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ON CONVERGENCE IN B^* - CONTINUOUS FUNCTIONS

BY

D.K. GANGULY AND CHANDRANI MITRA

Abstract. In this note it is shown that point-wise limit of transfinite sequence of B^*-continuous functions and uniform limit of a sequence of B^*-continuous functions is again B^*-continuous; but this property is not shared by the quasi-uniform convergence. Again it is establish that B^*-continuity of uniform limit of a sequence of B^*-continuous functions is transmitted to the oscillation-type function associated with the B^*-continuous function. Also quasi-uniform and quasi-normal convergence of sequence of real-valued B^*-continuous functions are compared.

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1. Introduction. The notion of quasi continuity for real functions of several real variables was introduced by S. Kempisty as a generalization of notion of continuity and it has been intensively studied since then. The other types of generalized continuity appear in the literature viz., cliquishness, simply continuity, B-continuity etc. We introduce another type of generalized continuity called B^*-continuity making slight generalization of the definition of quasi continuity. W. Sierpinski initiated the investigation of point-wise convergent of transfinite sequence of functions in the paper [5].

Let Ω be the first uncountable ordinal number; let X be a set and Y be a metric space. A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions from X into Y converges point-wise to a function $f : X \to Y$ if for every $x \in X$ and every neighborhood U of $f(x)$ there exists an ordinal number $\eta < \Omega$ such that $f_\xi(x) \in U$ for every $\eta < \xi < \Omega$. We shall denote this convergence by

$f_\xi \to f$ or more precisely by $\lim_{\xi<\Omega} f_\xi = f$.

Sierpinski himself proved that point-wise convergence of transfinite sequence of continuous function is continuous and that of Baire-1 function is Baire-1 again. In 1974, A Neubrunnova (see [4]) showed that quasi-continuity and cliquishness of functions from separable metric space to metric spaces are preserved under the point-wise limit of transfinite sequence.

In the present paper first we discuss the question of preserving property of being B^*-continuity. It is well known that the sets of quasi continuity and cliquish functions are closed with respect to the uniform convergence (see [3]). In this note we show that the set of B^*-continuity is closed under uniform limit. In the final part of this paper we shall present and investigate quasi-uniform and quasi-normal convergence as a convergence structure and compare. Our terminology is standard. No distinction is made between a function and its graph. Unless otherwise stated X and Y are topological spaces. We recall the definitions, which will be used in the paper.

- A function $f : X \to Y$ is quasi-continuous at a point $x \in X$ if, for each open neighborhood U of x and each open neighborhood V of $f(x)$, there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. If f is quasi-continuous at each point it is said to be quasi-continuous on X.
- A function $f : X \to Y$ (Y is a metric space with metric d) is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each open neighborhood U of x, there is a nonempty open set $G \subset U$ such that, $d(f(x_1, f(x_2)) < \varepsilon, \forall x_1, x_2 \in G$. If f is cliquish at each point, it is said to be so over the whole of X.
- A set $A \subset X$ is said to be B^*-set if it is not nowhere dense in X and have the property of Baire.
- A function $f : X \to Y$ is to be B^*-continuous at x if for each open set U containing x in X and each open set V containing $f(x)$ in Y, there is a B^*-set $B \subset U \cap f^{-1}(V)$. If f be B^*-continuous at each point of X, then f is B^*-continuous on the whole of X.
- Let X be a set and Y be a metric space (with metric d). Then the sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : X \to Y (n = 1, 2, \cdots)$ is said to convergence quasi-uniformly to a limit function $f : X \to Y$, if

  (i) f_n converges to f point-wise and

(ii) for every $\varepsilon > 0$, for each $n \in \mathbb{N}$, there exists $r(n) \in \mathbb{N}$ such that $\sup_x \min_{0 \le i \le r(n)} d(f_{n+i}(x), f(x)) < \varepsilon$.

- Let $f_n, f : X \to \mathbb{R}, n = 1, 2, \cdots$ We shall say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges quasi-normally to f on X, if there is a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of nonnegative reals converging to zero such that for every $x \in X$ there is an index k_x such that $|f_n(x) - f(x)| \le \varepsilon_n$ for every $n \ge k_x$.

We can show by example that point-wise limit of a sequence of B^*-continuous function may not be B^*-continuous.

Example 1. Let $f_n(x) = x^n, x \in [0, 1]$.

$$f(x) = \begin{cases} 0, & \text{if } x \ne 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Then $f_n \to f$ point-wise. Now each f_n being continuous it is B^*-continuous. But f is not B^*-continuous at $x = 1$.

But, point-wise limit of transfinite sequences of B^*-continuous functions remains B^*-continuous.

Theorem 1. *Let $f_a : X \to Y$ where X be a separable metric space and Y be any metric space, with metric $d, (a < \Omega)$. If the transfinite sequence $\{f_a\}$ of B^*-continuous functions point-wise converge to f, then f is also B^*-continuous.*

Proof. Let f be not B^*-continuous at the point $x_0 \in X$. Let $K(x_0, \delta)$ be a neighborhood of x_0, where $K(x_0, \delta)$ denotes the sphere with centre x_0 and radius δ. Then $T = \{t \in X : d(f(t), f(x_0)) \ge \varepsilon\}$ is dense in $K(x_0, \delta)$. Let S be a countable dense subset of T. There is a $\beta < \Omega$ such that $a > \beta, f_a(x) = f(x), \forall x \in S \cup \{x_0\}$; (this follows from the lemma in [4]).

Let $\gamma > \beta$ is any fixed ordinal number. Then, f_γ is B^*-continuous at the point x_0. Then, $P = \{x : d(f_\gamma(x), f_\gamma(x_0)) < \varepsilon\}$ is not nowhere dense in $K(x_0, \delta)$ and has Baire property. Therefore, $\exists$ a nonempty open set $U \subset K(x_0, \delta)$, such that $U \cap P \ne \emptyset$; where $P = (O \backslash I) \cup J, (O$ is open and I, J are sets of first category). Therefore, $P \cap S \ne \emptyset$. Then $\emptyset \ne (O \backslash I) \cap S \subset P \cap S, U \cap S \ne \emptyset$, as S is dense. Let $z \in U \cap P \cap S$, then, $d(f_\gamma(z), f_\gamma(x_0)) < \varepsilon$. But, $f_\gamma(z) = f(z), f_\gamma(x_0) = f(x_0)$. Therefore,

$d(f(z), f(x_0)) < \varepsilon$, i.e. $z \notin S$, which is a contradiction. Hence, f is B^*-continuous.

Like point-wise limit, the quasi-uniform limit of a sequence of B^*-continuous functions is also not B^*-continuous, which is clear from the following example.

Example 2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence, where $f_n : \mathbb{R} \to \mathbb{R}$, be defined by $f_n = \mathcal{X}_{\langle 0,1/n\rangle}((-1)^n.x), \forall n$. Each f_n being quasi continuous (see [2]), it is B^*-continuous. Also $f_n \to f = \mathcal{X}_{\{0\}}$. Let $\varepsilon > 0, m \in \mathbb{N}$. Denote $p = m + 2$. Then for $x \geq 0$, we have $|f_{m+p-1}(x) - f(x)| = 0$ and for $x < 0$ we have $|f_{m+p}(x) - f(x)| = 0$. Therefore, $\min\{|f_{m+1}(x) - f(x)|, \cdots, |f_{m+p}(x) - f(x)|\} < \varepsilon, \forall x \in \mathbb{R}$. So, $\{f_n\}$ quasi-uniformly converges to f. But f is not B^*-continuous.

The following theorem indicates that the class of B^*-continuous functions is closed under uniform limit.

Theorem 2. *Let $f_n : X \to (Y, d)$(Y is a metric space with metric d), be B^*-continuous functions ($n \in \mathbb{N}$), and $f_n \to f$ uniformly on X. Then f is B^*-continuous.*

Proof. Let $p \in X$ be an arbitrary point. Then, for any neighborhood U of the point p, $\exists$ a nonempty open set $G \subset U$ such that the set $H = \{x \in X : d(f_m(x), f_m(p)) < \varepsilon/4\}$ is a set having the property of Baire, which is dense in G. Let $x \in H$. Then, $d(f(x), f(p)) \leq d(f(x), f_m(x)) + d(f_m(x), f_m(p)) + d(f_m(p), f(p)) < 3\varepsilon/4 < \varepsilon$. Hence, $f(H) \subset \{y \in Y : d(f(p), y) < \varepsilon\}$, i.e., $H \subset U \cap f^{-1}(f(p) - \varepsilon, f(p) + \varepsilon)$. Hence, f is B^*-continuous at p.

We now introduce the concept of oscillation -type function related with a B^*-continuous function. Given a B^*- continuous function $f : X \to (Y, d)$, we associate a function $\Omega_f : X \to [0, \infty]$ as follows: Let $U(x)$ be a neighborhood of a point $x \in X$ and put $\Omega_f(x, U(x)) = \inf\{\sup\{d(f(x), f(z)) : z \in B\} : \exists$ a nonempty open set $G \subset U(x)$ and a set B having the property of Baire such that $B \subset G \subset cl(B)\}$; where inf is taken over all nonempty open subsets G of $U(x)$ and over all sets B having the property of Baire and dense in G.

Define $\Omega_f(x) = \sup\limits_{U(x)} \Omega_f(x, U(x))$. This function is an oscillation-type function.

The next theorem shows that B^*-continuity of uniform limit of a sequence of B^* - continuous functions is translated to that of the sequence

$\{\Omega_{f_n}\}$.

Theorem 3. *Let* $f : X \to (Y, d), f_n : X \to Y, (n = 1, 2, \cdots)$ *where* f_n *are* B^**-continuous. If* $f_n \to f$ *uniformly on* X*, then* $\Omega_{f_n} \to \Omega_f$ *uniformly on* X.

Proof. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that for $n \geq N d(f_n(x), f(x)) < \varepsilon, \forall x \in X$. Let $x \in X$ be any point of X and let $U(x)$ be a neighborhood of x. As x is the point of B^*-continuity for each f_n and $f, \exists$ a nonempty open set G and $B \subset G$ is dense in G having the property of Baire. Then $\forall y \in B$, we have by triangle inequality,

$$d(f(y), f(x)) \leq d(f(y), f_n(y)) + d(f_n(y), f_n(x)) + \\ + d(f_n(x), f(x)) < 2d(f_n(y), f_n(x)).$$

In the same way we can establish that $d(f_n(y), f_n(x)) < 2d(f(y), f(x))$. Therefore the above two inequalities imply that $|\Omega_{f_n}(x, U(x)) - \Omega_f(x, U(x))| \leq 2\varepsilon$.

Hence, $|\Omega_{f_n}(x) - \Omega_f(x)| \leq 2\varepsilon$ for $n \geq N$ and for every $x \in X$.

Note. It is shown in a paper (see [1]) that the quasi-uniform convergence as well as quasi-normal convergence imply point-wise convergence, but they are both implied by the uniform convergence.

The following examples show that neither quasi-uniform convergence nor quasi-normal convergence includes the other.

Remark 1. The sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in example 1, is quasi-normally convergent to f, as can be checked easily; but they are not quasi-uniformly convergent to f.

Remark 2. The sequence of functions in example 2, is quasi-uniformly convergent to f, but they are not quasi-normally convergent.

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Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road,
Calcutta - 700019,
INDIA