)], \\ 1/2[D(gx, Fy) + D(gy, Fx)]\} \tag{3.1}$$

for all $x, y \in K$, *where* $\alpha \geq 0, \beta \geq 0$ *satisfy*

$$2\alpha + \alpha\beta + \beta < 1. \tag{3.2}$$

Suppose that

(i) $F(K) \cap K \subseteq g(K)$ *and* $\partial K \subset g(K)$,

(ii) $\{F, g\}$ *are limit coincidentally commuting,*

(iii) any one of F *and* g *is continuous.*

Further if $gx \in \partial K$ *implies* $Fx \cap K \neq \Phi$ *for each* $x \in K$, *then* F *and* g *have a unique fixed point* $z \in K$ *i.e.,* $Fz = \{z\} = gz$, *and if* g *is continuous at* z, *then* F *is also continuous at* z *in the Hausdorff metric* H *on* X.

Proof. Let $y \in \partial K$ be arbitrary and define a sequence $\{y_n\}$ in X as follows. Take $y_0 = y$ and $y_0 = gx_0$. Now choose an element $y_1 \in Fx_0 \cap K$ as $Fx_0 \cap K \neq \Phi$. Since $Fx_0 \cap K \subseteq g(k)$ and $\partial K \subset g(K)$, there is a point $x_1 \in K$ such that $y_1 = gx_1$. Similarly pick a point $y_2 \in Fx_1 \cap K$ if $Fx_1 \cap K \neq \Phi$, otherwise choose a point $y_2 \in \partial K$ such that

$$d(y_1, y_2) + d(y_2, y_2') = d(y_1, Y_2')$$

for some $y_2' \in Fx_1 \subset X \backslash K$. Again since $Fx_1 \cap K \subseteq g(K)$ and $\partial K \subset g(K)$, there is a point $x_2 \in K$ such that $y_2 = gx_2$.

Continuing in this way, by induction, we obtain a sequence $y_n = \{gx_n\}$ such that $y_n \in Fx_{n-1} \cap K$ if $Fx_{n-1} \cap K \neq \Phi$, otherwise $y_n \in \partial K$ satisfying

$$d(y_{n-1}, y_n) + d(y_n, y_n') = d(y_{n-1}, y_n')$$

for some $y_n \in Fx_{n-1} \subset X \backslash K$, and $y_{n-1} \in Fx_{n-2} \cap K$.

Let $X_n = Fx_{n-1}, n \in N$. Then for any two consecutive sets $X_n = Fx_{n-1}$ and $X_{n+1} = Fx_n$ we have the following three possibilities:

(i) $X_n \cap K \neq \Phi$ and $X_{n+1} \cap K \neq \Phi$,

(ii) $X_n \cap K = \Phi$ and $X_{n+1} \cap K \neq \Phi$,

(iii) $X_n \cap K \neq \Phi$ and $X_{n+1} \cap K = \Phi$,

By the construction of the sequence $\{y_n\} = \{gx_n\}$, we have

$$\{y_n\} = P \cup Q \subset K \tag{3.3}$$

where $P = \{y_n \in \{y_n\} | y_n \in Fx_{n-1}, n \in \mathbb{N}\}$
and $Q = \{y_n \in \{y_n\} | y_n \in \partial K, y_n \notin Fx_{n-1}, n \in \mathbb{N}\}$.

Then for any two consecutive terms y_n, y_{n+1} of the sequence $\{y_n\} = \{gx_n\}$ we have the following three possibilities:

(i) $y_n, y_{n+1} \in P$,

(ii) $y_n \in P$ and $y_{n+1} \in Q$,

(iii) $y_n \in Q$ and $y_{n+1} \in P$.

We prove the conclusion of the theorem in the following two steps.

Step I. Suppose that $\delta(X_r, X_{r+1}) = 0$ for some $r \in \mathbb{N}$. Then $\{X_r\} = \{X_{r+1}\}$ is a singleton set, and hence by definition of $\{y_n\}$ we get

$$y_r = Fx_{r-1} = gx_r = Fx_r = gx_{r+1} = y_{r+1}. \tag{3.4}$$

Since $\{F, g\}$ are limit coincidentally commuting we have

$$Fgx_r = gFx_r = FFx_r. \tag{3.5}$$

Define $z = gx_r = Fx_r \in K$. We shall show that z is a common fixed point of F and g.

Indeed, by (3.1)

$$\begin{aligned}
\delta(z, Fz) = \delta(Fx_r, FFx_r) \leq \\
&\leq \alpha d(gx_r, gFx_r) + \beta \max\{1/2[D(gx_r, Fx_r) + D(gFx_r, FFx_r)], \\
&\qquad 1/2[D(gx_r, FFx_r) + D(gFx_r, Fx_r)]\} \\
&\leq \alpha\delta(z, Fz) + \beta \max\{0, 1/2[\delta(z, Fz) + \delta(z, Fz)]\} \\
&\leq (\alpha + \beta)\delta(z, Fz)
\end{aligned}$$

which is possible only when $Fz = \{z\}$. From (3.5) it follows that $gz = z$. Then z is a common fixed point of F and g.

Step II. Suppose that $\delta(X_n, X_{n+1}) \neq 0$ for each $n \in \mathbb{N}$. Then $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. We show that $\{y_n\}$ is a Cauchy sequence in K.

Now there are three cases:

Case I. Suppose that $y_n, y_{n+1} \in P$. Then by (3.1),

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \delta(Fx_{n-1}, Fx_n) \\ &\leq \alpha d(gx_{n-1}, gx_n) + \beta \max\{1/2[D(gx_n, Fx_{n-1}) + D(gx_n, Fx_n)], \\ &\qquad 1/2[D(gx_{n-1}, Fx_n) + D(gx_n, Fx_{n-1})]\} \\ &\leq \alpha d(y_{n-1}, y_n) + \beta \max\{1/2[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], \\ &\qquad 1/2[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]\} \\ &\leq \alpha d(y_{n-1}, y_n) + \beta \max\{1/2[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], \\ &\qquad 1/2[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\} \\ &\leq \alpha d(y_{n-1}, y_n) + 1/2\beta[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \\ &\leq \left(\frac{\alpha + 1/2\beta}{1 - 1/2\beta}\right) d(y_{n-1}, y_n). \end{aligned}$$

Hence

$$d(y_n, y_{n+1}) \leq k d(y_{n-1}, y_n) \tag{3.6}$$

where $k = \left(\dfrac{\alpha + 1/2\beta}{1 - 1/2\beta}\right) < 1$.

Therefore,

$$d(y_n, y_{n+1}) \leq k d(y_{n-1}, y_n) \leq \cdots \leq k^n d(y_0, y_1).$$

Case II. Let $y_n \in P$ and $y_{n+1} \in Q$. Then

$$d(y_n, y_{n+1}) + d(y_{n+1}, y'_{n+1}) = d(y_n, y'_{n+1}) \tag{3.7}$$

for some $y'_{n+1} \in Fx_n \subset X \backslash K$.

Clearly,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d(y_n, y'_{n+1}) \\ d(y_n, y_{n+1}) &\leq \delta(Fx_{n-1}, Fx_n) \end{aligned} \tag{3.8}$$

Now following the arguments similar to that in Case I with appropriate modifications, we obtain

$$d(y_n, y'_{n+1}) \leq k d(y_{n-1}, y_n) \tag{3.9}$$

where again $k = \left(\dfrac{\alpha + \beta/2}{1 - \beta/2}\right) < 1$.

From (3.8) and (3.9), it follows that

$$d(y_n, y_{n+1}) \leq k d(y_{n-1}, y_n)$$

for each $n \in \mathbb{N}$.

Hence

$$d(y_n, y_{n+1}) \leq k d(y_{n-1}, y_n) \leq \cdots \leq k^n d(y_0, y_1).$$

Case III. Suppose that $y_n \in Q$ and $y_{n+1} \in P$. Then $y_{n-1} \in P$ and there is a point $y'_n \in Fx_{n-1}$ such that

$$d(y_{n-1}, y_n) + d(y_n, y'_n) = d(y_{n-1}, y'_n) \tag{3.8}$$

Now

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq d(y_n, y'_n) + d(y'_n, y_{n+1}) \\
&\leq d(y_n, y'_n) + \delta(Fx_{n-1}, Fx_n) \\
&\leq d(y_n, y'_n) + \alpha d(gx_{n-1}, gx_n) + \beta \max\{1/2[d(gx_{n-1}, Fx_{n-1}) + \\
&\qquad + d(gx_n, Fx_n)], 1/2[d(gx_{n-1}, Fx_n) + d(gx_n, Fx_{n-1})]\} \\
&\leq d(y_n, y'_n) + \alpha d(y_{n-1}, y_n) + \beta \max\{1/2[d(y_{n-1}, y'_n) + \\
&\qquad + d(y_n, y_{n+1})], 1/2[d(y_{n-1}, y_{n+1}) + d(y_n, y'_n)]\} \\
&\leq d(y_n, y'_n) + \max\{(\alpha + 1/2\beta) d(y_{n-1}, y'_n) + 1/2\beta d(y_n, y_{n+1}), \\
&\qquad \alpha d(y_{n-1}, y_n) + 1/2\beta d(y_{n-1}, y'_n) + 1/2\beta d(y_n, y_{n+1})\} \\
&\leq d(y_n, y'_n) + \max\{(\alpha + 1/2\beta) d(y_{n-1}, y'_n) + 1/2\beta d(y_n, y_{n+1}), \\
&\qquad (\alpha + 1/2\beta) d(y_{n-1}, y'_n) + 1/2\beta d(y_n, y_{n+1})\} = \\
&= d(y_{n-1}, y'_n) + (\alpha + 1/2\beta) d(y_{n-1}, y'_n) + 1/2\beta d(y_n, y_{n+1}) = \\
&= \left(\frac{1 + \alpha + 1/2\beta}{1 - 1/2\beta}\right) d(y_{n-1}, y'_n) \leq \\
&\leq \left(\frac{1 + \alpha + 1/2\beta}{1 - 1/2\beta}\right) k d(y_{n-2}, y_{n-1}) = q d(y_{n-2}, y_{n-1})
\end{aligned}
$$

where $q = \left(\frac{1 + \alpha + 1/2\beta}{1 - 1/2\beta}\right)\left(\frac{\alpha + \beta/2}{1 - \beta/2}\right) < 1$. See DHAGE [3].

Now for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq q d(y_{2n-2}, y_{2n-1}) \\
&\vdots \\
&\leq q^n d(y_0, y_1).
\end{aligned}
$$

Since n is arbitrary, one has

$$
d(y_n, y_{n+1}) \leq q^n d(y_0, y_1). \tag{3.10}
$$

Thus from Cases I - III, it easily follows that $\{y_n\}$ is a Cauchy sequence in K. As K is closed subset of a complete metric space, it is complete and hence there is a point $u \in K$ such that $\lim_n y_n = u$. We shall show that

$$\lim_n gx_n = u = \lim_n Fx_n. \tag{3.11}$$

Clearly $\lim_n gx_n = \lim_n y_n = u$. So we only show that $\lim_n Fx_n = u$.

Without loss of generality, we may assume that $y_{n+1} \in Fx_n$ for some $n \in \mathbb{N}$. Then by (3.1)

$$\begin{aligned}\delta(y_{n+1}, Fx_n) &\leq \delta(Fx_n, Fx_n) \\ &\leq \alpha d(gx_n, gx_n) + \beta\delta(gx_n, Fx_n) \\ &= \beta\delta(gx_n, Fx_n).\end{aligned}$$

Passing the limit to $n \to \infty$, we get

$$\lim_n \delta(u, Fx_n) \leq (\alpha + \beta) \lim_n \delta(u, Fx_n)$$

which implies that $\lim_n \delta(u, Fx_n) = 0$ since $\alpha + \beta < 1$. This further implies that $\lim_n \delta(u, Fx_n) = u$. Hence (3.10) holds.

By the limit coincidentally commutativity of F and g,

$$\lim_n gFx_n = \lim_n Fgx_n.$$

First suppose that g is continuous on K. Then

$$gu = \lim_n gFx_n = u = \lim_n Fgx_n. \tag{3.11}$$

Now

$$\begin{aligned}\delta(Fu, gu) &= \lim_n \delta(Fu, Fgx_n) \\ &\leq \alpha \lim_n d(gu, gx_n) + \\ +\beta \lim_n &\max\{1/2[D(gu, Fu) + D(ggx_n, Fgx_n)], \\ &1/2[D(gu, Fgx_n) + D(ggx_n, Fu)]\} \\ &\leq \beta\delta(gu, Fu) \\ &= (\alpha + \beta)\delta(gu, Fu)\end{aligned}$$

which is possible only when $\delta(Fu, gu) = 0$ since $\alpha + \beta < 1$, i.e. $Fu = gu$.

Now suppose that F is continuous on K.
Then

$$Fu = \lim_n Fgx_n = \lim_n gFx_n = \lim_n FFx_n.$$

First we shall show that $Fu = \{u\}$. By (3.1),

$$\begin{aligned}\delta(u, Fu) &\leq \delta(u, Fx_n) + \delta(Fx_n, Fu)\\ &\leq \delta(u, Fx_n) + \alpha d(gx_n, gu) + \beta \max\{1/2[D(gx_n, Fx_n)+\\ &\qquad +D(gu, Fu)], 1/2[D(gx_n, Fu) + D(gu, Fx_n)]\}\end{aligned}$$

Passing the limit to $n \to \infty$,

$$\delta(u, Fu) \leq 0 + \alpha\delta(u, Fu) + \beta\delta(u, Fu) = (\alpha + \beta)\delta(u, Fu)$$

and so $\delta(u, Fu) = 0$, i.e. $Fu = \{u\}$.

Since $F(K) \cap K \subset g(K), Fu = \{u\} \in g(K)$ and so there is a point $v \in K$ such that $Fu = gv$. We show that $Fv = gv$. Suppose not. Then by (3.1),

$$\begin{aligned}\delta(Fv, gv) &= \delta(Fv, u)\\ &\leq \delta(u, Fx_n) + \delta(Fx_n, Fv)\\ &\leq \delta(u, Fx_n) + \alpha d(gx_n, gv)+\\ &\qquad +\beta \max\{1/2[D(gx_n, Fx_n) + D(gv, Fv)],\\ &\qquad 1/2[D(gx_n, Fv) + D(gv, Fx_n)]\}.\end{aligned}$$

Passing the limit to $n \to \infty$,

$$\begin{aligned}\delta(Fv, gv) &\leq 0 + 0 + \beta \max\{1/2[0 + \delta(Fv, gv)], 1/2[\delta(gv, Fv) + 0]\}\\ &= 1/2\beta\ \delta(Fv, gv)\end{aligned}$$

which is a contraction since $\alpha + \beta < 1$. Hence $Fv = gv$. Now following the arguments similar to that in Step I, it is proved that $z = Fv = gv$ is a common fixed point of F and g such that $Fz = \{z\} = gz$.

To prove the uniqueness, let $z^*(\neq z)$ be another common fixed point of F and g. Then by (3.1),

$$\begin{aligned}d(z, z^*) &= \delta(Fz, Fz^*)\\ &\leq \alpha d(gz, gz^*) + \beta \max\{1/2[D(gz, Fz^*) + D(gz^*, Fz^*)],\\ &\qquad 1/2[D(gz, Fz^* + D(gz^*, Fz)]\}\\ &= (\alpha + \beta)d(z, z^*).\end{aligned}$$

which is a contradiction and hence $z = z^*$.

Finally we prove the continuity of F at z. If F is assumed to be continuous on K, then it is continuous at the common fixed point z of F and g. Therefore, we prove the continuity of F only in the case when g is continuous on K.

Let $\{z_n\}$ be any sequence in K converging to the common fixed point z of F and g. Since g is continuous $d(gz_n, gz) \to 0$ as $n \to \infty$. To conclude it is enough to prove that $\lim_n H(Fz_n, Fz) = 0$. By definition of H, it is clear that

$$H(Fz_n, Fz) \leq \delta(Fx_n, Fz). \tag{3.12}$$

Now

$$\delta(Fz_n, Fz) \leq \alpha d(gz_n, gz) + \beta \max\{1/2[D(gz_n, Fz_n) + D(gz, Fz)], \\ 1/2[D(gz_n, Fz) + D(gz, Fz_n)]\}.$$

Therefore

$$\lim_n \delta(Fz_n, Fz) \leq 0 + \beta \ \max\{1/2[\lim_n \delta(Fz_n, Fz) + 0], \\ 1/2[0 + \lim_n \delta(Fz_n, Fz)]\} = 1/2\beta \ \delta(Fz_n, Fz)$$

which is possible only when $\lim_n \delta(Fz_n, Fz) = 0$. Hence from (3.1) it follows that $\lim_n H(Fz_n, Fz) = 0$, i.e., F is continuous at z in the Hausdorff metric on K. This completes the proof.

Theorem 3.2. *Let K be a nonempty, compact subset of the complete metric space X and let $F : K \to CB(X)$ and $g : K \to X$ be two continuous mappings satisfying*

$$\delta(Fx, Fy) < \alpha \ d(gz, gy) + \beta \ \max\{1/2[D(gx, Fx) + D(gy, Fy)], \\ 1/2[D(gx, Fy) + D(gy, Fx)]\} \tag{3.13}$$

for all $x, y \in K$, for which the right hand side is not zero, where $\alpha \geq 0, \beta \geq 0$ satisfy

$$0 < 2\alpha + \alpha\beta + \beta \leq 1. \tag{3.14}$$

Suppose also that

(i) $F(K) \cap K \subseteq g(K)$ and $\partial K \subset g(K)$,

(ii) $\{F, g\}$ *are limit coincidentally commuting.*

Further if $Fx \cap K \neq \Phi$ *whenever* $gx \in \partial K$ *for each* $x \in K$*, then* F *and* g *have a unique common fixed point* $z \in K$ *such that* $Fz = \{z\} = gz$.

Proof. If $2\alpha + \alpha\beta + \beta < 1$, then the desired result follows by a direct applications of Theorem 3.1. Therefore, we treat the case only when $2\alpha + \alpha\beta + \beta = 1$.

Denote for breavity,

$$M(x,y) = \alpha d(gx, gy) + \beta \max\{1/2[D(gx, Fx) + D(gy, Fy)], \\ 1/2[D(gx, Fy) + D(gy, Fx)]\}$$

for $x, y \in K$.

Define the function $T : K \times K \to (0, \infty)$ by

$$T(x,y) = \frac{\delta(Fx, Fy)}{M(x,y)} \tag{3.15}$$

for $x, y \in K$.

Clearly the function T is well defined, since $M(x,y) \neq 0$ for $x, y \in K$. As F and g are continuous, the function T is continuous and hence from the compactness of K, it follows that T attains its maximum value at some point, say $(u, v) \in K^2$. Denote $c = T(x,y)$. By (3.13), $0 < c < 1$. Now by the definition of T, we get

$$\frac{\delta(Fx, Fy)}{M(x,y)} = T(x,y) \leq T(u,v) = c$$

i.e.

$$\begin{aligned} \delta(Fx, Fy) &\leq cM(x,y) \\ &\leq \alpha' d(gx, gy) + \beta' \max\{1/2[D(gx, Fx) + D(gy, Fy)], \\ &\qquad 1/2[D(gx, Fy) + D(gy, Fx)]\} \end{aligned}$$

for all $x, y \in K$, where $\alpha' \geq 0, \beta' \geq 0$ satisfy

$$2\alpha' + \alpha'\beta' + \beta' = c(2\alpha + \alpha\beta + \beta) < 1.$$

Now the desired conclusion follows by an application of Theorem 3.1. The proof is complete.

4. Coincidentally commuting mappings. In this section we prove some common fixed point theorems for the pairs of maps $\{F, g\}$ under the weaker conditions of limit coincidentally commuting and continuity of any of the maps that used in section III, but in this case we require some strong condition on the range of the maps under consideration.

Definition 4.1. Two maps $F : X \to CB(X)$ and $g : X \to X$ are called *coincidentally commuting* if they commute at coincidence points, i.e. if $Fx = \{gx\}$ then $\{gFx\} = Fgx$, for some $x \in X$.

It is shown in Dhage [4] that every limit coincidentally commuting pair of maps is coincidentally commuting, but the converse may not be true.

Theorem 4.1. *Let K be a non-empty and bounded subset of a metrically convex metric space X and let $F : X \to CB(X)$ and $g : X \to X$ be two maps satisfying (3.1) - (3.2). Suppose that*

(i) $F(K) \cap K \subseteq g(K)$ and $\partial K \subset g(K)$,

(ii) any one $g(K)$ or $F(K) \cup \partial K$ is complete,

(iii) $\{F, g\}$ are coincidentally commuting.

Further if $gx \in \partial K$ implies $Fx \cap K \neq \Phi$ for each $x \in K$, then F and g have a unique common fixed point $z \in K$ such that $Fz = \{z\} = gz$, and if g is continuous at z, then so is also F in the Hausdorff metric on X.

Proof. Let $y \in K$ be arbitrary and define a sequence $\{y_n\} \subset K$ as in the proof of Theorem 3.1. Clearly $\{y_n\} \subseteq g(K)$.

Now consider the following two steps.

Step I. Suppose that $\delta(X_r, X_{r+1}) = 0$ for some $r \in \mathbb{N}$. Then $\{X_r\} = \{X_{r+1}\} = a$ singleton set, and hence

$$y_r = Fx_{r-1} = gx_r = Fx_r = gx_{r+1} = y_{r+1}.$$

By coincidentally commutativity of F and g, we get

$$Fgx_r = gFx_r = FFx_r.$$

Now following the arguments similar to that in Step I of Theorem 3.1, it is proved that $z = gx_r = Fx_r$ is a common fixed point of F and g.

Step II. Suppose that $\delta(X_n, X_{n+1}) \neq 0$ for each $n \in \mathbb{N}$. Then proceeding as in the Step II of the proof of Theorem 3.1, it is proved that $\{y_n\}$ is a Cauchy sequence in $g(K)$. Since $g(K)$ is complete, there is a point $z \in g(K)$ such that $\lim_n y_n = z$. As $z \in g(K)$, there is a point $u \in K$ such that $z = gu$. We shall show that $Fu = gu$. It is shown as in Step II of the proof of Theorem 3.1 that

$$\lim_n Fx_n = \lim_n gx_n.$$

Suppose that $Fu \neq gu$. Then by (3.1),

$$\begin{aligned}\delta(Fu, gu) = \delta(Fu, z) \leq \delta(z, Fx_n) + \delta(Fx_n, Fu) \leq \delta(z, Fx_n) + \\ +\alpha d(gx_n, gu) + \beta \max\{1/2[D(gx_n, Fx_n) + D(gu, Fu)], \\ 1/2[D(gx_n, Fu) + D(gu, Fx_n)]\}.\end{aligned}$$

Passing the limit to $n \to \infty$, we get

$$\delta(Fu, gu) \leq 0 + 0 + \beta \max\{1/2\delta(gu, Fu), 1/2\delta(gu, Fu)\} = 1/2\beta\delta(gu, Fu)$$

which is a contraction since $\alpha + \beta < 1$. Hence $Fu = gu$. Now following the arguments similar to Step I above, it is proved that $z = Fu = gu$ is a common fixed point of F and g.

Similarly if $F(X) \cup \partial K$ is complete, there is a point $z \in F(X) \cup \partial K$ such that $\lim_n y_n = z$. But $F(X) \subseteq g(X)$ and $\partial K \subset g(X)$, so $F(X) \cup \partial K \subseteq g(X)$. Consequently we have $z \in g(X)$. Now repeating the arguments similar to above, it is proved that z is a common fixed point of F and g such that $Fz = \{z\} = gz$.

Again conditions (3.1) implies the uniqueness of the common fixed point z and the continuity of F in the Hausdorff metric H, whenever g is continuous at z. This completes the proof.

Theorem 4.2. *Let K be a non-empty and compact subset of a metrically convex metric space X and let $F : K \to CB(X)$ and $g : K \to X$ be two continuous mappings satisfying (3.13) - (3.14). Suppose that*

(i) $F(K) \cap K \subseteq g(K)$ *and* $\partial K \subset g(K)$,

(ii) $\{F, g\}$ *are coincidentally commuting.*

Further if $gx \in \partial K$ implies $Fx \cap K \neq \Phi$ for each $x \in K$, then F and g have a unique common fixed point $z \in K$ such that $Fz = \{z\} = gz$.

Proof. The proof is similar to Theorem 3.2 and now the desired conclusion follows by and application of Theorem 4.1.

5. An example. In this section we give an example of the mappings showing the validity of the hypotheses of the fixed point theorems established in the previous section.

Example 5.1. Let $X = \mathbb{R}$, the set of all real numbers, be equipped with the usual Euclidean metric $d(x, y) = |x - y|$. Let $K = \{-1\} \cup [0, 1]$ be a set in $\mathbb{R}$ which is closed and bounded.

Define two mappings $F : K \to CB(\mathbb{R})$ and $g : K \to R$ by

$$Fx = \begin{cases} [-x/6, 0], & \text{if } x \in [0, 1] \\ 0, & \text{if } x = -1 \end{cases}$$

and

$$gx = \begin{cases} [-x], & \text{if } x \in [0, 1] \\ 1, & \text{if } x = -1 \end{cases}$$

Clearly $\partial K = \{-1, 0, 1\}$. As $g(K) = g(\{-1\} \cup [0, 1]) = [-1, 0] \cup \{1\}, \partial K \subset g(K)$. Again $F(K) \cap K = [-1/6, 0] \cap K = \{0\} \subset g(K)$.

Similarly

$$\begin{aligned} &g(0) = 0 \in \partial K \Rightarrow F(0) \cap K = \{0\} \neq \Phi, \\ &g(1) = -1 \in \partial K \Rightarrow F(-1) \cap K = \{0\} \neq \Phi \\ &g(-1) = 1 \in \partial K \Rightarrow F(1) \cap K = \{0\} = \Phi \end{aligned}$$

Also $\delta(g0, F0) = 0 \Rightarrow \delta(gF0, Fg0) = 0$ showing that $\{F, g\}$ are coincidentally commuting on K. Further the pair $\{F, g\}$ satisfy condition (3.1) with $\alpha = 1/3$ and $\beta = 1/12$. Thus all conditions of Theorem 4.1 are satisfied and hence $\{F, g\}$ have a unique fixed point $z = 0$ such that $F0 = \{0\} = g0$. It is easy to see that g is continuous at the fixed point 0 in the Hausdorff metric H on $CB(\mathbb{R})$.

We note that when $g = I$, the identity mapping on the metrically convex metric space X, then our results reduce to the results of Huang and Cho [6] under slightly stronger contraction condition but under a weaker boundary condition. Finally while concluding this paper, we mention that Theorems

3.1, 3.2, 4.1 and 4.2 also remain true if we replace the metric space X by a Banach space X because every Banach space is a metrically convex metric space.

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