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EXISTENCE AND UNIQUENESS RESULTS IN THE THEORY OF SWELLING POROUS ELASTIC SOILS

BY

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Abstract. This paper is concerned with the linear theory of swelling porous elastic soils. The theory of semigroups of linear operators is used to obtain the existence and uniqueness of solutions for the initial boundary value problems with homogeneous boundary conditions. Continuous dependence of the solutions upon the initial data and supply terms is also established.

1. Introduction. Eringen [6] pointed out the importance of the theory of mixtures (see [2,3,4,5] for a presentation of the work on the subject) to the applied field of swelling. In this connection, Eringen has developed a continuum theory of swelling porous elastic soils as a continuum theory of mixture for porous elastic solids filled with fluid and gas. The intended applications of the theory are in the field of swelling, oil exploration, slurries and consolidation problems. The theory is relevant to problems in the oil exploration industry, since oil is viscous and is usually accompanied by gas in underground rocks, porous solid in slurries and muddy river beds. Consolidation problems in the building industry, earthquake problems, swelling of plants and living tissues and a plethora of other problems fall into domain of mixture theory considered in [6]. It is also shown that the diffusion type theories [13], that are presently prominent among the applied workers in the field of swelling, are special cases of the present theory.

In the context of Eringen's mixture theory [6], some results concerning uniqueness, continuous dependence, instability, structural stability and spatial behavior of the solutions have been studied in several recent articles

[8,9,14,15]. In [8,9,14] the case when the fluid and the gas where compressible was studied. The case when the fluid is incompressible has been considered in [15].

The purpose of this paper is to establish the existence and uniqueness of solutions for the initial boundary value problems within the context of the linear theory of swelling porous elastic soils [6]. The paper also investigate the continuous dependence upon the initial data and supplies terms of the solutions. We transform the initial boundary value problem in a temporally abstract equation in a Hilbert space. Then we use the results of the semigroups theory of linear operators in order to obtain the existence, uniqueness and continuous dependence results.

2. Basic formulation. We refer the motion of a continuum to a fixed system of rectangular Cartesian axes $0x_k$ $(k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over integer $(1, 2, 3)$, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation.

We consider a body that at time $t = 0$ occupies the regular region Ω of Euclidean three–dimensional space whose boundary is the regular surface $\partial\Omega$.

We assume that Ω is occupied by a mixture consisting of three components: an elastic solid, a compressible fluid and a gas. We use superscripts s, f, g to denote respectively, the elastic solid, the fluid and the gas. We consider the fundamental equations for thermomechanical behavior of the mixture in the framework of the linearized theory (see [6])

$$
\begin{aligned}
\rho_0^s \ddot{u}_i^s &= \mu u_{i,jj}^s + (\lambda+\mu) u_{j,ji}^s - \sigma^f u_{j,ji}^f - \sigma^g u_{j,ji}^g + \\
&+ (\gamma^f + \gamma^g - \alpha_0) T_{,i} + (\xi^{ff} + \xi^{gf})(\dot{u}_i^f - \dot{u}_i^s) + \\
&+ (\xi^{fg} + \xi^{gg})(\dot{u}_i^g - \dot{u}_i^s) + \rho_0^s f_i^s, \qquad (2.1) \\
\rho_0^f \ddot{u}_i^f &= \mu_\nu \dot{u}_{i,jj}^f + (\lambda_\nu + \mu_\nu) \dot{u}_{j,ji}^f - \sigma^f u_{j,ji}^s - \sigma^{ff} u_{j,ji}^f - \sigma^{fg} u_{j,ji}^g - \\
&- (\alpha^f + \gamma^f) T_{,i} - \xi^{ff}(\dot{u}_i^f - \dot{u}_i^s) - \xi^{fg}(\dot{u}_i^g - \dot{u}_i^s) + \rho_0^f f_i^f,
\end{aligned}
$$

$$\begin{aligned}\rho_0^g\ddot{u}_i^g &= -\sigma^g u^s_{j,ji} - \sigma^{gf}u^f_{j,ji} - \sigma^{gg}u^g_{j,ji} - (\alpha^g+\gamma^g)T_{,i} - \\ &\quad -\xi^{gf}(\dot{u}_i^f - \dot{u}_i^s) - \xi^{gg}(\dot{u}_i^g - \dot{u}_i^s) + \rho_0^g f_i^g,\\ \alpha_1\dot{T} - \bar{\omega}\ddot{T} &= \Big(\frac{\zeta^f}{T_0} + \frac{\zeta^g}{T_0} - \alpha_0\Big)\dot{u}^s_{i,i} - \Big(\alpha^f + \frac{\zeta^f}{T_0}\Big)\dot{u}^f_{i,i} - \\ &\quad -\Big(\alpha^g + \frac{\zeta^f}{T_0}\Big)\dot{u}^g_{i,i} + \frac{k}{T_0}T_{,ii} + \frac{\rho_0 h}{T_0}.\end{aligned}$$

Equations (2.1) constitute a system of ten partial differential equations for the determination of ten unknown functions $(u_i^s, u_i^f, u_i^g; T)$ that represent the displacements of the elastic solid, fluid and gas respectively, and the temperature. The functions $(f_i^s, f_i^f, f_i^g; h)$ are the supply terms, ρ_0^s, ρ_0^f, ρ_0^g are the densities of each constituent, $\rho_0 = \rho_0^s + \rho_0^f + \rho_0^g$, $T_0 > 0$ is the constant absolute temperature of the body in its reference state and λ, μ, $\sigma^a, \alpha^a, \gamma^a$, ζ^a $(a = f, g)$, μ_ν, λ_ν, σ^{ab}, $\xi^{ab}(a, b = f, g)$, α_0, α_1, $\bar{\omega}$ and k are constitutive constants.

The coefficients ξ^{ab} $(a, b = f, g)$ have the following symmetries

$$\xi^{ab} = \xi^{ba}\ , \qquad a, b = f, g. \tag{2.2}$$

In the context of the linear theory, in view of the equations $(3.24)_2$, $(3.24)_3$ and $(3.9)_3$ and (3.26) respectively, reference [6], we get

$$\frac{\zeta^a}{T_0} = \gamma^a\ , \qquad \sigma^{ab} = \sigma^{ba}, \qquad a, b = f, g\ . \tag{2.3}$$

Further, it was shown that the local form of the Clausius–Duhem inequality implies that (see [6])

$$\mu_\nu \geq 0\ , \qquad 3\lambda_\nu + 2\mu_\nu \geq 0\ , \qquad \bar{\omega} \geq 0\ , \tag{2.4}$$

and the following symmetric matrix is positive semi–definite

$$\Delta = \begin{pmatrix} k/T_0 & \gamma^f & \gamma^g \\ \gamma^f & \xi^{ff} & \xi^{fg} \\ \gamma^g & \xi^{gf} & \xi^{gg} \end{pmatrix}. \tag{2.5}$$

Remark 1. From the relation $(2.4)_3$, the parameter $\bar{\omega}$ is non–negative. In this paper we consider that $\bar{\omega} = 0$.

Generally, the energy density associated with various models in continuum mechanics and which is usually assumed to be positive definite, defines a norm equivalent to the usual norm in the space of solutions. If we try to use the semigroup theory (or other methods) to prove an existence result for the field equations (2.1), we meet with difficult problems since in the space where we look for the solutions, the energy density (see [6,8])

$$\varepsilon = \frac{\lambda}{2}u^s_{i,i}u^s_{j,j}+\frac{\mu}{4}(u^s_{i,j}+u^s_{j,i})(u^s_{i,j}+u^s_{j,i})-\sum_{a=f,g}\sigma^a u^a_{i,i}u^s_{j,j}-\frac{1}{2}\sum_{a,b=f,g}\sigma^{ab}u^a_{i,i}u^b_{j,j}\ ,$$

not defines a norm. For this reason it is suitable to transform the system (2.1) in following way: We remark that it was obtained by taking into account the linearized and integrated form of the continuity equations

$$r^a = -u^a_{i,i} \qquad a = f, g, \tag{2.6}$$

where $r^a = \dfrac{\rho^a - \rho^a_0}{\rho^a_0}$, $(a = f, g)$ is the density variation from the constant density ρ^a_0 in the natural state. So, if we use the relation (2.6), and take into account the relation (2.3) and the remark 1, we obtain the system

$$\begin{aligned}
\rho^s_0\ddot{u}^s_i &= \mu u^s_{i,jj} + (\lambda+\mu)u^s_{j,ji} + \sigma^f r^f_{,i} + \sigma^g r^g_{,i} + \\
&+(\gamma^f + \gamma^g - \alpha_0)T_{,i} + (\xi^{ff} + \xi^{gf})(v^f_i - \dot{u}^s_i) + \\
&+(\xi^{fg} + \xi^{gg})(v^g_i - \dot{u}^s_i) + \rho^s_0 f^s_i, \\
\rho^f_0\dot{v}^f_i &= \mu_\nu v^f_{i,jj} + (\lambda_\nu + \mu_\nu)v^f_{j,ji} - \sigma^f u^s_{j,ji} + \sigma^{ff} r^f_{,i} + \sigma^{fg} r^g_{,i} - \\
&-(\alpha^f + \gamma^f)T_{,i} - \xi^{ff}(v^f_i - \dot{u}^s_i) - \xi^{fg}(v^g_i - \dot{u}^s_i) + \rho^f_0 f^f_i, \\
\rho^g_0\dot{v}^g_i &= -\sigma^g u^s_{j,ji} + \sigma^{gf} r^f_{,i} + \sigma^{gg} r^g_{,i} - (\alpha^g + \gamma^g)T_{,i} - \\
&-\xi^{gf}(v^f_i - \dot{u}^s_i) - \xi^{gg}(v^g_i - \dot{u}^s_i) + \rho^g_0 f^g_i, \\
\alpha_1\dot{T} &= (\gamma^f + \gamma^g - \alpha_0)\dot{u}^s_{i,i} + (\alpha^f + \gamma^f)\dot{r}^f + (\alpha^g + \gamma^g)\dot{r}^g + \\
&+\tfrac{k}{T_0}T_{,ii} + \tfrac{\rho_0 h}{T_0}, \\
\dot{r}^f &= -v^f_{i,i}, \\
\dot{r}^g &= -v^g_{i,i},
\end{aligned} \tag{2.7}$$

for the determination of the functions $(u^s_i, v^f_i, v^g_i, T, r^f, r^g)$, where v^f_i and v^g_i are the velocities of the fluid and gas. Because the problem is linear, if v^f_i and v^g_i are known the displacements of the fluid and gas can be obtained by a simple integration.

It is convenient to introduce the following dimensionless quantities

$$
\begin{gathered}
\bar{\mathbf{u}}^s = \frac{\mathbf{u}^s}{L}\ , \quad \bar{\mathbf{v}}^f = \frac{\tau_0 \mathbf{v}^f}{L}\ , \quad \bar{\mathbf{v}}^g = \frac{\tau_0 \mathbf{v}^g}{L}\ , \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L}\ , \quad \bar{t} = \frac{t}{\tau_0}\ , \\
\bar{T} = \frac{T}{\tilde{T}}\ , \quad \bar{T}_0 = \frac{T_0}{\tilde{T}}\ , \quad \bar{\rho}_0^\alpha = \frac{L^3 \rho_0^\alpha}{m_0}\ , \quad \bar{r}^\alpha = r^\alpha\ , \qquad \alpha = s, f, g
\end{gathered} \tag{2.8}
$$

where $L, \tau_0, \tilde{T}$ and m_0 are constants with dimension of length, time, temperature and mass respectively. With these quantities we can write the system (2.7) in the form

$$
\begin{aligned}
\bar{\rho}_0^s \ddot{\bar{u}}_i^s &= \bar{\mu} \bar{u}_{i,jj}^s + (\bar{\lambda} + \bar{\mu}) \bar{u}_{j,ji}^s + \bar{\sigma}^f \bar{r}_{,i}^f + \bar{\sigma}^g \bar{r}_{,i}^g + \\
&+ (\bar{\gamma}^f + \bar{\gamma}^g - \bar{\alpha}_0) \bar{T}_{,i} + (\bar{\xi}^{ff} + \bar{\xi}^{gf})(\bar{v}_i^f - \dot{\bar{u}}_i^s) + \\
&+ (\bar{\xi}^{fg} + \bar{\xi}^{gg})(\bar{v}_i^g - \dot{\bar{u}}_i^s) + \bar{\rho}_0^s \bar{f}_i^s, \\
\bar{\rho}_0^f \dot{\bar{v}}_i^f &= \bar{\mu}_\nu \bar{v}_{i,jj}^f + (\bar{\lambda}_\nu + \bar{\mu}_\nu) \bar{v}_{j,ji}^f - \bar{\sigma}^f \bar{u}_{j,ji}^s + \\
&+ \bar{\sigma}^{ff} \bar{r}_{,i}^f + \bar{\sigma}^{fg} \bar{r}_{,i}^g - (\bar{\alpha}^f + \bar{\gamma}^f) \bar{T}_{,i} - \\
&- \bar{\xi}^{ff}(\bar{v}_i^f - \dot{\bar{u}}_i^s) - \bar{\xi}^{fg}(\bar{v}_i^g - \dot{\bar{u}}_i^s) + \bar{\rho}_0^f \bar{f}_i^f, \\
\bar{\rho}_0^g \dot{\bar{v}}_i^g &= -\bar{\sigma}^g \bar{u}_{j,ji}^s + \bar{\sigma}^{gf} \bar{r}_{,i}^f + \bar{\sigma}^{gg} \bar{r}_{,i}^g - (\bar{\alpha}^g + \bar{\gamma}^g) \bar{T}_{,i} - \\
&- \bar{\xi}^{gf}(\bar{v}_i^f - \dot{\bar{u}}_i^s) - \bar{\xi}^{gg}(\bar{v}_i^g - \dot{\bar{u}}_i^s) + \bar{\rho}_0^g \bar{f}_i^g, \\
\bar{\alpha}_1 \dot{\bar{T}} &= (\bar{\gamma}^f + \bar{\gamma}^g - \bar{\alpha}_0) \dot{\bar{u}}_{i,i}^s + (\bar{\alpha}^f + \bar{\gamma}^f) \dot{\bar{r}}^f + \\
&+ (\bar{\alpha}^g + \bar{\gamma}^g) \dot{\bar{r}}^g + \frac{\bar{k}}{\bar{T}_0} \bar{T}_{,ii} + \frac{\bar{\rho}_0 \bar{h}}{\bar{T}_0}, \\
\dot{\bar{r}}^f &= -\bar{v}_{i,i}^f, \\
\dot{\bar{r}}^g &= -\bar{v}_{i,i}^g,
\end{aligned} \tag{2.7}
$$

where subscripts preceded by a comma denote partial differentiation respect to $\bar{\mathbf{x}}$, superposed dot denotes partial differentiation respect to $\bar{t}$ and

$$
\begin{gathered}
\bar{\mu} = \frac{L\tau_0^2}{m_0}\mu, \quad \bar{\lambda} = \frac{L\tau_0^2}{m_0}\lambda, \quad \bar{\mu}_\nu = \frac{L\tau_0}{m_0}\mu_\nu, \quad \bar{\lambda}_\nu = \frac{L\tau_0}{m_0}\lambda_\nu, \\
\bar{\sigma}^a = \frac{L\tau_0^2}{m_0}\sigma^a, \quad \bar{\sigma}^{ab} = \frac{L\tau_0^2}{m_0}\sigma^{ab}, \\
\bar{\gamma}^a = \frac{L\tau_0^2\tilde{T}}{m_0}\gamma^a, \quad \bar{\alpha}^a = \frac{L\tau_0^2\tilde{T}}{m_0}\alpha^a, \quad a, b = f, g \\
\bar{\alpha}_0 = \frac{L\tau_0^2\tilde{T}}{m_0}\alpha_0, \quad \bar{\alpha}_1 = \frac{L\tau_0^2\tilde{T}^2}{m_0}\alpha_1, \\
\bar{k} = \frac{\tau_0^3\tilde{T}}{Lm_0}k, \quad \bar{\xi}^{ab} = \frac{L^3\tau_0}{m_0}\xi^{ab}, \quad a, b = f, g \\
\bar{f}_i^\alpha = \frac{\tau_0^2}{L} f_i^\alpha, \quad \alpha = s, f, g \quad \bar{h}_0 = \frac{\tau_0^3}{L^2} h_0, \quad \bar{\rho}_0 = \bar{\rho}_0^s + \bar{\rho}_0^f + \bar{\rho}_0^g.
\end{gathered} \tag{2.10}
$$

To these equations we have to adjoin boundary conditions and initial conditions. We consider the following homogeneous boundary conditions

$$\bar{u}_i^s = 0\ , \quad \bar{v}_i^f = 0\ , \quad \bar{v}_i^g n_i = 0\ , \quad \bar{T} = 0\ , \qquad \text{on } \partial\Omega \times [0, t_1) \tag{2.11}$$

where n_i are the direction cosines of the outward normal to regular surface $\partial\Omega$. The condition $(2.11)_3$ is different from the others, and is associated to the assumption that gas can glide along the surface $\partial\Omega$. We will see later (lemma 2) that this is in connection with the field equations and an appropriate Hilbert space. The initial conditions are

$$\bar{u}_i^s(\mathbf{x}, 0) = a_i^s(\mathbf{x})\ , \qquad \dot{\bar{u}}_i^s(\mathbf{x}, 0) = b_i^s(\mathbf{x})\ , \tag{2.12}$$

$$\bar{v}_i^f(\mathbf{x}, 0) = b_i^f(\mathbf{x})\ , \qquad \bar{v}_i^g(\mathbf{x}, 0) = b_i^g(\mathbf{x})\ , \qquad \bar{T}(\mathbf{x}, 0) = \theta_0(\mathbf{x})\ ,$$

$$\bar{r}^f(\mathbf{x}, 0) = r_0^f(\mathbf{x})\ , \qquad \bar{r}^g(\mathbf{x}, 0) = r_0^g(\mathbf{x})\ , \qquad \mathbf{x} \in \Omega,$$

where $a_i^s, b_i^s, b_i^f, b_i^g, \theta_0, r_0^f$ and r_0^g are prescribed fields.

In this paper we shall assume the following:
(i) the densities $\bar{\rho}_0^s$, $\bar{\rho}_0^f$ and $\bar{\rho}_0^g$ are strictly positive;
(ii)

$$\bar{\alpha}_1 > 0\ ; \tag{2.13}$$

(iii)

$$\bar{\mu} > 0\ , \tag{2.14}$$

and the following symmetric matrix is positive definite

$$\bar{\delta} = \begin{pmatrix} \bar{\lambda} + \bar{\mu} & -\bar{\sigma}^f & -\bar{\sigma}^g \\ -\bar{\sigma}^f & -\bar{\sigma}^{ff} & -\bar{\sigma}^{fg} \\ -\bar{\sigma}^g & -\bar{\sigma}^{fg} & -\bar{\sigma}^{gg} \end{pmatrix}\ , \tag{2.15}$$

so,

$$\begin{aligned} &\bar{\mu}\bar{u}_{i,j}^s \bar{u}_{i,j}^s + (\bar{\lambda} + \bar{\mu})\bar{u}_{i,i}^s \bar{u}_{j,j}^s + 2\textstyle\sum_{a=f,g} \bar{\sigma}^a \bar{r}^a \bar{u}_{i,i}^s - \\ &- \sum_{a,b=f,g} \bar{\sigma}^{ab} \bar{r}^a \bar{r}^b \geq \bar{\delta}_m (\bar{u}_{i,j}^s \bar{u}_{i,j}^s + \bar{r}^f \bar{r}^f + \bar{r}^g \bar{r}^g)\ , \end{aligned} \tag{2.16}$$

for every $\bar{u}_i^s$, $\bar{r}^f$ and $\bar{r}^g$. Here $\bar{\delta}_m = \min\{\bar{\mu}$; the minimum eigenvalue of the matrix $\bar{\delta}\}$;
(iv)

$$\bar{\mu}_\nu > 0\ , \qquad \bar{\lambda}_\nu + \bar{\mu}_\nu \geq 0\ , \qquad \bar{k} > 0\ , \tag{2.17}$$

and the following symmetric matrix is positive semi–definite

$$\bar{\Delta}' = \begin{pmatrix} \bar{\xi}^{ff} & \bar{\xi}^{fg} \\ \bar{\xi}^{gf} & \bar{\xi}^{gg} \end{pmatrix}. \tag{2.18}$$

The mechanical interpretation of (i) and (ii) is obvious. The assumption (iv) represents a strengthening of the consequences of the entropy production inequality. The assumption (iii), usual in the study of the well posed problems, is in agreement with the restrictions imposed in [8,9,14,15] in order to study the uniqueness, continuous dependence and spatial behavior of solutions.

3. Existence and uniqueness results. In this section we use results of the semigroup theory of linear operators to obtain an existence theorem to the equations that characterize the linear theory of swelling porous elastic soils. First, we transform our initial boundary value problem defined by the relations (2.9), (2.11) and (2.12) to an abstract problem on a Hilbert space.

In order to simplify the notation we suppress the accent – in the previous section for the rest of the paper.

Let us define

$$\begin{aligned} X = \{\omega = (\mathbf{u^s}, \mathbf{v^s}, \mathbf{v^f}, \mathbf{v^g}, T, r^f, r^g) |\ \mathbf{u^s} \in \mathbf{H}_0^1(\Omega), \\ \mathbf{v^s}, \mathbf{v^f}, \mathbf{v^g} \in \mathbf{L}^2(\Omega), T, r^f, r^g \in L^2(\Omega)\} \end{aligned} \tag{3.1}$$

where $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^3$, $\mathbf{H}_0^m(\Omega) = [H_0^m(\Omega)]^3$ and $H_0^m(\Omega)$ is the well known Sobolev space [1,7]. Further, we introduce the operators

$$\begin{aligned} A_i^{s1}\omega &= v_i^s, \\ A_i^{s2}\omega &= \frac{1}{\rho_0^s}\Big[\mu u_{i,jj}^s + (\lambda+\mu)u_{j,ji}^s + \sigma^f r_{,i}^f + \sigma^g r_{,i}^g + \\ &\quad +(\gamma^f + \gamma^g - \alpha_0)T_{,i} + (\xi^{ff} + \xi^{gf})(v_i^f - v_i^s) + \\ &\quad +(\xi^{fg} + \xi^{gg})(v_i^g - v_i^s)\Big], \\ A_i^f\omega &= \frac{1}{\rho_0^f}\Big[\mu_\nu v_{i,jj}^f + (\lambda_\nu + \mu_\nu)v_{j,ji}^f - \sigma^f u_{j,ji}^s + \sigma^{ff} r_{,i}^f + \\ &\quad +\sigma^{fg} r_{,i}^g - (\alpha^f + \gamma^f)T_{,i} - \xi^{ff}(v_i^f - v_i^s) - \xi^{fg}(v_i^g - v_i^s)\Big], \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
A_i^g\omega &= \frac{1}{\rho_0^g}\Big[-\sigma^g u^s_{j,ji} + \sigma^{gf} r^f_{,i} + \sigma^{gg} r^g_{,i} - (\alpha^g+\gamma^g)T_{,i} - \\
&\quad -\xi^{gf}(v_i^f - v_i^s) - \xi^{gg}(v_i^g - v_i^s)\Big], \\
A\omega &= \frac{1}{\alpha_1}\Big[(\gamma^f+\gamma^g-\alpha_0)v^s_{i,i} - (\alpha^f+\gamma^f)v^f_{i,i} - (\alpha^g+\gamma^g)v^g_{i,i} + \frac{k}{T_0}T_{,ii}\Big], \\
A^f\omega &= -v^f_{i,i}, \\
A^g\omega &= -v^g_{i,i}.
\end{aligned}
$$

Let $\mathcal{A}$ be the operator

$$
\mathcal{A}\omega = (\mathbf{A}^{s1}\omega, \mathbf{A}^{s2}\omega, \mathbf{A}^f\omega, \mathbf{A}^g\omega, A\omega, A^f\omega, A^g\omega)\ , \tag{3.3}
$$

with the domain

$$
\begin{aligned}
\mathcal{D}(\mathcal{A}) = \{\omega = (\mathbf{u^s}, \mathbf{v^s}, \mathbf{v^f}, \mathbf{v^g}, T, r^f, r^g) \in X |\ \mathcal{A}\omega \in X, \\
\mathbf{v^f} = \mathbf{0}\ , \quad v_i^g n_i = 0\ , \quad T = 0 \quad \text{on} \quad \partial\Omega\}.
\end{aligned} \tag{3.4}
$$

We note that $\mathbf{C}_0^\infty(\Omega)\times\mathbf{C}_0^\infty(\Omega)\times\mathbf{C}_0^\infty(\Omega)\times\mathbf{C}_0^\infty(\Omega)\times C_0^\infty(\Omega)\times C_0^\infty(\Omega)\times C_0^\infty(\Omega)$, where $\mathbf{C}_0^\infty(\Omega) = [C_0^\infty(\Omega)]^3$, is a dense subset of X which is contained in $\mathcal{D}(\mathcal{A})$.

Thus, we reduce the initial boundary value problem (2.9), (2.11) and (2.12) to the abstract initial value problem on X

$$
\frac{d\omega}{dt}(t) = \mathcal{A}\omega(t) + \mathcal{F}(t)\ , \qquad \omega(0) = \omega_0\ , \tag{3.5}
$$

where $\mathcal{F}(t) = (\mathbf{0}, \mathbf{f}^s, \mathbf{f}^f, \mathbf{f}^g, \frac{\rho_0 h}{T_0\alpha_1}, 0, 0)$ and $\omega_0 = (\mathbf{a}^s, \mathbf{b}^s, \mathbf{b}^f, \mathbf{b}^g, \theta_0, r_0^f, r_0^g)$.

We introduce the following inner product in X:

$$
\begin{aligned}
&< (\mathbf{u}^s, \mathbf{v}^s, \mathbf{v}^f, \mathbf{v}^g, T, r^f, r^g), (\mathbf{u}^{\star s}, \mathbf{v}^{\star s}, \mathbf{v}^{\star f}, \mathbf{v}^{\star g}, T^\star, r^{\star f}, r^{\star g}) >_{\mathbf{X}} = \\
&= \int_\Omega \{\rho_0^s v_i^s v_i^{\star s} + \rho_0^f v_i^f v_i^{\star f} + \rho_0^g v_i^g v_i^{\star g} + \\
&+ \alpha_1 T T^\star + E[(\mathbf{u}^s, r^f, r^g), (\mathbf{u}^{\star s}, r^{\star f}, r^{\star g})]\} dx
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
&E[(\mathbf{u}^s, r^f, r^g), (\mathbf{u}^{\star s}, r^{\star f}, r^{\star g})] = \mu u^s_{i,j} u^{\star s}_{i,j} + (\lambda+\mu) u^s_{i,i} u^{\star s}_{j,j} + \\
&+ \textstyle\sum_{a=f,g} \sigma^a (r^a u^{\star s}_{i,i} + r^{\star a} u^s_{i,i}) - \sum_{a,b=f,g} \sigma^{ab} r^a r^{\star b}\ .
\end{aligned} \tag{3.7}
$$

In view of the assumption (iii) it follows that the norm induced by E is equivalent to the usual norm in $\mathbf{H}_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Thus $< \cdot, \cdot >_{\mathbf{X}}$ defines a norm equivalent to the usual norm in X.

Lemma 1. *The operator $\mathcal{A}$ is dissipative, that is*

$$< \mathcal{A}\omega, \omega >_{\mathbf{X}} \leq 0 \ , \qquad \text{for all} \quad \omega \in \mathcal{D}(\mathcal{A}) \ . \tag{3.8}$$

Proof. According to the relations (3.2), (3.3) and (3.6), we get

$$\begin{gathered} < \mathcal{A}\omega, \omega >_{\mathbf{X}} = \int_\Omega \{ v_i^s [\mu u_{i,jj}^s + (\lambda + \mu) u_{j,ji}^s + \sigma^f r_{,i}^f + \\ + \sigma^g r_{,i}^g + (\gamma^f + \gamma^g - \alpha_0) T_{,i}] + v_i^f [\mu_\nu v_{i,jj}^f + (\lambda_\nu + \mu_\nu) v_{j,ji}^f - \\ - \sigma^f u_{j,ji}^s + \sigma^{ff} r_{,i}^f + \sigma^{fg} r_{,i}^g - (\alpha^f + \gamma^f) T_{,i}] + \\ + v_i^g [-\sigma^g u_{j,ji}^s + \sigma^{gf} r_{,i}^f + \sigma^{gg} r_{,i}^g - (\alpha^g + \gamma^g) T_{,i}] + \\ + T[(\gamma^f + \gamma^g - \alpha_0) v_{i,i}^s - (\alpha^f + \gamma^f) v_{i,i}^f - \\ - (\alpha^g + \gamma^g) v_{i,i}^g + \frac{k}{T_0} T_{,ii}] - \sum_{a,b=f,g} \xi^{ab} (v_i^a - v_i^s)(v_i^b - v_i^s) + \\ + E[(\mathbf{v}^s, -v_{i,i}^f, -v_{i,i}^g), (\mathbf{u}^s, r^f, r^g)] \} dx \ . \end{gathered} \tag{3.9}$$

We now make use of the divergence theorem, boundary conditions, the fact that
$\mathcal{A}\omega \in X$ implies $\mathbf{v}^s = \mathbf{0}$ on $\partial\Omega$ and the relation (3.7) in order to obtain

$$\begin{gathered} < \mathcal{A}\omega, \omega >_{\mathbf{X}} = - \int_\Omega [\mu_\nu v_{i,j}^f v_{i,j}^f + (\lambda_\nu + \mu_\nu) v_{i,i}^f v_{j,j}^f + \tfrac{k}{T_0} T_{,i} T_{,i} + \\ + \textstyle\sum_{a,b=f,g} \xi^{ab} (v_i^a - v_i^s)(v_i^b - v_i^s)] dx \ . \end{gathered} \tag{3.10}$$

Lemma follows from the assumption (iv).

Lemma 2. *The operator $\mathcal{A}$ satisfies the range condition*

$$\mathrm{Rang}(I - \mathcal{A}) = X \ . \tag{3.11}$$

Proof. Let $\omega^* = (\mathbf{u}^{*s}, \mathbf{v}^{*s}, \mathbf{v}^{*f}, \mathbf{v}^{*g}, T^*, r^{*f}, r^{*g}) \in X$. Then we must show that the equation

$$\omega - \mathcal{A}\omega = \omega^* \ , \tag{3.12}$$

has a solution in $\mathcal{D}(\mathcal{A})$. From the definition of the operator $\mathcal{A}$ we obtain the system

$$
\begin{array}{c}
u_i^s - A_i^{s1}\omega = u_i^{*s},\ \ v_i^s - A_i^{s2}\omega = v_i^{*s}, \\
v_i^f - A_i^f\omega = v_i^{*f},\ \ v_i^g - A_i^g\omega = v_i^{*s}, \\
T - A\omega = T^*,\ \ r^f - A^f\omega = r^{*f},\ \ r^g - A^g\omega = r^{*g}.
\end{array} \tag{3.13}
$$

The first and the last two equations can be written, with the help of the relations (3.2), in the form

$$
v_i^s = u_i^s - u_i^{*s}\ , \qquad r^f = -v_{i,i}^f + r^{*f}\ , \qquad r^g = -v_{i,i}^g + r^{*g}\ . \tag{3.14}
$$

By eliminating the functions v_i^s, r^f and r^g in (3.13), we obtain for the determination of the functions u_i^s, v_i^f, v_i^g and T the following system

$$
\begin{aligned}
\mathcal{L}_i^s\mathbf{y} \ &\equiv u_i^s - \frac{1}{\rho_0^s}\Big[\mu u_{i,jj}^s + (\lambda+\mu)u_{j,ji}^s - \\
&\quad -\sigma^f v_{j,ji}^f - \sigma^g v_{j,ji}^g + (\gamma^f + \gamma^g - \alpha_0)T_{,i} + \\
&\quad +(\xi^{ff} + \xi^{gf})(v_i^f - u_i^s) + (\xi^{fg} + \xi^{gg})(v_i^g - u_i^s)\Big] = g_i^{*s}, \\
\mathcal{L}_i^f\mathbf{y} \ &\equiv v_i^f - \tfrac{1}{\rho_0^f}\Big[\mu_\nu v_{i,jj}^f + (\lambda_\nu + \mu_\nu)v_{j,ji}^f - \\
&\quad -\sigma^f u_{j,ji}^s - \sigma^{ff} v_{j,ji}^f - \sigma^{fg} v_{j,ji}^g - (\alpha^f + \gamma^f)T_{,i} - \\
&\quad -\xi^{ff}(v_i^f - u_i^s) - \xi^{fg}(v_i^g - u_i^s)\Big] = g_i^{*f}\ , \\
\mathcal{L}_i^g\mathbf{y} \ &\equiv v_i^g - \frac{1}{\rho_0^g}\Big[-\sigma^g u_{j,ji}^s - \sigma^{gf} v_{j,ji}^f - \sigma^{gg} v_{j,ji}^g - \\
&\quad -(\alpha^g + \gamma^g)T_{,i} - \xi^{gf}(v_i^f - u_i^s) - \xi^{gg}(v_i^g - u_i^s)\Big] = g_i^{*g}\ , \\
\mathcal{L}\mathbf{y} \ &\equiv T - \frac{1}{\alpha_1}\Big[(\gamma^f + \gamma^g - \alpha_0)u_{i,i}^s - (\alpha^f + \gamma^f)v_{i,i}^f - \\
&\quad -(\alpha^g + \gamma^g)v_{i,i}^g + \frac{k}{T_0}T_{,ii}\Big] = g^*,
\end{aligned} \tag{3.15}
$$

where $\mathbf{y} = (\mathbf{u^s}, \mathbf{v^f}, \mathbf{v^g}, T)$ and

$$
\begin{aligned}
& g_i^{*s} = v_i^{*s} + u_i^{*s} + \tfrac{1}{\rho_0^s}\Big[(\xi^{ff} + 2\xi^{fg} + \xi^{gg})u_i^{*s} + \sigma^f r_{,i}^{*f} + \sigma^g r_{,i}^{*g}\Big], \\
& g_i^{*f} = v_i^{*f} + \tfrac{1}{\rho_0^f}\Big[-(\xi^{ff} + \xi^{fg})u_i^{*s} + \sigma^{ff} r_{,i}^{*f} + \sigma^{fg} r_{,i}^{*g}\Big], \\
& g_i^{*g} = v_i^{*g} + \tfrac{1}{\rho_0^g}\Big[-(\xi^{fg} + \xi^{gg})u_i^{*s} + \sigma^{fg} r_{,i}^{*f} + \sigma^{gg} r_{,i}^{*g}\Big], \\
& g^{*} = T^{*} - \tfrac{1}{\alpha_1}(\gamma^f + \gamma^g - \alpha_0)u_{i,i}^{*s}
\end{aligned}
\tag{3.16}
$$

To study this system we introduce the Hilbert space

$$
\mathbf{Y} = \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{V}_0(\Omega) \times H_0^1(\Omega)\ , \tag{3.17}
$$

where $\mathbf{V}_0(\Omega) = \{v_i^g \in L^2(\Omega) |\ v_{i,i}^g \in L^2(\Omega)\ , \quad v_i^g n_i = 0 \text{ on } \partial\Omega\}$ is the Hilbert space equipped with the inner product (see [16])

$$
< \mathbf{v}^g, \widehat{\mathbf{v}}^g >_{\mathbf{V}} = \int_\Omega (v_i^g \widehat{v}_i^g + v_{i,i}^g \widehat{v}_{j,j}^g) dx\ . \tag{3.18}
$$

In fact, it can be show that $\mathbf{V}_0(\Omega)$ is the functional completion of the set $[C_0^\infty(\Omega)]^3$ with respect to the inner product (3.18). This property suggests to consider the boundary condition $(2.11)_3$.

Further, we introduce the bilinear form $B : \mathbf{Y} \times \mathbf{Y} \to R$

$$
\begin{aligned}
& B(\mathbf{y}, \widehat{\mathbf{y}}) =< (\mathcal{L}^s\mathbf{y}, \mathcal{L}^f\mathbf{y}, \mathcal{L}^g\mathbf{y}, \mathcal{L}\mathbf{y}), \\
& (\rho_0^s\widehat{\mathbf{u}}^s, \rho_0^f\widehat{\mathbf{u}}^f, \rho_0^g\widehat{\mathbf{u}}^g, \alpha_1\widehat{T} >_{\mathbf{L}^2\times\mathbf{L}^2\times\mathbf{L}^2\times L^2} = \\
& = \int_\Omega [\rho_0^s u_i^s \widehat{u}_i^s + \rho_0^f v_i^f \widehat{v}_i^f + \rho_0^g v_i^g \widehat{v}_i^g + \alpha_1 T\widehat{T} + \mu u_{i,j}^s \widehat{u}_{i,j}^s + \\
& + (\lambda+\mu) u_{i,i}^s \widehat{u}_{j,j}^s - \sum_{a=f,g} \sigma^a (v_{i,i}^a \widehat{u}_{j,j}^s + u_{i,i}^s \widehat{v}_{j,j}^a) - \\
& - \sum_{a,b=f,g} \sigma^{ab} v_{i,i}^a \widehat{v}_{j,j}^b + \mu_\nu v_{i,j}^f \widehat{v}_{i,j}^f + (\lambda_\nu + \mu_\nu) v_{i,i}^f \widehat{v}_{j,j}^f + \\
& + \frac{k}{T_0} T_{,i}\widehat{T}_{,i} + \sum_{a,b=f,g} \xi^{ab}(v_i^a - u_i^s)(\widehat{v}_i^b - \widehat{u}_i^s) + \\
& + (\gamma^f + \gamma^g - \alpha_0)(T\widehat{u}_{i,i}^s - u_{i,i}^s\widehat{T}) - \\
& - \sum_{a,b=f,g} (\alpha^a + \gamma^a)(T\widehat{v}_{i,i}^a - v_{i,i}^a\widehat{T})] dx,
\end{aligned}
\tag{3.19}
$$

where $\widehat{\mathbf{y}} = (\widehat{\mathbf{u}}^s, \widehat{\mathbf{v}}^f, \widehat{\mathbf{v}}^g, \widehat{T})$, and the linear operator $l : \mathbf{Y} \to R$

$$
\begin{aligned}
& l(\widehat{\mathbf{y}}) =< (\mathbf{g}^{*s}, \mathbf{g}^{*f}, \mathbf{g}^{*g}, g^*), \\
& (\rho_0^s \widehat{\mathbf{u}}^s, \rho_0^f \widehat{\mathbf{u}}^f, \rho_0^g \widehat{\mathbf{u}}^g, \alpha_1 \widehat{T} >_{\mathbf{L}^2 \times \mathbf{L}^2 \times \mathbf{L}^2 \times L^2} = \\
& = \int_\Omega [\rho_0^s (v_i^{*s} + u_i^{*s}) \widehat{u}_i^s + \rho_0^f v_i^{*f} \widehat{v}_i^f + \rho_0^g v_i^{*g} \widehat{v}_i^g + \alpha_1 T^* \widehat{T} + \\
& + (\xi^{ff} + 2\xi^{fg} + \xi^{gg}) u_i^{*s} \widehat{u}_i^s - \sum_{a=f,g} (\xi^{fa} + \xi^{ga}) u_i^{*s} \widehat{v}_i^a - \\
& - \sum_{a=f,g} \sigma^a r^{*a} \widehat{u}_{i,i}^s - \sum_{a,b=f,g} \sigma^{ab} r^{*a} \widehat{v}_{i,i}^b + (\gamma^f + \gamma^g - \alpha_0) u_i^{*s} \widehat{T}_{,i}] dx \ .
\end{aligned} \tag{3.20}
$$

It is easy to see that B is bounded, that is

$$
B(\mathbf{y}, \widehat{\mathbf{y}}) \leq m_1 \|\mathbf{y}\|_{\mathbf{Y}} \|\widehat{\mathbf{y}}\|_{\mathbf{Y}} \ , \qquad m_1 - \text{positive constant} \tag{3.21}
$$

and coercive

$$
\begin{aligned}
B(\mathbf{y}, \mathbf{y}) \ & \geq \int_\Omega [\rho_0^s u_i^s u_i^s + \rho_0^f v_i^f v_i^f + \rho_0^g v_i^g v_i^g + \alpha_1 T^2 + \\
& + \mu u_{i,j}^s u_{i,j}^s + (\lambda + \mu) u_{i,i}^s u_{j,j}^s - 2 \sum_{a=f,g} \sigma^a v_{i,i}^a u_{j,j}^s - \\
& - \sum_{a,b=f,g} \sigma^{ab} v_{i,i}^a v_{j,j}^b + \mu_\nu v_{i,j}^f v_{i,j}^f + (\lambda_\nu + \mu_\nu) v_{i,i}^f v_{j,j}^f + \\
& + \frac{k}{T_0} T_{,i} T_{,i}] dx \geq m_2 \|\mathbf{y}\|_{\mathbf{Y}}^2 \ , \quad m_2 - \text{positive constant}
\end{aligned} \tag{3.22}
$$

and the linear operator l is bounded

$$
l(\mathbf{y}) \leq m_3 \|\mathbf{y}\|_{\mathbf{Y}} \ , \qquad m_3 - \text{positive constant} \ . \tag{3.23}
$$

The Lax–Milgram theorem proves the existence of the solution $(\mathbf{u^s}, \mathbf{v^f}, \mathbf{v^g}, T) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{V}_0(\Omega) \times H_0^1(\Omega)$, of the system (3.15). By taking $v_i^s = u_i^s - u_i^{*s}$, $r^f = -v_{i,i}^f + r^{*f}$ and $r^g = -v_{i,i}^g + r^{*g}$ we deduce that $\omega = (\mathbf{u^s}, \mathbf{v^f}, \mathbf{v^g}, T, r^f, r^g) \in \mathcal{D}(\mathcal{A})$ is the solution of the system (3.13). The proof is complete.

Theorem 1. *The operator $\mathcal{A}$ defined by the relation* (3.3) *generates a C_0 contractive semigroup in X.*

Proof. The proof follows from the previous lemmas and the Lumer–Phillips theorem [12].

Using the results concerning the abstract Cauchy problem [12], we obtain the following:

Theorem 2. *Assume that* $f_i^s, f_i^f, f_i^g, h \in C^1([0,t_1); L^2(\Omega))$ *and* $\omega_0 \in \mathcal{D}(\mathcal{A})$. *Then there exists a unique solution* $\omega \in C^1((0,t_1); \mathbf{X}) \cap C^0([0,t_1); \mathcal{D}(\mathcal{A}))$ *of the boundary value problem* (3.5).

Since the semigroup defined by the operator $\mathcal{A}$ is contractive, we have the estimate

$$\begin{aligned} \|\omega(t)\|_{\mathbf{X}} \leq m_4 \Big(\|\omega_0(t)\|_{\mathbf{X}} + \int_0^t (\|\mathbf{f}^s\|_{\mathbf{L}^2(\Omega)} + \\ + \|\mathbf{f}^f\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}^g\|_{\mathbf{L}^2(\Omega)} + \|h\|_{L^2(\Omega)}) d\tau \Big) , \end{aligned} \tag{3.24}$$

m_4 – positive constant, which proves the continuous dependence of the solutions upon initial data and supply terms.

We recall that existence and uniqueness results in the theory of mixtures of elastic solids have been obtained using the semigroup theory by Ieşan and Quintanilla [10] and Martínez and Quintanilla [11].

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