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ON ℓ_p–STABILITY IN VARIATION OF DIFFERENCE EQUATIONS OF VOLTERRA TYPE

BY

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1. Introduction. The method of variation of parameters is well-known tool in the investigation of the properties of solutions of difference equations. It has been proved that the nonlinear version obtained by AGARWALL [1], [2] and BAINOV and SIMEONOV [3] is extremely useful and effective technique in the investigation of nonlinear problems.

In contrast to the usual Lyapunov function method, the use of a variation of parameters formula does not require that the unperturbed system satisfy very stringent hypotheses, and qualitative estimates are easily obtained.

In this paper, we shall investigate the stability properties of perturbed Volterra difference systems by combining the method of variation of parameters and the theory of difference inequalities.

We shall establish several criteria on so–called ℓ_p-stability and growth properties of solutions of the perturbed Volterra difference systems under the assumption that information on certain stability properties of unperturbed system is available.

The aim of the present paper is to extend parts of results from [4] concerning the uniform L^p-stability in variation of integro–differential equations to the case of certain difference systems of Volterra type. Actually we continue the study of the relationship between different kinds of behavior for difference equations begun in some of our earlier papers [5], [6].

2. Preliminaries. Now we explain some notations employed in this article. As usual, we denote by N the set of all nonnegative integers; $N(n_0) = \{n_0, n_0 + 1, ...,\}$, where $n_0 \in N$; R^q the q-dimensional real Euclidean space with norm $|x| = \sum_{i=1}^{q} |x_i|$, $x = (x_1, x_2, ..., x_q)$; for a $q \times q$ matrix $A = (a_{ij})$, define its norm $|A|$ by $|A| = \max_j \sum_{i=1}^{q} |a_{ij}|$.

In the sequel we are interested in the relation between solutions of the unperturbed difference system (equation)

$$(\mathrm{P}_0) \qquad x(n+1) = f(n, x(n))$$

and the solutions of its perturbed difference Volterra type systems

$$(\mathrm{P}_1) \qquad y(n+1) = f(n, y(n)) + \sum_{s=n_0}^{n-1} g(n, s, y(s)),$$

respectively,

$$(\mathrm{P}_0) \qquad y(n+1) = f(n, y(n)) + h(n, y(n)) + \sum_{s=n_0}^{n-1} g(n, s, y(s)),$$

where $f, h : N(n_0) \times R^q \to R^q$ and g is defined for $n \geq s \geq n_0$ and $|y| < H \leq \infty$ having its values in R^q. We assume that $f_x = \dfrac{\partial f}{\partial x}$ exists and is continuous and invertible on $N(n_0) \times R^q$ and also that $f(n, 0) = = h(n, 0) = g(n, s, 0) = 0$. Let $x(n) = x(n, n_0, x_0)$ be a solution of (P_0) with $x(n_0, n_0, x_0) = x_0$ and, respectively, $y(n) = y(n, n_0, y_0)$ a solution of (P_1) or (P_2) with $y(n_0, n_0, y_0) = y_0$.

At the same time, we consider the associated variational difference system

$$(\mathrm{V}) \qquad z(n+1) = f_x(n, x(n, n_0, x_0))z(n)$$

of (P_0) with respect to the solution $x(n, n_0, x_0)$, for which the fundamental matrix solution $\phi(n, n_0, x_0)$ is given by ([1])

$$\phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}, \ \phi(n_0, n_0, x_0) = I \text{ identity matrix}).$$

For two solutions $x(n,n_0,x_0)$, $x(n,n_0,y_0)$ of (P$_0$), existing for $n\in N(n_0)$ and such that x_0,y_0 belong to a convex subset of $\mathbb{R}^q$, we have

$$x(n,n_0,y_0)-x(n,n_0,x_0)=$$
$$=\left[\int_0^1 \phi(n,n_0,x_0+s(y_0-x_0))ds\right](y_0-x_0), \tag{2.1}$$

from where one obtains

$$x(n,n_0,x_0)=\left[\int_0^1 \phi(n,n_0,sx_0)ds\right]x_0. \tag{2.2}$$

We note that in case when $f(n,x)\equiv A(n)x$, where $A(n)$ is a $q\times q$ matrix on $N(n_0)$, we have $\phi(n,n_0,x_0)=X(n)X^{-1}(n_0)x_0$, $X(n)$ being a fundamental matrix for the corresponding system

$$x(n+1)=A(n)x(n). \tag{L}$$

If $x(n)$ and $y(n)$ are the solutions of (P$_0$), respectively, (P$_1$) with $x(n_0)=$ $=y(n_0)$, then the following formula of Alekseev type holds ([1], [2], [3])

$$y(n)=x(n)+\sum_{\ell=n_0}^{n-1}\left[\int_0^1 \phi(n,\ell+1,u(y(\ell),\tau))d\tau\right]\sum_{s=n_0}^{\ell-1} g(\ell,s,y(s)), \tag{2.3}$$

where $u(y(\ell),\tau)=f(\ell,y(\ell))+\tau\sum_{s=n_0}^{\ell-1} g(\ell,s,y(s)),\ 0\le\tau\le 1$. As is customary, in the following we use the convention $\sum_{i=s}^{k} u(i)=0$, $\prod_{i=s}^{k} v(i)=1$ when $s>k$.

An analogous formula to (2.3) is true when $y(n)$ is a solution of (P$_2$) with $y(n_0)=x(n_0)$.

We now give the definitions of different kinds of stability and of the concept of slowly–growing, in terms of behavior of solutions of (P$_0$) and of the variational system (V).

Definition 2.1. The solution $x=0$ of Eq. (P$_0$) is said to be:

a) *Uniformly stable in variation* (USV), if there exist $M\ge 1$ such that $|\phi(n,n_0,x_0)|\le M$ for all $n\in N(n_0)$, whenever $|x_0|<H$;

b) *Exponentially asymptotically stable in variation* (EASV), if there exist $L \geq 1$ and $\lambda > 0$ such that $|\phi(n, n_0, x_0)| \leq L \exp[-\lambda(n-n_0+1)]$ for all $n \in N(n_0)$, whenever $|x_0| < H$;

c) *Uniformly ℓ_p–stable in variation* ($p \geq 1$) (Uℓ_p-SV), if it is (USV) and $\sum_{\ell=n_0}^{\infty} |x(\ell, n_0, x_0)|^p < \infty$ for all $|x_0| < H$;

d) *Generalized slowly–growing in variation* (GSGV), if $|\phi(n, n_0, x_0)| \leq$ $\leq L(n_0) \exp\left(\sum_{s=n_0}^{n} \alpha(s)\right)$ for all $n \in N(n_0)$, whenever $|x_0| < H$, where $L(n)$ is a positive function and $\alpha(n)$ is a real function defined on $N(n_0)$;

e) When $L(n) \equiv L \equiv$ const. > 1, the solution $x = 0$ is said to be *generalized uniformly slowly–growing in variation* (GUSGV).

With regard to the definitions of different kinds of stability of the solution $y = 0$ of Eq. (P_1) or (P_2) we shall use the classical ones.

Definition 2.2. The solution $y = 0$ of Eq. (P_1) or (P_2) is said to be:

a) *Stable* (S), if for every $\varepsilon > 0$ and any $n_0 \in N$ there exists $\delta = \delta(n_0, \varepsilon) > 0$ such that $|y_0| < \delta$ and $n \in N(n_0)$ imply $|y(n, n_0, y_0)| < \varepsilon$;

b) *Uniformly stable* (US), if it is (S) and the above δ is independent of n_0;

c) *Exponentially asymptotically stable* (EAS), if there exists $\lambda > 0$ and for every $\varepsilon > 0$ there exists $\delta(n_0, \varepsilon) > 0$ such that $n_0 \in N$, $|y_0| < \delta$ and $n \in N(n_0)$ imply $|y(n, n_0, y_0)| < \varepsilon \exp[-\lambda(n - n_0)]$;

d) *Globally exponentially asymptotically stable* (GEAS), if there exists $\lambda >$ 0 and for any $\alpha > 0$ there exists $L(\alpha) \geq 1$ such that $|y_0| < \alpha$ and $n \in N(n_0)$ imply $|y(n, n_0, y_0)| \leq L(\alpha) \exp[-\lambda(n - n_0)]|y_0|$;

e) *Uniformly ℓ_p–stable* (Uℓ_p-S), $p \geq 1$, if it is (US) and $\sum_{\ell=n_0}^{\infty} |y(\ell, n_0, y_0|^p < \infty$ for all $|y_0| < H$.

Finally we give

Definition 2.3. The solutions of Eq. (P_1) or (P_2) are said to be *uniformly bounded* (UB), if for every $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that $n_0 \in N$, $|y_0| \leq \alpha$ and $n \in N(n_0)$ we have $|y(n, n_0, y_0)| \leq \beta(\alpha)$.

Regarding the functions $g(n, s, y)$ and $h(n, y)$, throughout this paper, we shall make growth hypotheses of the following kind:

Assume that there exist positive functions $a(n)$ and $b(n, s)$ defined for $n \geq s \geq n_0$ such that for $|y| < H$

(H1): $|g(n, s, y)| \leq b(n, s)|y|$;

(H2): $|h(n, y)| \leq a(n),\ |g(n, s, y)| \leq b(n, s)|y|$;

(H3): $|g(n, s, y)| \leq b(n, s)|y|^m,\ 0 < m \leq 1$;

(H4): $|h(n, y)| \leq a(n)|y|^m,\ |g(n, s, y)| \leq b(n, s)|y|^m,\ 0 < m \leq 1$;

(H5): $|g(n, s, y)| \leq a(n) + b(n, s)|y|^m,\ 0 < m \leq 1$.

In view to develop different stability or growth relationships between solutions of systems (P_0) and (P_1) or (P_2), we need some auxiliary results.

Lemma 2.1. ([5; Lemma 2.1]) *Let there exist functions $u(n), v(n)$, $h(n, s)$ and $\omega(\rho)$ such that:*

a) *$u(n), v(n), h(n, s)$ are nonnegative for $n \geq s \geq n_0$;*

b) *$\omega(\rho)$ is positive, continuous and nondecreasing for $\rho > 0$;*

c) *for any $n \in N(n_0)$ we have the inequality*

$$u(n) \leq c + \sum_{\ell=n_0}^{n-1} \left(v(\ell)\omega(u(\ell)) + \sum_{s=n_0}^{\ell} h(\ell, s)\omega(u(s)) \right),$$

where $c \geq n(n_0)$ is a positive constant. Then, for any $n \in N(n_0)$ we have

$$\Omega(u(n)) \leq \Omega(c) + \sum_{\ell=n_0}^{n-1} \left[v(\ell) + \sum_{s=n_0}^{\ell} h(\ell, s) \right], \tag{2.4}$$

where $\Omega(z) = \displaystyle\int \frac{dz}{\omega(z)}$.

Remark 2.1. When $\omega(\rho) = \rho$, (2.4) becomes

$$u(n) \le c \exp \sum_{\ell=n_0}^{n-1} \left[v(\ell) + \sum_{s=n_0}^{\ell} h(\ell, s) \right]. \tag{2.5}$$

Using the properties of ω it follows that there exists Ω^{-1} and (2.4) can be written as

$$u(n) \le \Omega^{-1} \left(\Omega(c) + \sum_{\ell=n_0}^{n-1} \left[v(\ell) + \sum_{s=n_0}^{\ell} h(\ell, s) \right] \right), \ n_0 \le n \le T, \tag{2.6}$$

where $T = \sup \left\{ n \ge n_0 : \Omega(c) + \sum_{\ell=n_0}^{n-1} \left[v(\ell) + \sum_{s=n_0}^{\ell} h(\ell, s) \right] \in \operatorname{dom} \Omega^{-1} \right\}$.

In the particular case when $\omega(\rho) = \rho^m$, $0 < m < 1$, we have $\Omega : [0, \infty) \to [0, \infty)$, and one obtains the estimate

$$u(n) \le \left[c^{1-m} + (1-m) \sum_{\ell=n_0}^{n-1} \left(v(s) + \sum_{s=n_0}^{\ell} h(\ell, s) \right) \right]^{1/(1-m)}, \quad n \in N(n_0). \tag{2.7}$$

When $\omega(\rho) = \rho^m$, $m > 1$, we have $\Omega : (0, \infty) \to (-\infty, 0)$, and one obtains

$$u(n) \le c \left[1 - (m-1) c^{m-1} \sum_{\ell=n_0}^{n-1} \left(v(\ell) + \sum_{s=n_0}^{\ell} h(\ell, s) \right) \right]^{-1/(m-1)}, \quad n \in N(n_0). \tag{2.8}$$

Lemma 2.2. ([6; Lemma 2.1]) *Let there exist nonnegative functions $u(n), v(n), h(n, s)$ such that for any $n \in N(n_0)$ we have*

$$u(n) \le v(n) + \sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) u^m, \ 0 < m < 1.$$

Then, for any $n \in N(n_0)$ are true the estimates:

$$\begin{aligned} u(n) \le v(n_0) \exp \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right] + \\ + \sum_{\ell=n_0}^{n-1} \Delta v(\ell) \exp \left[\sum_{k=\ell+1}^{n-1} \sum_{s=n_0}^{k} h(k, s) \right], \end{aligned} \tag{2.9}$$

if $m = 1$, *respectively*

$$u(n) \leq (1-m)^{-1} v(n) + \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right]^{1/(1-m)}, \tag{2.10}$$

if $0 < m < 1$.

We note that when $v(n)$ is assumed to be nondecreasing, then (2.9) reduces to

$$u(n) \leq v(n) \exp \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right], \; n \in N(n_0). \tag{2.11}$$

Indeed, because $\Delta v(\ell) \geq 0$, from (2.9) one obtains

$$u(n) = v(n_0) \exp \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right] + \exp \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right] \sum_{\ell=n_0}^{n-1} \Delta v(\ell) =$$

$$= v(n) \exp \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell} h(\ell, s) \right].$$

Lemma 2.3. ([6; Lemma 2.2]) *Let α be a real function defined on $N(n_0)$. If there exists a constant $K \geq 1$ such that*

$$\sum_{\ell=n_0}^{n} \exp \left[\sum_{s=\ell}^{n} \alpha(s) \right] \leq K, \quad \textit{for all } n \in N(n_0),$$

then

$$\lim_{n \to \infty} \exp \left[\sum_{\ell=n_0}^{n} \alpha(\ell) \right] = 0. \tag{2.12}$$

We note that (2.7) is possible only if $\alpha(s) < 0$.

Lemma 2.4. ([6; Lemma 2.3]) *Let λ be a positive constant and $g(n) \geq 0$ a function defined on $n \in N(n_0)$ and $g(n) \geq 0$ a function defined on $n \in N(n_0)$*

and $g(n) \in \ell_1(n_0)$. *Then* $\sum_{\ell=n_0}^{n} \exp[-\lambda(n-\ell)]g(\ell) \in \ell_p(n_0)$, $p \geq 1$, *and therefore it tends to zero as* $n \to \infty$.

As immediate consequences of Lemmas 2.1 and 2.2 we have the following results:

Corollary 2.1. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (USV), *that* (H5) *holds with* $sa(s) \in \ell_1(n_0)$ *and* $\sum_{s=n_0}^{n} b(\ell, s) \in \ell_1(n_0)$. *Then the solutions of* (P_1) *are* (UB).

Indeed, if $0 < m < 1$, according to (2.2) and (2.3) we have

$$|y(n)| \equiv |y(n, n_0, y_0| \leq M \left[|y_0| + \sum_{\ell=n_0}^{n-1} \ell a(\ell) + \sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell-1} b(\ell, s)|y(s)|^m \right],$$

for all $n \in N(n_0)$. By using Lemma 2.2, formula (2.10), we get

$$|y(n, n_0, y_0)| \leq M(1-m)^{-1} \left[|y_0| + \sum_{\ell=n_0}^{n-1} \ell a(\ell) \right] +$$

$$+ \left[M \sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell-1} b(\ell, s) \right]^{1/(1-m)},$$

from where the desired result.

To prove our assertion if $m = 1$, we shall use Lemma 2.2, formula (2.11), since in this case $v(n) = M \left[|y_0| + \sum_{\ell=n_0}^{n-1} \ell a(\ell) \right]$ is nondecreasing.

Corollary 2.2. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (USV), *that* (H4) *holds and* $a(\ell) + \sum_{s=n_0}^{\ell} b(\ell, s) \in \ell_1(n_0)$. *Then the solutions of* (P_2) *are* (UB).

To prove this corollary we use Lemma 2.1, formula (2.7). We note that when (H4) is satisfied with $m = 1$ we use (2.5) to prove the formulated assertion.

3. Relationship between the (Uℓ_p-SV) and the (Uℓ_p-S). This section is devoted to the study of the relationship between the (Uℓ_p-SV) of $x = 0$ of Eq. (P_0) and the (Uℓ_p-S) of $y = 0$ of Eq. (P_1) or (P_2).

In the following we shall assume that there exists a positive constant K such that

$$\sum_{\ell=n_0}^{\infty} \sum_{s=n_0}^{\ell} b(\ell, s) \leq K < \infty. \tag{3.1}$$

Theorem 3.1. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (Uℓ_p-SV) *and that* (H1) *and* (3.1) *hold. Furthermore, we assume that*

$$v(n) \equiv \left(\sum_{\ell=n_0}^{n} \left[\sum_{s=n_0}^{\ell} b(\ell, s) \right]^q \right)^{p/q} \in \ell_1(n_0), \;\; p^{-1} + q^{-1} = 1. \tag{3.2}$$

Then $y = 0$ *of Eq.* (P_1) *is* (Uℓ_p-S).

Proof. From the (Uℓ_p-SV) of $x = 0$ of Eq. (P_0), it is also (USV) and since (3.1) holds, by [5; Theorem 3.3] it follows that $y = 0$ of Eq. (P_1) is (US). To show that $|y(n, n_0, y_0)| \in \ell_p(n_0)$ whenever $|y_0| < H$, we shall use formula (2.3). So, we have

$$\begin{aligned}
&|y(n, n_0, y_0)|^p \leq \\
&\leq 2^{p-1} \left\{ |x(n, n_0, y_0)|^p + M^p \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell-1} b(\ell, s) |y(s, n_0, y_0)| \right]^p \right\} \leq \\
&\leq 2^{p-1} |x(n, n_0, y_0)|^p + 2^{p-1} M^p \left[\sum_{\ell=n_0}^{n-1} \sum_{s=n_0}^{\ell-1} b(\ell, s) \gamma(s) \right]^p,
\end{aligned}$$

where $\gamma(n) = \sup\{|y(s, n_0, y_0)|;\ n = n_0, n_0+1, ..., n\}$. Hence

$$|y(n, n_0, y_0)|^p \le$$
$$\le \gamma^p(n) \le 2^{p-1}|x(n, n_0, y_0)|^p + 2^{p-1}M^p \left[\sum_{\ell=n_0}^{n-1}\left(\sum_{s=n_0}^{\ell-1} b(\ell, s)\right)^q\right]^{p/q} \sum_{\ell=n_0}^{n-1}\gamma^p(\ell) =$$
$$= 2^{p-1}|x(n, n_0, y_0)|^p + 2^{p-1}M^p v(n) \sum_{\ell=n_0}^{n-1}\gamma^p(\ell).$$

For arbitrary $k \in N(n_0)$, we obtain

$$\sum_{n=n_0}^{k}\gamma^p(n) \le 2^{p-1}\sum_{n=n_0}^{k}|x(n, n_0, y_0)|^p + 2^{p-1}M^p \sum_{n=n_0}^{k} v(n)\sum_{\ell=n_0}^{n-1}\gamma^p(\ell) \le$$
$$\le K_1 + K_2 \sum_{n=n_0}^{m} v(n) \sum_{\ell=n_0}^{n-1}\gamma^p(\ell),$$

where $K_1 = 2^{p-1}\sum_{n=n_0}^{\infty}|x(n, n_0, y_0)|^p,\ K_2 = 2^{p-1}M^p.$

From here, by Bihari's inequality (formula (2.5) for $h(n, s) \equiv 0$), one obtains

$$\sum_{n=n_0}^{k}\gamma^p(n) \le K_1 \exp\left[K_2 \sum_{n=n_0}^{k} v(n)\right] \le K_1 \exp\left[K_2 \sum_{n=n_0}^{\infty} v(n)\right],$$

for any $k \in N(n_0)$, and therefore $\sum_{n=n_0}^{\infty}|y(n, n_0, y_0)|^p \le \sum_{n=n_0}^{\infty}\gamma^p(n) < \infty$, i.e., Theorem 3.1 is proved.

Remark 3.1. An analogous result can be formulated for $y = 0$ of Eq. (P_2). To this end, we shall assume that $x = 0$ of Eq. (P_0) is (Uℓ_p-SV), (H2) and (3.1) hold and $\sum_{n=n_0}^{n} a(\ell) \in \ell_1(n_0)$, $v(n) \in \ell_1(n_0)$.

Corollary 3.1. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (Uℓ_p-SV) *and* (H3) *holds. Moreover, we assume that*

$$w(n) \equiv \left(\sum_{\ell=n_0}^{n} \left[\sum_{s=n_0}^{\ell} b(\ell, s) \right]^r \right)^{p-m} \in \ell_1(n_0), \ \textit{where } r = p/(p-m).$$

Then all solutions of (P_1) *belong to* $\ell_p(n_0)$*.*

Proof. By using hypothesis (H3) and the Minkowski–Hölder inequality, we have as above

$$\gamma^p(n) \leq 2^{p-1} \left[|x(n, n_0, y_0)|^p + M^p \sum_{\ell=n_0}^{n-1} \left(\sum_{s=n_0}^{n-1} b(\ell, s) \right)^r \right]^{p/r} \left[\sum_{\ell=n_0}^{n-1} \gamma^p(\ell) \right]^m .$$

Again, for arbitrary $k \in N(n_0)$, we obtain

$$\sum_{n=n_0}^{k} \gamma^p(n) \leq K_1 + K_2 \sum_{n=n_0}^{k} w(n) \left(\sum_{\ell=n_0}^{n-1} \gamma^p(\ell) \right)^m ,$$

and, by the inequality (2.7), it follows

$$\sum_{n=n_0}^{k} \gamma^p(n) \leq \left[K_1^{1-m} + (1-m) K_2 \sum_{n=n_0}^{k} w(n) \right]^{1/(1-m)} ,$$

for any $k \in N(n_0)$. Therefore $|y(n, n_0, y_0)| \in \ell_1(n_0)$.

Remark 3.2. The conclusion of Corollary 3.1 remains valid when $m = 1$ if we assume that $w(n) \equiv \left[\sum_{\ell=n_0}^{n} \sum_{s=n_0}^{\ell} b(\ell, s) \right]^p \in \ell_1(n_0)$ and $2^{p-1} M^p K < 1$, where $K = \sup \left\{ \left[\sum_{\ell=n_0}^{n} \sum_{s=n_0}^{\ell} b(\ell, s) \right]^p : n \in N(n_0) \right\}$.

Remark 3.3. A result similar to Corollary 3.1 can be formulated for Eq. (P_2) if we assume that (H4) holds and, in addition,

$$w(n) \equiv \left(\sum_{\ell=n_0}^{n} \left[a(\ell) + \sum_{s=n_0}^{\ell} b(\ell, s) \right]^r \right)^{p-m} \in \ell_1(n_0).$$

Similar results can be formulated when $p = 1$.

Theorem 3.2. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (Uℓ_1-SV)*, that* (H1) *holds and* (3.1) *is satisfied with* $K < M^{-1}$. *Then* $y = 0$ *of Eq.* (P_1) *is* (Uℓ_1-S).

Proof. The (US) of $y = 0$ of Eq. (P_1) follows like in the proof of the first part of Theorem 3.1. Further more, we have

$$\gamma(n) \le |x(n, n_0, y_0)| + MK\gamma(n) \text{ for all } n \in N(n_0),$$

and therefore,

$$\sum_{n=n_0}^{\infty} |y(n, n_0, y_0)| \le \sum_{n=n_0}^{\infty} \gamma(n) \le (1 - MK)^{-1} \sum_{n=n_0}^{\infty} |x(n, n_0, y_0)| < \infty.$$

Remark 3.4. If we assume that $x = 0$ of Eq. (P_0) is (Uℓ_1-SV), that (H2) holds with $K < M^{-1}$ (K given by (3.1)) and, in addition, $\sum\limits_{\ell=n_0}^{n} a(\ell) \in \ell_1(n_0)$, then $y = 0$ of Eq. (P_2) is (Uℓ_1-S).

Corollary 3.2. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (Uℓ_1-SV) *and that* (H3) *holds. Furthermore, we assume that*

$$w(n) \equiv \left(\sum_{\ell=n_0}^{n} \left[\sum_{s=n_0}^{\ell} b(\ell, s) \right]^r \right)^{1-m} \in \ell_1(n_0), \ \ \textit{where } r = (1 - m)^{-1}.$$

Then all solutions of (P_1) *belong to* $\ell_1(n_0)$.

Indeed, this time we have

$$|y(n, n_0, y_0)| \le |x(n, n_0, y_0)| + M \left[\sum_{\ell=n_0}^{n-1} \left(\sum_{s=n_0}^{\ell-1} b(\ell, s) \right)^r \right]^{1/r} \left[\sum_{\ell=n_0}^{n-1} \gamma(\ell) \right]^{1/m},$$

from where, by using the same argument as in the proof of Corollary 3.1, one obtains the desired assertion.

Remark 3.5. If we assume that $x = 0$ of Eq. (P_0) is $(\mathrm{U}\ell_1\text{-SV})$, that (H4) holds and, in addition, $w(n)$ given in Remark 3.3 (with $r = (1-m)^{-1}$) belongs to $\ell_1(n_0)$, then all solutions of Eq. (P_2) belong to $\ell_1(n_0)$.

4. Relationship between the (GUSGV) and the ($\mathbf{U}\ell_p$-S). In the sequel we shall study the relationship between the (GUSGV) of $x = 0$ of Eq. (P_0) and the $(\mathrm{U}\ell_p\text{-S})$ of $y = 0$ of Eq. (P_1) or (P_2). To this end, with respect to the function $\alpha(n)$ from the definition of (GUSGV), we make the assumption that there exists a number $K \geq 1$ such that

$$\sum_{\ell=n_0}^{n} \exp\left[\sum_{s=\ell}^{n} \alpha(s)\right] \leq K, \text{ for all } n \in N(n_0). \tag{4.1}$$

We note that (4.1) and Lemma 2.3 yield to

$$\lim_{n\to\infty} \exp\left[\sum_{\ell=n_0}^{n} \alpha(\ell)\right] = 0. \tag{4.2}$$

Furthermore, we shall assume that

$$\sup\left\{\sum_{s=n_0}^{\ell} b(\ell, s) : \ell \in N(n_0)\right\} < (KL)^{-1}. \tag{4.3}$$

Theorem 4.1. *Assume that $x = 0$ of Eq.* (P_0) *is* (GUSGV) *and that* (H1), (4.1) *and* (4.3) *hold. Furthermore, we assume that*

$$\exp\left[p \sum_{s=n_0}^{n} \alpha(s)\right] \in \ell_1(n_0), \tag{4.4}$$

$$\sum_{\ell=n_0}^{n} \exp\left(-q \sum_{s=n_0}^{\ell} \alpha(s)\right) \left(\sum_{s=n_0}^{\ell} b(\ell, s)\right)^{q} \leq K_1, \tag{4.5}$$

for all $n \in N(n_0)$, $p^{-1} + q^{-1} = 1$.

Then $y = 0$ of Eq. (P_1) *is* $(\mathrm{U}\ell_p\text{-S})$. *Moreover, for every $\varepsilon > 0$ there exists $\delta(\varepsilon, n_0) > 0$ such that for a solution $y(n, n_0, y_0)$ of* (P_1) *for which $|y_0| < \delta$ the relation $\sum_{\ell=n_0}^{\infty} |y(n, n_0, y_0)|^p < \varepsilon$ holds.*

Proof. By the definition of (GUSGV) and hypotheses (H1), (4.1) and (4.3), in accordance with proof of the first part of Theorem 4.1 [6], it follows that $y = 0$ of Eq. (P_1) is (US). Then, by using the same type of reasoning as in the proof of Theorem 3.1, we have

$$\gamma^p(n) \leq 2^{p-1}\left\{L^p \exp\left(p\sum_{s=n_0}^{n}\alpha(s)\right)|y_0|^p + L^p\exp\left(p\sum_{s=n_0}^{n}\alpha(s)\right)\cdot\right.$$

$$\left.\cdot\left[\sum_{\ell=n_0}^{n-1}\exp\left(-q\sum_{s=n_0}^{\ell}\alpha(s)\right)\left(\sum_{s=n_0}^{\ell-1}b(\ell,s)\right)^q\right]^{p/q}\sum_{\ell=n_0}^{n-1}\gamma^p(\ell)\right\}.$$

From here, for arbitrary $k \in N(n_0)$, one obtains

$$\sum_{n=n_0}^{k}\gamma^p(n) \leq K_2|y_0|^p + K_3\sum_{n=n_0}^{k}\exp\left(p\sum_{s=n_0}^{n}\alpha(s)\right)\sum_{\ell=n_0}^{n-1}\gamma^p(\ell),$$

where $K_2 = 2^{p-1}L^p\displaystyle\sum_{n=n_0}^{\infty}\exp\left(p\sum_{s=n_0}^{n}\alpha(s)\right)$, $K_3 = 2^{p-1}L^pK_1^{p/q}$. Therefore,

$$\sum_{n=n_0}^{\infty}|y(n,n_0,y_0)|^p \leq \sum_{n=n_0}^{\infty}\gamma^p(n) \leq K_2|y_0|^p\exp K_3\sum_{n=n_0}^{\infty}\exp\left(p\sum_{s=n_0}^{n}\alpha(s)\right) < \infty.$$

The last assertion of the theorem immediately follows.

Remark 4.1. By using (2.2), it is a simple matter to show that (GUSGV) and (4.4) imply the ($\mathrm{U}\ell_p$-SV) of $x = 0$ of Eq. (P_0).

Remark 4.2. A similar result can be formulated for (P_2) if we accept (H2), (4.4), (4.5) and the (GUSGV) of $x = 0$ of Eq. (P_0). Moreover, we shall assume that

$$\sum_{\ell=n_0}^{n}a(\ell) \in \ell_1(n_0) \quad \text{and} \quad \sup\left\{a(\ell) + \sum_{s=n_0}^{\ell}b(\ell,s) : \ell \in N(n_0)\right\} < KL)^{-1}.$$

Theorem 4.1 takes a particularly simple form when $\alpha(n) = -\lambda$, where λ is a positive constant and $b(n,s) = \exp[-(\lambda+\sigma)n + \sigma s]c(s)$, $\sigma > 0$ and

$c(s)$ a positive function on $N(n_0)$. We note that in this case the (GUSGV) of $x = 0$ of Eq. (P_0) reduces to its (EASV). Thus we have

Corollary 4.1. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (EASV) *and* $c(n) \in \ell_1(n_0)$. *Then the conclusions of Theorem* 4.1 *hold.*

Proof. Clearly that (4.1) and (4.4) are satisfied. Then, we have

$$\sum_{s=n_0}^{\ell} b(\ell, s) = e^{-(\lambda+\sigma)\ell} \sum_{s=n_0}^{\ell} e^{\sigma s} c(s) \leq e^{-\sigma \ell} \sum_{s=n_0}^{\ell} e^{\sigma s} c(s) \to 0$$

as $\ell \to \infty$, according to Lemma 2.4, so that (4.3) holds.

Again by using Lemma 2.4, since $c(n) \in \ell_1(n_0)$ and $c(n) > 0$, it follows that $\sum\limits_{\ell=n_0}^{n} \exp[-\sigma(n-s)c(s) \in \ell_q(n_0)$, $q \geq 1$, and therefore (4.5) is satisfied. So, all hypotheses of Theorem 4.1 being accomplished, the conclusion follows from above.

The results corresponding to those in Theorem 4.1, in the special case $p = 1$, take the following form.

Theorem 4.2. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (GUSGV) *and that* (H1), (4.1) *and* (4.3) *hold. Furthermore, we assume that*

$$\exp\left[\sum_{\ell=n_0}^{n} \alpha(\ell)\right] \in \ell_1(n_0), \tag{4.6}$$

$$\exp\left[-\sum_{\ell=n_0}^{n} \alpha(\ell)\right] \sum_{s=n_0}^{n} b(n, s) \leq K_1 < \infty. \tag{4.7}$$

Then $y = 0$ *of Eq.* (P_1) *is* $(\mathrm{U}\ell_1\text{-S})$. *Moreover, for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon, n_0) > 0$ *such that for each solution* $y(n, n_0, y_0)$ *of* (P_1) *for which* $|y_0| < \delta$ *the relation* $\sum\limits_{\ell=n_0}^{\infty} |y(n, n_0, y_0)| < \varepsilon$ *holds.*

Proof. The (US) of $y = 0$ of Eq. (P_1) follows by the same arguments made in the proof of Theorem 4.1. Then we have

$$|y(n, n_0, y_0)| \leq L \exp\left[\sum_{\ell=n_0}^{n} \alpha(\ell)\right] |y_0| +$$

$$+ L \exp\left[\sum_{\ell=n_0}^{n} \alpha(\ell)\right] \left[\sum_{\ell=n_0}^{n-1} \exp\left(-\sum_{s=n_0}^{\ell} \alpha(s)\right)\right] \left(\sum_{s=n_0}^{\ell-1} b(\ell, s)\right) \gamma(\ell).$$

From here, for an arbitrary $k \in N(n_0)$, one obtains

$$\sum_{n=n_0}^{k} \gamma(n) \leq K_2 |y_0| + L K_1 \sum_{n=n_0}^{k} \exp\left(\sum_{\ell=n_0}^{n} \alpha(\ell)\right) \sum_{\ell=n_0}^{n-1} \gamma(\ell),$$

where $K_2 = \sum_{n=n_0}^{\infty} \exp\left(\sum_{\ell=n_0}^{n} \alpha(\ell)\right)$.

By Bihari's inequality one obtains

$$\sum_{n=n_0}^{\infty} |y(n, n_0, y_0)| \leq$$

$$\leq \sum_{n=n_0}^{\infty} \gamma(n) \leq K_2 |y_0| \exp\left[L K_1 \sum_{n=n_0}^{\infty} \exp\left(\sum_{\ell=n_0}^{n} \alpha(s)\right)\right] \leq$$

$$\leq K_2 |y_0| \exp(L K_1 K_2) < \infty.$$

From this inequality it follows the last part of the theorem.

A result analogous to Corollary 3.1 is given by

Theorem 4.3. *Assume that* $x = 0$ *of Eq.* (P_0) *is* (GUSGV) *and that* (H3) *and* (4.4) *hold. Furthermore, we assume that*

$$\sum_{\ell=n_0}^{n} \exp\left(-r \sum_{s=n_0}^{\ell} \alpha(s)\right) \left(\sum_{s=n_0}^{\ell} b(\ell, s)\right)^{r} \leq K_1, \text{ for all } n \in N(n_0), \tag{4.8}$$

where $r = p/(p-m)$. Then all solutions of (P$_1$) *belong to* $\ell_p(n_0)$.

Proof. By using hypothesis (H3) and Hölder inequality, we have

$$|y(n,n_0,y_0)|^p \leq 2^{p-1}\left\{L^p \exp\left(p\sum_{\ell=n_0}^{n}\alpha(\ell)\right)|y_0|^p + L^p \exp\left(p\sum_{\ell=n_0}^{n}\alpha(\ell)\right)\cdot\right.$$

$$\left.\cdot\left[\sum_{\ell=n_0}^{n-1}\exp\left(-r\sum_{s=n_0}^{\ell}\alpha(s)\right)\left(\sum_{s=n_0}^{\ell-1}b(\ell,s)\right)^r\right]^{p/r}\left[\sum_{\ell=n_0}^{n-1}\gamma(\ell)\right]^m\right\}.$$

So, one obtains

$$\sum_{n=n_0}^{k}\gamma^p(n) \leq K_2|y_0|^p + K_3\sum_{n=n_0}^{k}\exp\left(p\sum_{\ell=n_0}^{n}\alpha(\ell)\right)\left(\sum_{\ell=n_0}^{n-1}\gamma^p(\ell)\right)^m,$$

for any $k \in N(n_0)$, where

$$K_2 = 2^{p-1}L^p\sum_{n=n_0}^{\infty}\exp\left(p\sum_{\ell=n_0}^{n}\alpha(\ell)\right),\ K_3 = 2^{p-1}L^pK_1^{p/r} = 2^{p-1}K_1^{p-m}.$$

An application of the inequality (2.7), with $v(n) \equiv 0$, yields to

$$\sum_{n=n_0}^{k}\gamma^p(n) \leq \left[(K_2|y_0|^p)^{1-m} + (1-m)K_3\sum_{n=n_0}^{k}\exp\left(p\sum_{\ell=n_0}^{n}\alpha\ell)\right)\right]^{1/(1-m)},$$

from where one obtains the desired result.

Remark 4.3. Analogous results can be formulated as regard the solutions of (P$_1$) when $p = 1$, respectively, for the solutions of (P$_2$) under hypothesis (H4), $p \geq 1$.

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