ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIX, s.I a, Matematică, 2003, f.1.

BARRIER METHODS FOR MULTICRITERIAL OPTIMIZATION

BY

CONSTANTIN ILIOI

Let us consider the multicriterial minimization problem:

$$\min_{v\in C} f(v),\ C=\{v\in\mathbb{R}^n | g(v)\leq 0\} \tag{1}$$

with $f:\mathbb{R}^n\to\mathbb{R}^p,\ p\geq 2;\ g:\mathbb{R}^n\to\mathbb{R}^m, m\geq 1$ where f and g are continuous functions with convex components, $C\neq\emptyset$ and f is such that

$$(\exists)(f_0\in\mathbb{R}^p), f(v)\geq f_0,\ (\forall)(v\in\mathbb{R}^n) \tag{2}$$

The vectorial partial order is the usual (by components), the strict order being for all components.

Definition 1. A vector $u\in C$ is called a Slater (Pareto) solution of the problem (1), if there does not exist any admissible vector $v\in C$, such that $f(v)<f(u)$.

Definition 2. A vector $w\in\mathbb{R}^p$ is called a barrier vector for our problem if there does not exist any admissible vector v such that $f(v)<w$.

We propose two iterative methods of the penalty - Morrison type for approximating a Slater solution of the problem.

Definition 3. A continuous function $P:\mathbb{R}^q\to\mathbb{R}$, is called a forcing function if it satisfies the following conditions:

i) $(\forall)(x\in\mathbb{R}^q),\ P(x)=0$ if $x\leq 0$ and $P(x)>0$ otherwise;

ii) P is increasing when $P(x)>0$, that is:

$$x_1\leq x_2, (x_1\neq x_2), P(x_1)>0, P(x_2)>0 \text{ imply } P(x_1)<P(x_2);$$

iii) P is positively homogeneous, that is

$$(\forall)(x \in \mathbb{R}^q), (\forall)(\lambda > 0),\ P(\lambda x) = \lambda P(x)$$

A typical example of a forcing function is given by:

$$(\forall)(x \in \mathbb{R}^q), x = (x^1, x^2, \cdots, x^q), P(x) = \|x_+\|$$

where $x_+ = (x_+^1, x_+^2, \cdots, x_+^q), x_+^i = \max\{0, x^i\}$, and the norm is any of the usual norms (Euclidean norm is prefered since it is differentiable).

Let $q = p+m$ and let us use the notation $x = [y, z]$, with $y \in \mathbb{R}^p, z \in \mathbb{R}^m$. For each $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^p$ we shall define $x : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$:

$$x(v, w) = [f(v) - w, g(v)] \tag{3}$$

and $J : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ by

$$J(v, w) = P(x(v, w)) = P[f(v) - w, g(v)] \tag{4}$$

Theorem 1. *If $w \in \mathbb{R}^p$ is barrier vector for the problem (1) and $u \in \mathbb{R}^n$ is such that*

$$J(u, w) = 0, \tag{5}$$

then u is a Slater solution of the problem. Conversely, if u is a Slater solution, then $J(u, f(u)) = 0$ and $w = f(u)$ is a barrier vector.

Proof. The equality $J(u, w) = 0$ implies $x(u, w) \leq 0$ that is $f(u) - w \leq 0, g(u) \leq 0$. Hence $u \in C$. In order to prove that u is a Slater solution, let us suppose that would exist $v \in C$ such that $f(v) < f(u)$. Then $f(v) - w < f(u) - w \leq 0$, which is absurd, because w is a barrier vector. Therefore u is a Slater solution of our problem. The converse statement is quite obvious.

Remark. Under the hypotheses of the theorem, in the relation $f(u) \leq w$, at least a component of $f(u)$ is equal to the corresponding component of w, since w is a barrier vector.

We describe now a first iterative method for approximating a Slater solution of the problem by constructing a sequence of barrier vectors.

Algorithm 1.
1. Let w_0 be a barrier vector; $h > 0, h \in \mathbb{R}^p; k = 0$.
2. Solve the problem of unconstrained minimization

$$\min_{v \in \mathbb{R}^n} J(v, w_k)$$

and find a solution u_k.

3. If $J(u_k, w_k) = 0$ then stop; otherwise take

$$w_{k+1} = w_k + \lambda_k h,$$

$$\lambda_k = J(u_k, w_k)/P(h, 0). \tag{6}$$

4. Take $k = k + 1$ and go to step 2.

Theorem 2. *If w_k is a barrier vector and $J(u_k, w_k) > 0$, then w_{k+1}, as constructed by Algorithm 1, is also a barrier vector.*

Proof. Suppose that w_{k+1} would not be a barrier vector. Then there will exist $v \in C$ such that $f(v) < w_{k+1}$. Therefore,

$$J(v, w_k) = P[f(v) - w_k, g(v)] \leq P[f(v) - w_k, 0] =$$

$$P[f(v) - w_{k+1} + \lambda_k h, 0] < P[\lambda_k h, 0] = \lambda_k P[h, 0] = J(u_k, w_k),$$

which is absurd by step 2 of the algorithm. Hence w_{k+1} is a barrier vector.

Remark. By the Theorem 1 and 2 the algorithm constructs a sequence of barrier vectors (w_k) starting with w_0. If it happens that $J(u_k, w_k) = 0$, then the algorithm stops and, according to Theorem 1, u_k is a Slater solution. Otherwise the algorithm never stops and it generates two infinite sequences (u_k) in $\mathbb{R}^n$ and (w_k) in $\mathbb{R}^p$. In order to stop the algorithm we have to show the convergence, so that, after a finite number of iterations, to obtain a good approximation.

Theorem 3. *Suppose that the domain C of constraints satisfies the following hypotheses.*

$$(\exists)(g_0 \in \mathbb{R}^m),\ g_0 < 0 \ \text{ such that } \ g(v) \geq g_0, (\forall)(v \in C) \tag{7}$$

$$\begin{gathered}(\exists)(\delta \in \mathbb{R}), \delta > 0 \text{ such that the set} \\ M(\delta) = \{v \in \mathbb{R}^n |\ g^i(v) \leq \delta,\ 1 \leq 1 \leq m\} \text{ is bounded}\end{gathered} \tag{8}$$

Then the limit points of the sequence (u_k) are Slater solutions for the problem (1), (w_k) is convergent to a barrier vector and

$$\lim_{k \to \infty} J(u_k, w_k) = 0.$$

Proof. As w_0 is a barrier vector, there does not exist any $v \in C$ such that $f(v) < w_0$. By our hypotheses C is bounded and, since the functions f and g are continuous, f is bounded on C. Then let us denote:

$$\overline{\lambda} = \inf\{\lambda \geq 0 \mid (\exists)(v \in C),\ f(v) \leq w_0 + \lambda h\} \tag{9}$$

According to the algorithm we have:

$$w_{k+1} = w_0 + (\sum_{s=0}^{k} \lambda_s)h \tag{10}$$

and w_{k+1} is a barrier vector. So $\sum\limits_{s=0}^{k} \lambda_s < \overline{\lambda}$, where from the series $\sum\limits_{s=0}^{\infty} \lambda_s$ is convergent. It is following that $\lim\limits_{k\to\infty} w_k = w \leq w_0 + \overline{\lambda}h$ and $\lim\limits_{k\to\infty} \lambda_k = 0$, where from $\lim\limits_{k\to\infty} J(u_k, w_k) = 0$, that is

$$\lim_{k\to\infty} P[f(u_k) - w_k, g(u_k)] = 0$$

$(f(u_k) - w_k)$ and $(g(u_k))$ are lower bounded and also upper bounded according to the properties of P. But any limit point of $[f(u_k) - w_k, g(u_k)]$ is a nonpositive vector. Then, for k large enough, we have $g(u_k) \leq \delta$ so (u_k) will be bounded.

Let u be a limit point of (u_k). We have $f(u) - w \leq 0$ and $g(u) \leq 0$, that is $u \in C$ and $J(u, w) = 0$. Since w_k are barrier vectors, then there does not exist any $v \in C$ such that $f(v) < w_k < w$. Hence w is a barrier vector. According to Theorem 1, u is a solution of the problem (1).

The starting barrier vector w_0 could be f_0 for example, if it is known; or w_0 could be obtain by solving the problems of minimization for each component of the objective function f.

The choice of the direction $h > 0$ is quite arbitrary. But we expect to obtain different solutions for each choice of h, since we did not assume the uniqueness under our hypotheses.

The problem of unconstrained minimization from the step 2, cannot be solved efficiently in general hypotheses. For obtaining better results we must assume differentiability of the forcing function P on $\mathbb{R}^q$, with the possible exception of the boundary of $\mathbb{R}^q$. This condition is satisfied for the Euclidean norm. Also we suppose that P is subadditive (hence convex) which is also satisfied for the Euclidean norm.

In that case the step 2 of Algorithm 1 is solvable by a gradient method and also the step 3 can be modified. So that we have a new variant:

Algorithm 2.
1. Let w_0 a barrier vector, $h > 0, k = 0$;
2. Solve the unconstrained problem

$$\min J(v, w_k)$$

and find $u_k \in \mathbb{R}^n$;
3. If $J(u_k, w_k) = 0$ stop; otherwise

$$w_{k+1} = w_k + \mu_k h$$

$$\mu_k = \langle P'(x(u_k, w_k)), x(u_k, w_k)\rangle_q / \langle P'_y(x(u_k, w_k)), h\rangle_p \tag{11}$$

4. $k = k + 1$ go to step 2.

In (11) we denote by $\langle\cdot,\cdot\rangle_q$ and $\langle\cdot,\cdot\rangle_p$ the scalar products in $\mathbb{R}^q$, respectively $\mathbb{R}^p$; by P', P'_y the gradients of P in $\mathbb{R}^q$, respectively with respect to the first p components.

Theorem 4. *If w_k is a barrier vector and $J(u_k, w_k) > 0$, then $0 < \mu_k < \infty$.*

Proof. Let us denote $x_k = x(u_k, w_k), \delta_k = J(u_k, w_k) = P(x_k)$. Since the foreign function P is convex, the level set

$$L(\delta_k) = \{x \in \mathbb{R}^q | P(x) \leq \delta_k\}$$

is convex. Consider also the set

$$M_k = \{x = [y, z] \in \mathbb{R}^q / (\exists)(v \in \mathbb{R}^n), y \geq f(v) - w_k, z \geq g(v)\}$$

It is easy to see that M_x is also convex by using the convexity of f and g. For $[y, z] \in M_k$ we have

$$P[y, z] \geq P[f(v) - w_k, g(v)] = J(v, w_k) \geq J(u_k, w_k) = \delta_k.$$

The intersection of $L(\delta_k)$ and M_k consists only in the vector $x_k = x(u_k, w_k)$. Then according to the Hahn - Banach theorem of separation, $L(\delta_k)$ and M_k are separated by a supporting hyperplane passing by x_k. The minimal value of P is not achieved at x_k, because $P(x_k) = \delta_k > 0$. Hence, $P'(x_k) \neq 0$ and the supporting hyperplane will be $\langle P'(x_k), x - x_k\rangle = 0$. x_k is the minimal point on M_k, so we get

$$\langle P'(x_k), x - x_k\rangle \geq 0 \text{ for } x \in M_k \tag{12}$$

and

$$\langle P'(x_k), x - x_k\rangle < 0 \text{ for } x \in L(\delta_k) \tag{13}$$

Since $P(0) = 0 < \delta_k$ we have $0 \in L(\delta_k)$. So by taking $x = 0$ in (13) we obtain

$$\langle P'(x_k), x_k \rangle > 0. \tag{14}$$

Let us remark that each $x \leq x_k$ belongs to $L(\delta_k)$ because $P(x) \leq P(x_k) = \delta_k$. Therefore by (13) we get $\langle P'(x_k), x - x_k \rangle < 0$ for each $x - x_k \leq 0$. Hence $P'(x_k) > 0$ and

$$\langle P'_y(x_k), h \rangle > 0. \tag{15}$$

Theorem 5. *For Algorithm 2, if w_k is a barrier vector and $J(u_k, w^k) > 0$ then w_{k+1} is also a barrier vector.*

Proof. Let us suppose that w_{k+1} would not be a barrier vector. Then it will exist $v \in C$ such that $f(v) < w_{k+1}$, that is $x(v, w_{k+1}) \leq 0$ and $\langle P'(x_k), x(v, w_{k+1}) \rangle < 0$. But

$$x(v, w_{k+1}) = [f(v) - w_{k+1}, g(v)] = [f(v) - w_k - \mu_k h, g(v)] =$$

$$= [f(v) - w_k, g(v)] - \mu_k [h, 0] = x(v, w_k) - \mu_k [h, 0]$$

So

$$\langle P'(x_k), x(v, w_{k+1}) \rangle = \langle P'(x_k), x(v, w_k) \rangle - \mu_k \langle P'_y(x_k), h \rangle =$$

$$= \langle P'(x_k), x(v, w_k) \rangle - \langle P'(x_k), x_k \rangle \geq 0.$$

This contradiction yields the conclusion.

Theorem 6. *In the hypotheses of Theorem 3 any limit point of the sequence (u_k) constructed by the second iterative method is a Slater solution of the problem.*

Proof. As in the proof of Theorem 3 by the convergence of the series $\sum_{k=0}^{\infty} \mu_k$ we obtain $\lim_{k \to \infty} \mu_k = 0$ where from, under our hypotheses

$$\lim_{k \to \infty} \langle P'(x(u_k, w_k)), x(u_k, w_k) \rangle = 0.$$

From the convexity of P we get

$$P(0) - P(x(u_k, w_k)) \geq \langle P'(x(u_k, w_k)), -x(u_k, w_k) \rangle$$

where from

$$J(u_k, w_k) \leq \langle P'(x(u_k, w_k)), x(u_k, w_k)\rangle$$

Therefore

$$\lim_{k\to\infty} J(u_k, w_k) = 0$$

From here the proof is identical with that of the Theorem 3.

REFERENCES

1. Kurcyusz, S. and Majchrzak, J. – *Morrison type algorithms for constrained optimization, Control and Cybernetics*, vol. 8, 1979, 131-140, 227-240.
2. Podinovski, V.V. and Nogin, B.D. – *Pareto optimalnîie reşenia mnogokriterialnîi zadaci,* Nauka, Moscow, 1982.
3. Rockafellar, R.T. – *Convex analysis,* Princeton University Press, 1979.

Received: 03.II.2003

Faculty of Mathematics,
University of Iaşi,
Iaşi 6600,
ROMANIA