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RIEMANNIAN MANIFOLDS CARRYING A PAIR OF SKEW SYMMETRIC KILLING VECTOR FIELDS

BY

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Dedicated to Prof. Dr. Radu MIRON on the occasion of his 75th birthday

Abstract. We deal with a Riemannian manifold M carrying a pair of skew symmetric Killing vector fields (X, Y). The existence of such a pairing is determined by an exterior differential system in involution (in the sense of Cartan). In this case, M is foliated by 3-dimensional submanifolds of constant scalar curvature.

Additional geometric properties are proved.

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Introduction. Killing vector fields X (or *infinitesimal isometries*) play a distiguished role in Differential Geometry. They also play an important role when dealing with manifolds having indefinite metrics (as, for instance, connected Lorentzian manifolds or para-Kaehlerian manifolds [ER]), or also when X is a null vector field in general relativity.

Skew symmetric Killing (abbr. SSK) vector fields have been defined in [R1], and also studied in [MMR], [DRV] and [R2].

In the present paper we consider the case when a Riemannian C^∞-manifold M carries a pairing (X, Y) of SSK vector fields, such that the generator U of (X, Y) is a *torse forming* (abbr. TF) [Y1]. It is first proved

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that the existence of such a pairing is determined by an exterior differential system in involution (in the sense of CARTAN [C]), which depends on 2 arbitrary functions of 5 arguments.

Further it is shown that all three vector fields X, Y and U posses the property to be *exterrior concurrent* (abbr. EC) vector fields [R1], [PRV], that is. satisfy the relation

$$\nabla^2 Z = \lambda Z^\flat \otimes dp, \tag{0.1}$$

where λ is a conformal scalar (which in the case under consideration is constant), $\lambda \in C^\infty M$, $Z \in \Gamma TM$ and dp is the soldering form of M.

In consequence of these facts, one may state the following.

Any Riemannaian manifold M which carries SSK vector fields pairing is foliated by 3-dimensional submanifolds M_U (tangent to the set $\{X, Y, U\}$) of constant sectional curvature $-\lambda$. The immersion $x : M_U \to M$ is *totally geodesic*.

In addition, it is shown that:

i) U defines an *infinitesimal conformal transformation* (abbr. ICT) of X and Y and X, Y, U define an ICT of the volume element of M_U;

ii) the following properties are mutually equivalent:

a) U is of constant lenght;

b) U commutes with X and Y;

c) U is a geodesic.

iii) if M is compact, then necessarly $\lambda = -a^2$ and the following Yano integral formulas [Y2]

$$\int [(n-1)a^2 \|X\|^2 - 2(\|U\|^2 \|X\|^2 - g(U,X)^2)] = 0,$$

$$\int [(n-1)a^2 \|Y\|^2 - 2(\|U\|^2 \|Y\|^2 - g(U,Y)^2)] = 0$$

hold good, where $n = \dim M$.

1. Preliminaries. Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume that M is oriented and that ∇ is the Levi-Civita connection. Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ the classical musical isomorphisms defined by g.

Following [P], we set $A^q(M,TM) = \Gamma Hom(\Lambda^q TM, TM)$ and notice that elements of $A^q(M,TM)$ are vector valued q-forms, $q \leq \dim M$. Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$ the exterior covariant derivative operator with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$).

If $p \in M$, then the vector valued 1-form $dp \in A^1(M,TM)$ is called *the soldering form* of M (dp is the canonical vector valued 1-form of M and one has $d^\nabla(dp) = 0$). The operator

$$d^\omega = d + e(\omega) \tag{1.1}$$

acting on ΛM is called *the cohomology operator*, where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, i.e.

$$d^\omega u = du + \omega \wedge u, \tag{1.2}$$

for any $u \in \Lambda M$. We have $d^\omega \circ d^\omega = 0$ and if $d^\omega u = 0$, u is said to be d^ω-*closed* [GL].

Any vector field $X \in \Gamma TM$ such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M,TM) \tag{1.3}$$

is said to be an *exterior concurrent* vector field.

The 1-form π, which is called the *concurrence form*, is given by

$$\pi = f X^\flat, \ f \in C^\infty M. \tag{1.4}$$

If $\mathcal{R}$ denotes the Ricci tensor of ∇, we have

$$\mathcal{R}(X,Y) = -(n-1) f g(X,Z), \ Z \in \Gamma TM, \ n = \dim M, \tag{1.5}$$

and consequently

$$f = -\frac{1}{n-1}\mathrm{Ric}(X), \tag{1.6}$$

where $\mathrm{Ric}(X)$ means the Ricci curvature of M with respect to X.

A vector field $\mathcal{T}$ on M such that

$$\nabla \mathcal{T} = \lambda dp + \gamma \otimes \mathcal{T}, \ \lambda \in C^\infty M, \tag{1.7}$$

is defined as a *torse forming* (abbr. TF) vector field.

The 1-form γ is called the *generating form* of $\mathcal{T}$ [PRV] and we may prove that γ is related to the dual form $\mathcal{T}^\flat$ of $\mathcal{T}$ by

$$d\mathcal{T}^\flat = \gamma \wedge \mathcal{T}^\flat. \tag{1.8}$$

We notice that the above equation shows that $\mathcal{T}^\flat$ is an exterior recurrent form [D], having ω as recurrence form.

Let $\mathcal{O} = vect\{e_A | A \in \{1, ..., n\}\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = covect\{\omega^A\}$ be its associated coframe.

With respect to $\mathcal{O}$ and $\mathcal{O}^*$, the soldering form dp and E. Cartan's structure equations in indexless form are

$$dp = \omega^A \otimes e_A \in A^1(M, TM), \tag{1.9}$$

$$\nabla e = \theta \otimes e \in A^1(M, TM), \tag{1.10}$$

$$d\omega = -\theta \wedge \omega, \tag{1.11}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.12}$$

In the above equations, θ, respectively Θ, are the local connection forms in the bundle $\mathcal{O}(M)$, respectively the curvature forms on M.

A vector field X such that

$$\nabla X = V \wedge U, \tag{1.13}$$

where $\wedge$ denotes as usual the wedge product of vector fields ($\wedge$ is a linear operator which is skew symmetric with respect to g), has been defined in [MRV] as a SSK vector field. As is known, one may develop (1.13) as

$$\nabla X = U^\flat \otimes V - V^\flat \otimes U. \tag{1.14}$$

2. Main Result. Let (M, g) be an n-dimensional oriented Riemannian C^∞-manifold. Then be reference to [BR] (see also (1.14)), we agree to say that two vector fields X and Y such that

$$\begin{cases} \nabla X = X \wedge U, \\ \nabla Y = Y \wedge U \end{cases} \tag{2.1}$$

define a *skew symmetric Killing vector fields pairing.*

In (2.1), U is a certain vector called the *generator* of X and Y. In order to simplify, we set

$$X^\flat = \alpha,\ Y^\flat = \beta,\ U^\flat = u. \tag{2.2}$$

In terms of the above notations, equations (2.1) may be expanded as:

$$\begin{cases} \nabla X = u \otimes X - \alpha \otimes U, \\ \nabla Y = u \otimes Y - \beta \otimes U. \end{cases} \tag{2.3}$$

In the present paper, we will assume that the genarator U is a TF vector field [Y1] (see also (1.7)). Therefore, the covariant derivative ∇U of U is expressed by

$$\nabla U = \lambda dp + \gamma \otimes U,\ \lambda \in C^\infty M, \tag{2.4}$$

where γ is the recurrence form of u [MRV] (see (1.8)), i.e.

$$du = \gamma \wedge u. \tag{2.5}$$

Setting

$$X = X^A e_{A,}\, Y = Y^A e_{A,}\, U = U^A e_A, \tag{2.6}$$

then using the structure equations (1.10), one derives from (2.3) by a standard calculation

$$\begin{cases} d\alpha = 2u \wedge \alpha, \\ d\beta = 2u \wedge \beta, \end{cases} \tag{2.7}$$

which are coherent with the general properties of SSK vector fields.

In consequence of (2.7), it follows by a simple argument that one has

$$du = 0, \tag{2.8}$$

and so, by reference to [GL], one may write

$$\begin{cases} d^{-2u}\alpha = 0, \\ d^{-2u}\beta = 0, \end{cases} \tag{2.9}$$

that is α and β are both d^{-2u}-closed.

Next, by (2.5) and (2.8), one derives

$$\gamma = tu,\ t \in C^{\infty}M. \tag{2.10}$$

In order to simplify we set

$$\begin{cases} 2x = \|X\|^2,\ 2y = \|Y\|^2,\ 2l = \|U\|^2, \\ s_1 = g(X,U),\ s_2 = g(Y,U). \end{cases} \tag{2.11}$$

From (2.3) and (2.4) one finds

$$\begin{cases} dx = 2xu - s_1\alpha, \\ dy = 2yu - s_2\beta, \\ dl = (\lambda + 2lt)u \end{cases} \tag{2.12}$$

and

$$\begin{cases} ds_1 = (1+t)s_1u + (\lambda - 2l)\alpha, \\ ds_2 = (1+t)s_2u + (\lambda - 2l)\beta. \end{cases} \tag{2.13}$$

Next, by exterior differentiation of (2.12) and (2.13) and taking account of (2.7) and (2.8), one infers carrying out the calculations

$$t + 1 = 0,\ \lambda = const. \tag{2.14}$$

Denote now by $\sum$ the exterior differential system which defines the SSK vector fields pairing (X,Y). By (2.7), (2.12) and (2.13) it is seen that the *characteristic numbers* (or *Cartan's numbers*) are $r = 7$, $s_0 = 5$ and $s_1 = 2$.

Since $r = s_0 + s_1$, $\sum$ is in *involution* and, following Cartan [C], it follows that the existence of the pairing (X,Y) is determinated by 2 arbitrary functions of 5 arguments.

We notice that by (2.10) and (2.14) the covariant differential ∇U given by (2.4) moves to

$$\nabla U = \lambda dp - u \otimes U, \tag{2.15}$$

and by (2.15) and (2.3) one check the Ricci identity

$$\mathcal{L}_U g(X,Y) = g(\nabla_U X, Y) + g(X, \nabla_U Y).$$

Denote by $D_U = \{X, Y, U\}$ the 3-distribution spanned by the set $\{X, Y, U\}$. If $U^{'}$, $U^{''}$ are any vector fields of D_U it is easly seen that one has $\nabla_{U^{''}} U^{'} \in D_U$, then D_U is an *autoparallel distribution* and the leaves M_U and D_U are totally geodesic submanifolds of M.

Next, operating by d^∇ on ∇X, ∇Y and ∇U, one derives talking account of (2.7) and (2.15)

$$\begin{cases} d^\nabla(\nabla X) = \nabla^2 X = \lambda\alpha \wedge dp, \\ d^\nabla(\nabla Y) = \nabla^2 Y = \lambda\beta \wedge dp, \\ d^\nabla(\nabla U) = \nabla^2 U = \lambda u \wedge dp. \end{cases} \tag{2.16}$$

This shows that X, Y and U are EC vector fields.

Moreover, since the conformal scalar λ is constant and since the property of EC is invariant by linearity, then by reference to [PRV] and [R3], we conclude that the submanifold M_U is of constant scalar curvature $-\lambda$.

One easly finds by (2.16) that the sectional curvature

$$K_{X\wedge Y} = \frac{g(\mathcal{R}(X,Y)Y,X)}{\|X\|^2\|Y\|^2 - g^2(X,Y)} = -\lambda.$$

If M is compact, then necessarly $\lambda = -a^2$ and by (2.16) Yano formulas are written as

$$\int [(n-1)a^2\|X\|^2 - 2(\|U\|^2\|X\|^2 - g(U,X)^2)] = 0, \tag{2.17}$$

$$\int [(n-1)a^2\|Y\|^2 - 2(\|U\|^2\|Y\|^2 - g(U,Y)^2)] = 0. \tag{2.18}$$

Further, we recall that there exists an isomorphism between the Lie algebra of the Lie groups and the space of Killing vector fields. Consequently, if $\omega \in \Lambda^1 M$ is any 1-form, one has

$$d\omega(X,Y) = -\omega[X,Y] = \omega(g(U,Y)X - g(U,X)Y). \tag{2.19}$$

It is worth to notice that by (2.3) and (2.15) one finds:

$$[U,X] = (2l-1)X,\ [U,Y] = (2l-1)Y, \tag{2.20}$$

which shows that U defines an ICT of X and Y.

Next, by (2.20) and the 3-rd equation (2.12), one concludes that the following properties are mutually equivalent:

a) U is of constant lenght;

b) U commutes with X and Y;

c) U is a geodesic.

On the other hand, since, in general, div Z = trace (∇Z), one derives from (2.15)

$$\operatorname{div} U = \lambda n - 2l \tag{2.21}$$

and

$$\nabla_X X = s_1 X - 2lU = -A_X X. \tag{2.22}$$

Next, by (2.13), one gets from (2.22)

$$\operatorname{div} A_X X = \lambda(n-1)\|X\|^2 + 2\|X \wedge U\|^2 \tag{2.23}$$

and, as is known, one has (see also [KN]):

i) div $A_X X \geq 0$ at any point $p \in M$ where $\frac{\|X\|^2}{2}$ attains a relative minimum;

ii) div $A_X X \leq 0$ at any point $p \in M$ where $\frac{\|X\|^2}{2}$ attains a relative maximum.

Clearly, the same considerations hold good for the SSK vector field Y.

Finally denote by

$$\varphi = \alpha \wedge \beta \wedge u \tag{2.24}$$

the volume element of M_U.

Talking account of (2.12) and (2.13), one derives

$$\mathcal{L}_X \varphi = g(U, X)\varphi,$$

$$\mathcal{L}_Y \varphi = g(U, Y)\varphi,$$

$$\mathcal{L}_U \varphi = 4(\lambda - 2l)\varphi.$$

Therefore, one may say that the triple of vector fields X, Y, U define an ICT of φ.

Summarizing, we state the following.

Theorem. *Let M be an n-dimensional Riemannian C^∞-manifold carrying a pair of SSK vector fields (X, Y) with torse forming generator U.*

Then X, Y, U are exterior concurrent vector fields with the same constant conformal scalar λ.

The existence of such a pairing is determined by an exterior differential system in involution which depends on 2 arbitrary functions of 5 arguments.

Any such manifold is foliated by 3-dimensional space forms M_U of constant sectional curvature $-\lambda$ and the immersion $x : M_U \to M$ is totally geodesic.

In addition:

i) U defines an ICT of X and Y;

ii) the following properties are mutually equivalent:

a) U is of constant lenght;

b) U commutes with X and Y;

c) U is a geodesic;

iii) X, Y, U define ICT of the volume element of M_U;

iv) if M is compact, then $\lambda = -a^2$ and the integral formulas

$$\int [(n-1)a^2 \|X\|^2 - 2(\|U\|^2 \|X\|^2 - g(U, X)^2)] = 0,$$

$$\int [(n-1)a^2 \|Y\|^2 - 2(\|U\|^2 \|Y\|^2 - g(U, Y)^2)] = 0$$

hold good.

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