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A GOULD TYPE INTEGRAL WITH RESPECT TO A MULTIMEASURE II

BY

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Introduction. In an earlier paper [3] we have introduced a Gould type integral for bouded functions $f : (S, \mathcal{A}) \to R$ (S is a nonempty set and $\mathcal{A}$ is an algebra of subsets of S) with respect to a finite additive multimeasure $\varphi : \mathcal{A} \to \mathcal{P}_{kc}(X)$ of finite variation ($\mathcal{P}_{kc}(X)$ is the family of nonvoid compact convex subsets of a Banach space X). In the present paper we define a Gould type integral in the general case when f may be not bounded and the multimeasure φ may be not of finite variation. We also study some basic properties of this integral and we establish a Lebesgue type theorem of dominated convergence.

1. Terminology and Notations. Let $S \neq \phi$ be an arbitrary set and $\mathcal{A}$ an algebra of subsets of S. Denote by $(\mathcal{S}, \mathcal{A})$ a measurable space and by $(X, \|\cdot\|)$ a real Banach space. $\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ is the family of all nonempty compact convex subsets of X and D is the usual Hansdorff metric defined on $\mathcal{P}_{kc}$. We define $|A| = D(A, \theta)$, $A \in \mathcal{P}_{kc}$, where $\theta = \{0\}$. If $A, B \in \mathcal{P}_{kc}$, $\lambda \in \mathbb{R}$, then

$$A + B = \{x + y \mid x \in A, y \in B\}$$
$$\lambda A = \{\lambda x \mid x \in A\}.$$

Definition 1.1.

a) A **partition** of S is a finite family $P = \{E_i\}_{i=1}^n \subset \mathcal{A}$, $E_i \cap E_j = \phi (i \neq j)$ such that $\bigcup_{i=1}^{n} E_i = S$.

b) Let P, P' be two partitions of S, $P = \{E_i\}_{i=1}^n$, $P' = \{B_j\}_{j=1}^m$. P' is **finer** than P (denoted $P' \geq P$) if for any $j \in \{1, ..., m\}$, there is $i \in \{1, ..., n\}$ such that $B_j \subseteq E_i$.

c) The common refinement of two partitions $P = \{E_i\}_{i=1}^n, P' = \{B_j\}_{j=1}^m$ is the partition $P \wedge P' = \{E_i \cap B_j\}$.

Definition 1.2. If $\mu : \mathcal{A} \to X$ is a finitely additive measure, then we define:

$$\widetilde{\mu}(E) = \sup\{\|\mu(A)\|; A \in \mathcal{A}, A \subseteq E\}, (\forall) E \in \mathcal{A}$$

and

$$\widetilde{\mu}^*(B) = \inf\{\widetilde{\mu}(E); E \in \mathcal{A}, B \subseteq E\}, (\forall) B \subseteq S.$$

Remark 1.3. The positive set functions $\widetilde{\mu}$ and $\widetilde{\mu}^*$ are monotone and subadditive. If μ is countably additive, then $\widetilde{\mu}$ is countably subadditive.

Definition 1.4. Let $\mu : \mathcal{A} \to X$ be a finitely additive measure. Let $f, (f_n)_{n \in \mathbb{N}}$ be real functions defined on S. The sequence $(f_n)_{n \in \mathbb{N}}$ is said to be **convergent in $\mu-$measure to** f (denoted $f_n \xrightarrow{\mu} f$) if $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} \widetilde{\mu}^*(\{s \in S; |f_n(s) - f(s)| \geq \varepsilon\}) = 0.$$

Definition 1.5. Let $\mu : \mathcal{A} \to X$ be a finitely additive measure.

a) A set $E \in \mathcal{A}$ is said to be **$\mu-$bounded** if $\{\|\mu(B)\|; B \in \mathcal{A}, B \subseteq E\}$ is a bounded set.

b) $(S, \mathcal{A}, \mu)$ is said to be **non-degenerate** if for any $\mu-$unbounded set $E \in \mathcal{A}$ there exists $\alpha > 0$ such that every $\mu-$unbounded set $B \subseteq E, B \in \mathcal{A}$, contains a $\mu-$ bounded set $C \subseteq B, C \in \mathcal{A}$, such that $\|\mu(C)\| > \alpha$.

In what follows we consider multifunctions $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ for which $\varphi(\phi) = 0$.

Definition 1.6.

a) A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is said to be a **multimeasure** if

$$\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2), \forall E_1, E_2 \in \mathcal{A}, E_1 \cap E_2 = \phi. \tag{1}$$

b) A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is said to be **countable additive** or σ-**additive** if for every sequence $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$ with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ and $E_n \cap E_m = \phi (n \neq m)$,

$$\varphi(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \varphi(E_n), \tag{2}$$

where the sum in the right side of equality is the limit in Hausdorff distance of the sequence $\left\{\sum_{k=1}^{n} \varphi(E_k)\right\}_{n \in N}$.

Definition 1.7.

a) Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multifunction. The **total variation** of φ is the non-negative (possibly infinite) set function $\upsilon(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$\upsilon(\varphi, A) = \nu(A) = \sup \left\{ \sum_{i=1}^{n} \left| \varphi(E_i) \right| ; (E_i)_{i=1}^{n} \subset \mathcal{A}, \right. \\ \left. E_i \cap E_j = \phi (i \neq j), \bigcup_{n}^{i=1} E_i = A \right\}, \forall A \in \mathcal{A}. \tag{3}$$

b) If $\nu(S) < \infty$ we say that the multifunction φ is of **finite variation**.

If φ is a multimeasure, then $v(\varphi, \cdot)$ is finitely additive and if φ is σ-additive the same is ν.

Definition 1.8. For a multimeasure $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$,
a) $\widetilde{\varphi} : \mathcal{A} \to [0, +\infty]$ we associate two real functions defined by:

$$\widetilde{\varphi}(A) = \sup\{ \left| \varphi(E) \right| ; E \in \mathcal{A}, E \subseteq A\}, \forall A \in \mathcal{A} \tag{4}$$

b) and $\nu^* : \mathcal{P}(S) \to [0, +\infty]$ defined by

$$\nu^*(E) = \inf\{\nu(B); B \in \mathcal{A}, E \subseteq B\}, \tag{5}$$

where ν is the total variation of φ.

Remark. $\widetilde{\varphi}$ is monotone, subadditive and $\widetilde{\varphi}(E) \leq \nu(\varphi; E)$, $(\forall)(E) \in \mathcal{A}$.

Definition 1.9. Let $\mu : \mathcal{A} \to X$ be a finitely additive measure and let $f : S \to \mathbb{R}$ be a given function.

a) f is said to be μ-**totally measurable** over $(S, \mathcal{A}, \mu)$ (briefly μ-totally measurable) if for each $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ such that $\widetilde{\mu}(E_0) < \varepsilon$ and $\sup\limits_{s,t \in E_i} |f(s) - f(t)| = osc(f, E_i) < \varepsilon, \forall i \in \{1, ..., n\}$.

b) If $E \in \mathcal{A}$, f is called μ-**totally measurable** over E if the restriction $f|_E$ is μ-totally measurable over $(E, \mathcal{A}_E, \mu_E)$, where $\mathcal{A}_E = \{A \cap E \mid A \in \mathcal{A}\}$ and $\mu_E = \mu|_{\mathcal{A}_E}$.

2. The multivalued Gould integral. Bounded case. In this section we recall some results concerning the multivalued Gould integral for the bounded case [3]. We denote by $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ a multimeasure with total variation ν for which $\nu(S) < +\infty$.

Definition 2.1. Let $f : S \to \mathbb{R}$ be a real function.

a) f is said to be φ-**totally measurable over** $(S, \mathcal{A}, \varphi)$(briefly φ-totally measurable) if for each $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ such that $\widetilde{\varphi}(E_0) < \varepsilon$ and $osc(f, E_i) < \varepsilon, \forall i \in \{1, ..., n\}$.

b) If $E \in \mathcal{A}, f$ is called φ-**totally measurable on** E if the restriction $f|_E$ is φ-totally measurable on $(E, \mathcal{A}_E, \varphi_E)$, where $\varphi_E = \varphi|_{\mathcal{A}_E}$.

Definition 2.2. Let $f : S \to \mathbb{R}$ be a bounded function.

a) f is said to be φ-**integrable over** $(S, \mathcal{A}, \varphi)$ (briefly φ-integrable) if there exists $A \in \mathcal{P}_{kc}$ such that the following holds: given $\varepsilon > 0$ there is a partition P_0 of S such that for any partition $P = \{E_i\}_{i=1}^n$ of S, with $P \geq P_0$ and for any sum $\sigma(P) = \sum_{i=1}^{n} f(s_i)\varphi(E_i), \forall s_i \in E_i, i \in \{1, ..., n\}$,

$$D(\sigma(P), A) < \varepsilon \tag{6}$$

b) For $E \in \mathcal{A}, f$ is said to be φ-**integrable over** E if the restriction $f|_E$ of f to E is φ-integrable over $(E, \mathcal{A}_E, \varphi_E)$.

Definition 2.3. The element A from Definition 2.2 defined by:

$$A = \int_S f d\varphi$$

is said to be the **Gould integral of** f **over** S with respect to φ.

Definition 2.4. Let $f : S \to \mathbb{R}$ be a bounded function and $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ a multimeasure with finite variation:

a) f is said to be **strong**-φ-**integrable** with respect to φ on S (briefly s-φ-integrable) if there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$ there exists a partition $P = \{E_i\}_{i=1}^n$ of S with $P \geq P_0$ and $\forall s_i \in E_i, i = 1, 2...n$, we have

$$\left| \sum_{i=1}^{n} f(s_i) \cdot \varphi(E_i) - A \right| < \varepsilon.$$

b) The element $A \in \mathcal{P}_{kc}$ defined in a) is said to be the **strong integral** (of Gould type) of f with respect to φ on S. This element A is denoted by $(S)\int_S f d\varphi$.

Remark 2.5. We can easily see that the element A is uniquely determined.

We proved in [3] the following properties of Gould integrable functions.

I) If $f : S \to \mathbb{R}$ is a bounded function such that $f = 0$, φ-almost everywhere, then f is φ-integrable and $\int_S f d\varphi = 0$.

II) If $f : S \to \mathbb{R}$ is a φ-integrable function on S, then for every $T \in \mathcal{A}$, f is φ-integrable on T.

III) If $f, g : S \to \mathbb{R}$ are φ-integrable functions, then $D\left(\int_S f d\varphi, \int_S g d\varphi\right) \leq \sup_{s\in S} |f(s) - g(s)| \cdot \nu(S)$.

IV) If $f : S \to R$ is φ-integrable, then for every $\alpha \in \mathbb{R}$ the function αf is φ-integrable and

$$\int_S \alpha f d\varphi = \alpha \int_S f d\varphi.$$

V) If $f, g : S \to R$ are two φ-integrable functions on S such that $f(s) \cdot g(s) \geq 0$ for all $s \in S$, then $f + g$ is φ-integrable and

$$\int_S (f + g) d\varphi = \int_S f d\varphi + \int_S g d\varphi.$$

VI) If $f : S \to R$ is φ-integrable, then the multifunction $\Gamma : \mathcal{A} \to \mathcal{P}_{kc}$ defined by

$$\Gamma(T) = \int_T f d\varphi, \forall T \in \mathcal{A}, \tag{7}$$

is a multimeasure.

VII) If $f : S \to R$ is φ-integrable and $\Gamma(T)$ is defined by (7), then

$$\left|\int_S f d\varphi\right| \leq 2 \sup_{s\in S} |f(s)| \cdot \nu(S) \tag{8}$$

and Γ is absolutely continuous with respect to $\nu(\Gamma \ll \nu)$, that is for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $E \in \mathcal{A}$ with $\nu(E) < \delta \Rightarrow \Gamma(E) < \varepsilon$.

3. A multivalued Gould type integral. Unbounded case. Throughout this section it is assumed that $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is a multimeasure with total variation ν and $(S, \mathcal{A}, \nu)$ is non-degenerate.

For a function $f : S \to R$ we define

$$M(f) = \{K \in \mathcal{A}; \nu(K) < \infty \text{ and } f \text{ is bounded over } K\} \tag{9}$$

and

$$M_E(f) = M(f) \cap \mathcal{A}_E, \forall E \in \mathcal{A}. \tag{10}$$

Remark 3.1. If $f : S \to R$ is ν-totally measurable, then $M(f) \neq \phi$. Indeed, since f is ν-totally measurable, for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{E_i\}_{i=0}^n$ of S such that $\nu(E_0) < \varepsilon$ and $\operatorname{osc}(f, E_i) = \sup_{s,t\in E_i} |f(s) - f(t)| < \varepsilon, \forall i = 1, 2, ...n$.

Let $i \in \{1, 2, ...n\}$ and $y_0 \in E_i$ arbitrary fixed. Then for every $x \in E_i$ we have

$$|f(x)| = |f(x) - f(y_0)| + |f(y_0)| < \varepsilon + |f(y_0)| = m,$$

that is f is bounded on every $E_i, \forall i \in \{1, 2, ...n\}$.

Now we have two situations:

I) If $\nu(S) < \infty$, then $\nu(E_i) < +\infty, \forall i \in \{1, 2, ..., n\}$ and consequently $E_i \in M(f), \forall i \in \{1, 2, ..., n\}$ and hence $M(f) \neq \phi$.

II) If $\nu(S) = +\infty$, since $(S, \mathcal{A}, \nu)$ is non-degenerate there exists $\alpha > 0$ and $C \in \mathcal{A}, C \subseteq S$ such that $\alpha < \nu(C) < \infty$. Because f if ν-totally measurable on C and $\nu(C) < \infty$ we can apply the case I) and hence again we have $M(f) \neq \phi$.

Definition 3.2. Let $f : S \to R$ be a real function. f is said to be **generalized φ-integrable** (briefly g-φ-integrable) on S if the following two conditions are satisfied:

a) f is ν-totally measurable;

b) there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ with the property that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$ we have

$$D\left(\overline{\int_K} f d\varphi, A\right) < \varepsilon,$$

where by $\overline{\int_K} f d\varphi$ we mean the integral in the sense of Definition 2.2.

Remark 3.3. It is easy to see that if f is generalized φ-integrable, then the set A from the definition 3.2 is uniquely determined.

Definition 3.4. If $f : S \to R$ is generalized φ-integrable on S, the element $A \in \mathcal{P}_{kc}$ from the definition 3.2 is denoted by $A = \int_S f d\varphi$ and is called the **generalized Gould integral** of f on S with respect to φ.

Using the definition 3.2 it is easy to prove.

Proposition 3.5 *If $f : S \to R$ is generalized φ-integrable and bounded and $\nu(S) < \infty$, then $\overline{\int_S} f d\varphi = \int_S f d\varphi$.*

Remark 3.6. According to this proposition we can use the same symbol $\int_S f d\varphi$ for the Gould integral in both situations "bounded" or "unbounded".

Theorem 3.7 *The function $f : S \to R$ is g-φ-integrable if and only if the two conditions are satisfied:*

a) *f is ν-totally measurable*

b') *for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for every $K \in M(f)$ with $K \cap K_\varepsilon = \phi$ we have*

$$\left|\int_K f d\varphi\right| < \varepsilon. \tag{11}$$

Proof. In fact it is sufficient to prove the equivalence of b') to the condition b) from definition 3.2.

Let us show b) $\Rightarrow$ b')

By virtue of b) there exists $A \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0, (\exists) K_\varepsilon \in M(f)$ such that $(\forall) K \in M(f)$ with $K \supseteq K_\varepsilon$ we have

$$D\left(\int_K f d\varphi, A\right) < \frac{\varepsilon}{2}. \tag{12}$$

Let us take $(\forall) K \in M(f)$ with $K \cap K_\varepsilon = \phi$. Then $K \cup K_\varepsilon \in M(f)$ and $K \cup K_\varepsilon \supseteq K_\varepsilon$ and from (10) we have

$$D\left(\int_{K\cup K_\varepsilon} f d\varphi, A\right) < \frac{\varepsilon}{2}$$

which, by virtue of Theorem 2.11 - ii) [3], is equivalent to

$$D\left(\int_K f d\varphi + \int_{K\cup K_\varepsilon} f d\varphi, A\right) < \frac{\varepsilon}{2}. \tag{13}$$

Then, using (12) and (13) we have

$$\left|\int_K f d\varphi\right| = D\left(\int_K f d\varphi + \int_{K_\varepsilon} f d\varphi, \int_{K_\varepsilon} f d\varphi\right) \leq$$

$$D\left(\int_K f d\varphi + \int_{K_\varepsilon} f d\varphi, A\right) + D\left(A, \int_K f d\varphi\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

that is b') is satisfied.

Conversely, if b') holds, then for every $K \in M(f)$, let us denote by

$$F_K = \left\{\int_L f d\varphi; L \in M(f), K \subseteq L\right\} \subset \mathcal{P}_{kc}$$

and by

$$\mathcal{B} = \{F_K; K \in M(f)\} \subseteq \mathcal{P}(\mathcal{P}_{kc}).$$

It is easy to see that $\mathcal{B}$ is a filter base. Let $\mathcal{F} = \{\mathcal{C} \subset \mathcal{P}_{kc}; (\exists)K \in M(f)$ such that $F_K \subseteq \mathcal{C}\}$ be the filter generated by $\mathcal{B}$. We shall show that $\mathcal{F}$ is a Cauchy filter.

Indeed, if $\varepsilon > 0$ is arbitrary, by virtue of b') $(\exists)K_\varepsilon \in M(f)$ such that for every $K \in M(f)$ with $K \cap K_\varepsilon = \phi$ we have (11).

Let us take $\mathcal{C} = F_{K_\varepsilon} \in \mathcal{F}$. Then for every $E, B \in F_{K_\varepsilon}$ there exist $L_1, L_2 \in M(f)$ with $K_\varepsilon \subseteq L_1, K_\varepsilon \subseteq L_2$ such that $E = \int_{L_1} f d\varphi, B = \int_{L_1} f d\varphi$. Next we obtain

$$\mathcal{D}(E, B) = \mathcal{D}\left(\int_{K_\varepsilon \cup (L_1 \setminus K_\varepsilon)} f d\varphi, \int_{K_\varepsilon \cup (L_2 \setminus K_2)} f d\varphi\right)$$

and using Theorem 2.11-ii [3] we have

$$\mathcal{D}(E, B) = \mathcal{D}\left(\int_{K_\varepsilon} f d\varphi + \int_{L_1|K_\varepsilon} f d\varphi, \int_{K_\varepsilon} f d\varphi + \int_{L_2|K_\varepsilon} f d\varphi\right) \tag{14}$$

Since $K_\varepsilon, L_1, L_2 \in M(f)$, we obtain that $L_1 \setminus K_\varepsilon, L_2 \setminus K_\varepsilon \in M(f)$ and hence from (14) we have

$$\begin{aligned} \mathcal{D}(E, B) &\leq \mathcal{D}\left(\int_{L_1|K_\varepsilon} f d\varphi, \int_{L_2 \setminus K_\varepsilon} f d\varphi\right) \leq \\ &\leq \left|\int_{L_1 \setminus K_\varepsilon} f d\varphi\right| + \left|\int_{L_2 \setminus k_\varepsilon} f d\varphi\right|. \end{aligned} \tag{15}$$

Since $L_1 \setminus K_\varepsilon, L_2 \setminus K_\varepsilon \in M(f)$ and $(L_1 \setminus K_\varepsilon) \cap K_\varepsilon = \phi, L_1 \setminus K_\varepsilon) \cap K_\varepsilon = \phi$ from (11) it follows

$$\left|\int_{L_1 \setminus K_\varepsilon} f d\varphi\right| < \frac{\varepsilon}{3}, \quad \left|\int_{L_2 \setminus K_\varepsilon} f d\varphi\right| < \frac{\varepsilon}{3}. \tag{16}$$

From (15) and (16) we have

$$\mathcal{D}(E, B) < \frac{2\varepsilon}{3}$$

for every $E, B \in F_{K_\varepsilon} = \mathcal{C}$ and hence the diameter of $\mathcal{C}$, diam $\mathcal{C} = \sup_{E,B \in \mathcal{C}} \mathcal{D}(E, B) < \varepsilon$, which assures that $\mathcal{F}$ is a Cauchy filter. Because

$(\mathcal{P}_{kc}, \mathcal{D})$ is complet [2] it follows that $\mathcal{F}$ converges to an element $A \in \mathcal{P}_{kc}$, that is $\mathcal{F}$ is finer then $\mathcal{V}(A)$, the filter of neighborhoods of A.

For $\varepsilon > 0$, let $V = S(A, \varepsilon) \in \mathcal{V}(A)$ ($S(A, \varepsilon)$ is the ball of center A and radius ε). Since $\mathcal{V}(A) \subseteq \mathcal{F}$ there exists $\mathcal{C}' \in \mathcal{F}$ with $\mathcal{C}' \subseteq V$.

Because $\mathcal{F}$ is a Cauchy filter, there exists $\mathcal{C}'' \in \mathcal{F}$ with diam $\mathcal{C}'' < \varepsilon$.

Let us take $\mathcal{C} = \mathcal{C}' \cap \mathcal{C}'' \in \mathcal{F}$. Then there exists $K_\varepsilon \in M(f)$ such that $F_{k_\varepsilon} \subseteq \mathcal{C}$. But $\mathcal{C} \subseteq \mathcal{C}' \subseteq V$ and hence $F_{K_\varepsilon} \subset V = S(A, \varepsilon)$. Now, if $K \in M(f)$ and $K \supseteq K_\varepsilon$ we have $\int_K f d\varphi \in F_{K_\varepsilon} \subseteq S(A, \varepsilon)$ and therefore $\mathcal{D}\left(\int_K f d\varphi, A\right) < \varepsilon$. Consequently for every $\varepsilon > 0$, there exists $K_\varepsilon \in M(f)$ such that for every $K \in M(f)$ with $K \supseteq K_\varepsilon$ we have $\mathcal{D}\left(\int_K f d\varphi, A\right) < \varepsilon$, that is b) holds. ■

Theorem 3.8 *If $f : S \to R$ is such that $f = 0$, φ-almost everywhere, then for every $E \in \mathcal{A}$ f is generalized φ-integrable on E and $\int_E f d\varphi = 0$.*

Proof. We give the proof for $E = S$ (the proof for arbitrary $E \in \mathcal{A}$ is similar).

According to Theorem 2.6 [3], f is ν-totally measurable. By virtue of remark 3.6 we see that $M(f) \neq \phi$. Let $K_0 \in M(f)$. Let us take $\varepsilon > 0$ and $K \in M(f)$ with $K \supseteq K_0$. Since $K \in M(f)$ we have $\nu(K) < \infty$ and f bounded on K.

Now, using theorem 2.6 [3] we obtain $\int_K f d\varphi = 0$.

Hence

$$\mathcal{D}\left(\int_K f d\varphi, 0\right) = \mathcal{D}(0, 0) = 0 < \varepsilon.$$

Consequently, $(\exists)\{0\} \in \mathcal{P}_{kc}$ such that for every $\varepsilon > 0$ exists $K_0 \in M(f)$ such that for every $K \in M(f)$ with $K \supseteq K_0$, we have $\mathcal{D}\left(\int_K f d\varphi, 0\right) < \varepsilon$, which assures that f is g-φ-integrable and $\int_S f d\varphi = 0$ ■

Definition 3.9 The function $f : S \to R$ is said to be **generalized φ-integrable** (briefly g-φ-integrable) on the set $E \in \mathcal{A}$ if the restriction $f|_E$ of

f to the set E is g-φ_E-integrable on $(E, \mathcal{A}_E, \varphi_E)$, where $\mathcal{A}_E = \{T \cap E; T \in \mathcal{A}\}$ and $\varphi_E = \varphi|_{\mathcal{A}_E}$.

Theorem 3.10 *If $f : S \to R$ is a function generalized φ-integrable on S, then:*

i) *f is g-φ-integrable on E, for every $E \in \mathcal{A}$.*

ii) *The multifunction $\Gamma : \mathcal{A} \to \mathcal{P}_{kc}$ defined by $\Gamma(E) = \int_E f d\varphi, (\forall) E \in \mathcal{A}$, is a multimeasure.*

iii) *Γ is absolutely continuous with respect to ν, that is for $(\forall)\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $E \in \mathcal{A}$ with $\nu(E) < \delta$ we have $\left| \Gamma(E) \right| < \varepsilon$ (denoted $\Gamma \ll \nu$).*

Proof.

i) Let $E \in \mathcal{A}$ be fixed and let us denote by $M_E(f) = \{K \in \mathcal{A}_E; \nu(K) < +\infty$ and f is bounded on $K\}$. Since f is g-φ-integrable on S, it is ν-totally measurable on S and hence on E.

Now, to prove the g-φ-integrability of f on E it is sufficient, according to theorem 3.7, to show the condition b') corresponding to $E \in \mathcal{A}$.

From the g-φ-integrability of f on S, using theorem 3.7-b'), for every $\varepsilon > 0$ there exists a set $K_\varepsilon \in M(f)$ such that for every $K \in M(f)$ with $K \cap K_\varepsilon = \phi$ we have

$$\left| \int_K f d\varphi \right| < \varepsilon. \tag{17}$$

If $K'_\varepsilon = K_\varepsilon \cap E$, then $K'_\varepsilon \in M_E(f)$ and for every $K' \in M_E(f)$ with $K' \cap K'_\varepsilon = \phi$ we have $(K' \cap E) \cap K_\varepsilon = \phi$.

From (7) we obtain

$$\left| \int_{K'} f d\varphi \right| < \varepsilon \tag{18}$$

which assures the condition b') from theorem 3.7 with respect to the set $E \in \mathcal{A}$.

ii) We have to prove that for every $E_1, E_2 \in \mathcal{A}$ with $E_1 \cap E_2 = \phi$ it follows $\Gamma(E_1 \cup E_2) = \Gamma(E_1) + \Gamma(E_2)$.

Since f is g-φ-integrable on $E_1, E_2 \in \mathcal{A}$ we have that for $(\forall)\varepsilon > 0$, there exists $K'_E \subseteq E, K'_E \in M_E(f)$, and $K''_E \subseteq E, K''_E \in M_E(f)$, such that for $(\forall)K \in M_E(f)$ with $K \supseteq K'_E$ we obtain

$$\mathcal{D}\left(\int_K f d\varphi, \int_{E_1} f d\varphi\right) < \frac{\varepsilon}{2} \tag{19}$$

and for $(\forall)K \in M_E(f)$ with $K \supseteq K''_\varepsilon$

$$\mathcal{D}\left(\int_K f d\varphi, \int_{E_2} f d\varphi\right) < \frac{\varepsilon}{2}. \tag{20}$$

Since $E_1 \cap E_2 = \phi$ it follows $K'_\varepsilon \cap K''_E = \phi$.

Now, if $K_\varepsilon = K'_\varepsilon \cup K''_\varepsilon \in M_{E_1 \cup E_2}(f)$, taking $(\forall)K \in M_{E_1 \cup E_2}(f)$ with $K \supseteq K_\varepsilon$ we have $K = K'_\varepsilon \cup (K \setminus K'_\varepsilon) = K' \cup K''$, where $K' \cap K'' = \emptyset$ and hence for $K'_\varepsilon = K', K''_\varepsilon \subseteq K''$ we have

$$\mathcal{D}\left(\int_{K'} f d\varphi, \int_{E_1} f d\varphi\right) < \frac{\varepsilon}{2} \text{ and } \mathcal{D}\left(\int_{K''} f d\varphi, \int_{E_2} f d\varphi\right) < \frac{\varepsilon}{2} \tag{21}$$

Now, using Theorem 2.11 [3] and the relations 20 we obtain

$$\mathcal{D}\left(\int_K f d\varphi, \int_{E_1} f d\varphi + \int_{E_2} f d\varphi\right) =$$

$$= \mathcal{D}\left(\int_{K' \cup K''} f d\varphi, \int_{E_1} f d\varphi + \int_{E_2} f d\varphi\right) =$$

$$= \mathcal{D}\left(\int_{K'} f d\varphi + \int_{K''} f d\varphi, \int_{E_1} f d\varphi + \int_{E_2} f d\varphi\right) \leq$$

$$\leq \mathcal{D}\left(\int_{K'} f d\varphi, \int_{E_1} f d\varphi\right) +$$

$$+\mathcal{D}\left(\int_{K''} f d\varphi, \int_{E_2} f d\varphi\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, for every $\varepsilon > 0$ there exists $K_\varepsilon \in M_{E_1 \cup E_2}(f)$ such that for every $K \in M_{E_1 \cup E_2}(f)$ with $K \supseteq K_2$ we have

$$\mathcal{D}\left(\int_{E_1} f d\varphi, \int_{E_1} f d\varphi + \int_{E_2} f d\varphi\right) < \varepsilon,$$

which leads to

$$\int_{E_1 \cup E_2} f d\varphi = \int_{E_1} f d\varphi + \int_{E_2} f d\varphi.$$

iii) By virtue of theorem 3.7, for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for $(\forall) T \in M(f)$ with $T \cap K_\varepsilon = \phi$ we have

$$\left|\int_T f d\varphi\right| < \frac{\varepsilon}{4} \tag{22}$$

If $E \in M(f)$, denoting $\Gamma'(C) = \int_C f d\varphi, (\forall) C \in \mathcal{A}$ with $C \subseteq E$, we obtain, applying theorem 20-b) [3], a $\delta(\varepsilon) > 0$ such that $(\forall) C \in \mathcal{A}, C \subseteq E$, f bounded on C and $\nu(C) < \delta$ it follows that

$$v(\Gamma', C) < \frac{\varepsilon}{2} \tag{23}$$

Let us take $(\forall) B \in \mathcal{A}$ with $B \cap K_\varepsilon = \phi$. According to theorem 3.10-i) f is g-φ-integrable on B and then, by virtue of definition 3.2, $\exists K'_\varepsilon \in M_B(f)$ such that for $(\forall) K \in M_B(f), K \supseteq K'_\varepsilon$ we have

$$\mathcal{D}\left(\int_K f d\varphi, \int_B f d\varphi\right) < \frac{\varepsilon}{4} \tag{24}$$

Then, using (24) we obtain

$$\begin{aligned} &\left|\textstyle\int_B f d\varphi\right| \leq \mathcal{D}\left(\textstyle\int_B f d\varphi, \int_{K_\varepsilon} f d\varphi\right) + \\ &+\mathcal{D}\left(\textstyle\int_{K'_\varepsilon} f d\varphi, 0\right) < \frac{\varepsilon}{4} + \left|\displaystyle\int_{K'_\varepsilon} f d\varphi\right|. \end{aligned} \tag{25}$$

Since $K'_\varepsilon \in M_B(f)$ it follows that $K'_\varepsilon \subseteq B$. But $B \cap K_\varepsilon = \phi$ and hence $K'_\varepsilon \cap K_\varepsilon = \phi$. From (22) we have

$$\left|\int_{K'_\varepsilon} f d\varphi\right| < \frac{\varepsilon}{4}. \tag{26}$$

Using (25) and (26) we obtain

$$\left|\int_B f d\varphi\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Hence for $(\forall)B \in \mathcal{A}$ with $B \cap K_\varepsilon = \phi$ we have

$$\left|\int_B f d\varphi\right| < \frac{\varepsilon}{2}. \tag{27}$$

Let us take $E \in \mathcal{A}$ with $\nu(E) < \delta$. We can write $E = (E \cap K_\varepsilon) \cap (E|K_\varepsilon)$. According to theorem 3.7-ii) we have

$$\int_E f d\varphi = \int_{E\cap K_\varepsilon} f d\varphi + \int_{E|K_\varepsilon} f d\varphi$$

from which we obtain, using (27),

$$\left|\int_E f d\varphi\right| = \left|\int_{E\cap K_\varepsilon} f d\varphi + \int_{E|K_\varepsilon} f d\varphi\right| \leq$$

$$\leq \left|\int_{E\cap K_\varepsilon} f d\varphi\right| + \left|\int_{E|K_\varepsilon} f d\varphi\right| \leq \tag{28}$$

$$\leq \left|\Gamma'(E \cap K_\varepsilon)\right| + \frac{\varepsilon}{2} \leq v(\Gamma'(E \cap K_\varepsilon)) + \frac{\varepsilon}{2}.$$

Since $\nu(E \cap K_\varepsilon) \leq \nu(E) < \delta$ and f is bounded on $E \cap K_\varepsilon$, from (23) we have

$$v(\Gamma', E \cap K_\varepsilon) < \frac{\varepsilon}{2}. \tag{29}$$

Now, from (28) and (29) we obtain

$$\left|\int_E f d\varphi\right| < \varepsilon,$$

which assures that Γ is absolutely continuous with respect to ν.■

Theorem 3.11. *If $f, g : S \to R$ are generalized φ-integrable with $f(s) \cdot g(s) \geq 0$ for $(\forall)s \in S$, then $f + g$ is generalized φ-integrable and*

$$\int_E (f+g) d\varphi = \int_E f d\varphi + \int_E g d\varphi, (\forall)E \in \mathcal{A} \tag{30}$$

Proof. Since f, g are g-φ-integrable it results that f, g are ν-totally measurable. Hence for $\varepsilon > 0$ there exists two partitions $P'_\varepsilon, P''_\varepsilon$ of S, $P'_\varepsilon = \{A_i\}_{i=0}^n$ $P''_\varepsilon = \{B_j\}_{j=0}^m$ such that $\nu(A_0) < \frac{\varepsilon}{2}, \nu(B_0) < \frac{\varepsilon}{2}, osc(f, A_i) < \frac{\varepsilon}{2}$, $i = 1, 2, ..., n$ and $osc(g, B_j) < \frac{\varepsilon}{2}, j = 1, 2, ..., m$.

Let us take $P_\varepsilon = P'_\varepsilon \cap P''_\varepsilon = \{A_i \cap B_j\}_{\substack{i=0,1...n \\ j=0,1...m}} = \{E_K\}_{k=0}^p$, where $E_0 = A_0 \cap B_0$. Because $E_0 \subset A_0$ we have $\nu(E_0) < \varepsilon$. On the other hand,

$$osc(f + g, E_k) \leq osc(f, A_i) + osc(g, B_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, (\forall)k = 1, 2...p;$$

hence $f + g$ is ν-totally measurable.

Now, using the theorem 3.10-iii), for $\varepsilon > 0$ then exists $\delta(\varepsilon) > 0$ such that for $(\forall)E \in \mathcal{A}$ with $\nu(E) < \delta$ we have

$$\left| \int_E f d\varphi \right| < \varepsilon, \quad \left| \int_E g d\varphi \right| < \varepsilon \tag{31}$$

Because f, g are ν-totally measurable, for the above $\delta > 0$, $(\exists)N(\delta) > 0$ and $F_N, G_N \in \mathcal{A}$ such that

$$\nu(S \setminus F_N) < \frac{\delta}{2}, \ \ \nu(S \setminus G_N) < \frac{\delta}{2}, \sup_{s \in F_N} |f(s)| < N, \sup_{s \in G_N} |g(s)| < N \tag{32}$$

Then, for $(\forall)A \in \mathcal{A}$ with $A \cap (F_N \cap G_N) = \phi$ we have $\nu(A) < \delta$.

From (31) we obtain

$$\left| \int_A f d\varphi \right| < \varepsilon \text{ and } \left| \int_A g d\varphi \right| < \varepsilon \tag{33}$$

Since f, g are g-φ-integrable there exists $K_1^\varepsilon \in M(f), K_2^\varepsilon \in M(f)$ such that for $(\forall)K \in M(f)$ with $K \cap K_1^\varepsilon = \phi$ we have $\left| \int_K f d\varphi \right| < \varepsilon$ and respectively for $(\forall)K \in M(g)$ with $K \cap K_2^\varepsilon = \phi$ we have $\left| \int_K g d\varphi \right| < \varepsilon$.

Now, reasoning as in the proof of iii)-Theorem 3.10, we obtain that for $(\forall)A \in \mathcal{A}$ with $A \cap K_1^\varepsilon = \phi$ it follows

$$\left| \int_A f d\varphi \right| < \varepsilon \tag{34}$$

and for $(\forall)A \in \mathcal{A}$ with $A \cap K_2^\varepsilon = \phi$ it follows

$$\left| \int_A g d\varphi \right| < \varepsilon \tag{35}$$

Let us denote $K_0^\varepsilon = K_1^\varepsilon \cup K_2^\varepsilon$. We see that $\nu(K_0^\varepsilon) < +\infty$ and for $(\forall)A \in \mathcal{A}$ with $A \cap K_0^\varepsilon = \phi$, from (34) and (35) we have

$$\left| \int_A f d\varphi \right| < \varepsilon, \quad \left| \int_A g d\varphi \right| < \varepsilon \tag{36}$$

Let us take $K = F_N \cap G_N \cap K_0^\varepsilon$. We observe that $\nu(K) < +\infty$ and from (32) we see that $f + g$ is bounded on K. Let $T \in \mathcal{A}$ with $\nu(T) < \infty, T \cap K = \phi$ and $f + g$ bounded on T. It follows that T can be written as $T = T_1 \cup T_2$, where $T_1 \cap T_2 = \phi, T_1 \cap (F_N \cap G_N) = \phi, T_2 \cap K_0^\varepsilon = \phi$.

Now it is sufficient to prove that $\left|\int_T (f+g)d\varphi\right| < \varepsilon$. According to theorem 2.11-ii), [3] we have

$$\int_T (f+g)d\varphi = \int_{T_1} (f+g)d\varphi + \int_{T_2} (f+g)d\varphi.$$

Since f is ν-totally measurable and the integral is ν-absolutely continuous on the subsets of T it follows that there exists $B_1 \in \mathcal{A}, B_1 \subseteq T_1$ such that

$$\left| \int_{T_1 \setminus B_1} (f+g)d\varphi \right| < \varepsilon \tag{37}$$

and f is bounded on B_1. Then we see that g is also bounded on B_1.

Now we have

$$\int_{T_1} (f+g)d\varphi = \int_{T_1 \cup (A_1|T_1)} (f+g)d\varphi = \int_{T_1} (f+g)d\varphi + \int_{T_1 \setminus B_1} (f+g)d\varphi \tag{38}$$

Since $f + g, f, g$ are all bounded on B_1 we obtain

$$\int_{B_1} (f+g)d\varphi = \int_{B_1} f d\varphi + \int_{B_1} g d\varphi \tag{39}$$

From (38) and (39) we find

$$\int_{T_1} (f+g)d\varphi = \int_{B_1} f d\varphi + \int_{B_1} g d\varphi + \int_{T_1 \setminus B_1} (f+g)d\varphi \tag{40}$$

Since $T_1 \cap (F_N \cap G_N) = \phi$ we have $B_1 \cap (F_N \cap G_N) = \phi$. Using (36) we obtain

$$\left| \int_{B_1} f d\varphi \right| < \varepsilon \text{ and } \left| \int_{B_1} g d\varphi \right| < \varepsilon \tag{41}$$

and then

$$\left|\int_{T_1}(f+g)d\varphi\right| \leq \left|\int_{T_1} f d\varphi\right| + \left|\int_{T_1} g d\varphi\right| + \left|\int_{T_1\setminus B_1}(f+g)d\varphi\right| \tag{42}$$

Analogously we can prove

$$\left|\int_{T_2}(f+g)d\varphi\right| < 3\varepsilon \tag{43}$$

From (42) and (43) we obtain

$$\left|\int_T (f+g)d\varphi\right| = \left|\int_{T_1}(f+g)d\varphi + \int_{T_2}(f+g)d\varphi\right| \leq$$

$$\leq \left|\int_{T_1}(f+g)d\varphi\right| + \left|\int_{T_2}(f+g)d\varphi\right| < 6\varepsilon.$$

Consequently for $(\forall)T \in \mathcal{A}$ with $\nu(T) < \infty$, $f+g$ bounded on T, $T \cap K = \phi$ we have

$$\left|\int_T (f+g)d\varphi\right| < 6\varepsilon,$$

which together with the ν-total measurability of $f+g$ assures, according to theorem 3.7, that $f+g$ is g-φ-integrable.

Now we prove (30) for S (similarly we can prove the relation for $(\forall)E \in \mathcal{A}$).

Writing successively that f, g and $f+g$ are g-φ-integrable we have that for $(\forall)\varepsilon > 0, (\exists)K'_\varepsilon \in M(f)$ such that for $(\forall)K \in M(f)$ with $K \supseteq K'_\varepsilon$ holds

$$\mathcal{D}\left(\int_K f d\varphi, \int_S f d\varphi\right) < \frac{\varepsilon}{3}, \tag{44}$$

$(\exists)K''_\varepsilon \in M(f)$ such that for $(\forall)K \in M(g)$ with $K \supseteq K''_\varepsilon$

$$\mathcal{D}\left(\int_K g d\varphi, \int_S g d\varphi\right) < \frac{\varepsilon}{3}, \tag{45}$$

$(\exists)K_\varepsilon \in M(f+g)$ such that for $(\forall)K \in M(f+g)$ with $K \supseteq K_\varepsilon$

$$\mathcal{D}\left(\int_K (f+g)d\varphi, \int_S (f+g)d\varphi\right) < \frac{\varepsilon}{3} \tag{46}$$

Now for every $K \in \mathcal{A}$ with $\nu(K) < \infty, f, g$ bounded on K and $K \supseteq K'_\varepsilon \cup K''_\varepsilon \cup K_\varepsilon$ we have $K \in M(f) \cap M(g) \cap M(f+g)$.

From the bounded case we obtain

$$\int_K (f+g)d\varphi = \int_K f d\varphi + \int_K g d\varphi.$$

If $\widetilde{A}$ denotes the $\int_K (f+g)d\varphi$ we have

$$\mathcal{D}\left(\int_S (f+g)d\varphi, \int_S f d\varphi + \int_S g d\varphi\right) \leq \mathcal{D}\left(\int_S (f+g)d\varphi, \widetilde{A}\right) +$$

$$+\mathcal{D}\left(\widetilde{A}, \int_S f d\varphi + \int_S g d\varphi\right) = \mathcal{D}\left(\int_S (f+g)d\varphi, \int_K (f+g)d\varphi\right) +$$

$$+\mathcal{D}\left(\int_K f d\varphi + \int_K g d\varphi, \int_S f d\varphi + \int_S g d\varphi\right).$$

Using successively (46), (44) and (45) we obtain

$$\mathcal{D}\left(\int_S (f+g)d\varphi, \int_S f d\varphi + \int_S g d\varphi\right) < \frac{\varepsilon}{3} +$$

$$+\mathcal{D}\left(\int_K f d\varphi, \int_S f d\varphi\right) + \mathcal{D}\left(\int_K g d\varphi, \int_S g d\varphi\right)$$

which assures that (30) is satisfied. ∎

Theorem 3.12. *If $f, g : S \to R$ are g-φ-integrable and $|f| \leq |g|$, then*

$$\left| \int_E f d\varphi \right| \leq 2 \left| \int_E g d\varphi \right|, (\forall) E \in \mathcal{A} \tag{47}$$

Proof. We give the proof for $E = S$. Writing that f, g are g-φ-integrable, we have for $(\forall)\varepsilon > 0, (\exists) K^1_\varepsilon \in M(f), (\exists) K^2_\varepsilon \in M(g)$ such that for $(\forall) K \in M(f)$ with $K \supseteq K^1_\varepsilon$ we have

$$\mathcal{D}\left(\int_K f d\varphi, \int_S f d\varphi\right) < \frac{\varepsilon}{3} \tag{48}$$

and for $(\forall)K \in M(g)$ with $K \supseteq K_\varepsilon^2$ we have

$$\mathcal{D}\left(\int_K gd\varphi, \int_S gd\varphi\right) < \frac{\varepsilon}{3} \tag{49}$$

Because $|f| \leq |g| \Longrightarrow M(g) \subseteq M(f)$.

Let us take $K \in M(g)$ with $K \supseteq K_\varepsilon^1 \cup K_\varepsilon^2$.

For this K, hold both relations (47) and (48). Now using (48), (49) and (47) for bounded case we have

$$\left|\int_S fd\varphi\right| \leq \mathcal{D}\left(\int_S fd\varphi, \int_K fd\varphi\right) + \left|\int_K fd\varphi\right| < \frac{\varepsilon}{3} + \left|\int_K fd\varphi\right| \leq$$

$$\leq \frac{\varepsilon}{3} + 2\left|\int_K gd\varphi\right| \leq \frac{\varepsilon}{3} + 2\mathcal{D}\left(\int_K gd\varphi, \int_S gd\varphi\right) + 2\left|\int_S gd\varphi\right| <$$

$$< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} + 2\left|\int_S gd\varphi\right| = \varepsilon + 2\left|\int_S gd\varphi\right|$$

for every $\varepsilon > 0$. Hence (47) is proved. ■

Theorem 3.13. *Let $g : S \to R$ be a generalized φ-integrable function, $f_n : S \to R$ a sequence of ν-totally measurable functions such that $|f_n| \leq |g|, (\forall)n \in N$. Then:*

i) *f_n is g-φ-integrable for $(\forall)n \in \mathbf{N}$.*

ii) *For $(\forall)\varepsilon > 0$ there exists $K_0 \in \mathcal{A}$ such that $(f_n)_{n\in\mathbf{N}}$ is uniformly bounded on K_0 and for $(\forall)E \in \mathcal{A}$ with $E \cap K_0 = \phi$ we have $\left|\int_E f_n d\mu\right| < \varepsilon, (\forall)n \in \mathbf{N}$.*

iii) *Denoting by $\Gamma_n(E) = \int_E f_n d\varphi, (\forall)E \in \mathcal{A}, (\forall)n \in \mathbf{N}$, for $(\forall)\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $(\forall)E \in \mathcal{A}$ with $\nu(E) < \delta$ it follows $\left|\Gamma_n(E)\right| < \varepsilon, (\forall)n \in \mathbf{N}$.*

Proof.

i) Since g is g-φ-integrable, if $n_0 \in N$ is arbitrary fixed and $\varepsilon > 0$ given, then there exists $K_\varepsilon \in M(g)$ such that for $(\forall)T \in M(g)$with $T \cap K_\varepsilon = \phi$ we have

$$\left| \int_T g d\varphi \right| < \frac{\varepsilon}{4} \tag{50}$$

Because $|f_{n_0}| \leq |g|$, we obtain $M(g) \subseteq M(f_{n_0})$ and hence $K_\varepsilon \in M(f_{n_0})$.

Let $K \in M(f_0)$ be such that $K \cap K_\varepsilon = \phi$. Since g is generalized φ-integrable on K, there exists $K'_\varepsilon \in M_K(g)$ such that for $(\forall)T \in M_K(g)$ with $T \supseteq K'_\varepsilon$ holds

$$\mathcal{D}\left(\int_T g d\varphi, \int_K g d\varphi \right) < \frac{\varepsilon}{4} \tag{51}$$

Now we have

$$\begin{aligned} \left| \int_K g d\varphi \right| &\leq \mathcal{D}\left(\int_K g d\varphi, \int_{K'_\varepsilon} g d\varphi \right) + \\ &+ \left| \int_{K'_\varepsilon} g d\varphi \right| < \frac{\varepsilon}{4} + \left| \int_{K'_\varepsilon} g d\varphi \right| \end{aligned} \tag{52}$$

But $K'_\varepsilon \in M_K(g)$ and hence $K'_\varepsilon \subseteq K$. Since $K \cap K_\varepsilon = \phi$ it follows that $K'_\varepsilon \cap K_\varepsilon = \phi$ and, according to (50) we have

$$\left| \int_{K'_\varepsilon} g d\varphi \right| < \frac{\varepsilon}{4} \tag{53}$$

Now using (52) and (53) we obtain

$$\left| \int_K g d\varphi \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \tag{54}$$

and next, applying theorem 3.12 we have

$$\left| \int_K f_{n_0} d\varphi \right| \leq 2 \left| \int_K g d\varphi \right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon,$$

which together with the ν-total measurability of f_{n_0} assures the g-φ-integrability of f_{n_0}.

ii) Since g is g-φ-integrable, for $(\forall)\varepsilon > 0$ there exists $K_\varepsilon \in M(g)$ such that for $(\forall)T \in M(g)$ with $T \cap K_\varepsilon = \phi$ we have (50). Because g is bounded on K_ε, from the inequality $|f_n| \leq g, (\forall)n \in N$, it follows that $(f_n)_{n\in\mathbf{N}}$ is uniformly bounded on K_ε. If $E \in \mathcal{A}$ with $E \cap K_\varepsilon = \phi$, applying theorem 3.12 we obtain

$$\left| \int_K f_n d\varphi \right| \leq 2 \left| \int_E g d\varphi \right| , (\forall)n \in \mathbf{N} \tag{55}$$

Because g g-φ-integrable on E, there exists $K'_\varepsilon \in M(g)$ such that

$$\mathcal{D}\left(\int_{K'_\varepsilon} g d\varphi, \int_E g d\varphi \right) < \frac{\varepsilon}{4} \tag{56}$$

But $K'_\varepsilon \subseteq E$ and since $E \cap K_\varepsilon = \phi$ it follows $K'_\varepsilon \cap K_\varepsilon = \phi, K'_\varepsilon \in M(g)$. From (50) we have

$$\left| \int_{K'_\varepsilon} g d\varphi \right| < \frac{\varepsilon}{4} \tag{57}$$

Now, using (56) and (57) we obtain

$$\begin{aligned} \left| \int_E g d\varphi \right| &\leq \mathcal{D}\left(\int_E g d\varphi, \int_{K'_\varepsilon} g d\varphi \right) + \\ &+ \mathcal{D}\left(\int_{K'_\varepsilon} g d\varphi, 0 \right) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \end{aligned} \tag{58}$$

From (55) and (58) we have

$$\left| \int_E f_n d\varphi \right| \leq 2 \left| \int_E g d\varphi \right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon, (\forall)n \in N.$$

Hence ii) is proved.

iii) Let us denote $\Gamma(E) = \int_E g d\varphi, (\forall)E \in \mathcal{A}$. According to theorem 3.10-iii) for $(\forall)\varepsilon > 0, (\exists)\delta(\varepsilon) > 0$ such that for $(\forall)E \in \mathcal{A}$ with $\nu(E) < \delta$ we have

$$\left| \int_E g d\varphi \right| < \frac{\varepsilon}{2} \tag{59}$$

Since $|f_n| \leq |g|, (\forall)n \in \mathbf{N}$, by virtue of theorem 3.12 one obtains

$$\left| \int_E f_n d\varphi \right| \leq 2 \left| \int_E g d\varphi \right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon, (\forall)n \in N$$

and iii) is proved. ■

Theorem 3.14. (The Lebesgue dominated convergence theorem)
If $f_n, g : S \to R, (\forall)n \in \mathbf{N}$ are g-φ-integrable functions with $|f_n| \leq |g|, (\forall)n \in \mathbf{N}$ and $f : S \to R$ is such that $|f| \leq |g|$ and f_n converges in ν-measure to f then f is g-φ-integrable and

$$\int_E f d\varphi = \lim_{n\to\infty} \int_E f_n d\varphi, (\forall)E \in \mathcal{A}.$$

Proof. Since f_n is ν-totally measurable for $(\forall)n \in \mathbf{N}$ (f_n is g-φ-integrable) it follows that f is ν-totally measurable. From $|f| \leq |g|$ and g-φ-integrability of g we obtain, using theorem 3.13-i), that f is also g-φ-integrable.

According to theorem 3.13-ii), for $(\forall)\varepsilon > 0, (\exists)K_0 \in \mathcal{A}$ with $\nu(K_0) < \infty$ and $(\exists)M > 0$ such that

$$|f_n(s)| \leq M, (\forall)s \in K_0, (\forall)n \in \mathbf{N} \text{ and } |f(s)| \leq M, (\forall)s \in K_0 \tag{60}$$

and for $(\forall)T \in \mathcal{A}$ with $T \cap K_0 = \phi$ it follows

$$\left| \int_T f_n d\varphi \right| < \frac{\varepsilon}{4}, (\forall)n \in \mathbf{N} \tag{61}$$

and

$$\left| \int_T f d\varphi \right| < \frac{\varepsilon}{4}, (\forall)n \in \mathbf{N} \tag{62}$$

Now, we have for $(\forall)E \in \mathcal{A}$

$$\mathcal{D}\left(\int_E f_n d\varphi, \int_E f d\varphi \right) =$$

$$= \mathcal{D}\left(\int_{E\setminus K_0} f_n d\varphi + \int_{E\cap K_0} f_n d\varphi, \int_{E\setminus K_0} f d\varphi, \int_{E\cap K_0} f d\varphi \right) \leq \tag{63}$$

$$\leq \mathcal{D}\left(\int_{E|K_0} f_n d\varphi, \int_{E\setminus K_0} f d\varphi \right) + \mathcal{D}\left(\int_{E\cap K_0} f_n d\varphi, \int_{E\cap K_0} f d\varphi \right).$$

For $(\forall)\delta > 0$, denote

$$B_n(\delta) = \{s \in S; |f_n(s) - f(s)| \geq \delta\}.$$

Since $f_n \xrightarrow{\nu} f$ it follows that $\lim_{n\to\infty} \nu^*(B_n(\delta)) = 0, (\forall)\delta > 0$, that is $(\exists)n_0 \in N$ such that for $(\forall)n \geq n_0$

$$\nu^*\left(B_n\left(\frac{\varepsilon}{4 \cdot \nu(K_0)}\right)\right) < \frac{\varepsilon}{8M}.$$

From (5) it follows that $(\exists)C \in \mathcal{A}$ such that $B_n\left(\frac{\varepsilon}{4\nu(K_0)}\right) \subseteq C$ and

$$\nu(C) < \frac{\varepsilon}{8M} \tag{64}$$

Now, using remark 2.5-III and (64) we have

$$\begin{aligned}
&\mathcal{D}\left(\int_{E\cap K_0} f_n d\varphi, \int_{E\cap K_0} f d\varphi\right) = \\
&= \mathcal{D}\left(\int_{(E\cap K_0)\cap C} f_n d\varphi + \int_{(E\cap K_0)|C} f_n d\varphi,\right. \\
&\left.\int_{(E\cap K_0)\cap C} f d\varphi + \int_{(E\cap K_0)|C} f d\varphi\right) \leq \\
&\leq \mathcal{D}\left(\int_{(E\cap K_0)\cap C} f_n d\varphi, \int_{(E\cap K_0)\cap C} f d\varphi\right) + \\
&+\mathcal{D}\left(\int_{(E\cap K_0)|C} f_n d\varphi, \int_{(E\cap K_0)|C} f d\varphi\right) \leq \\
&\leq \sup_{s\in E\cap K_0\cap C} |f_n(s) - f(s)| \cdot \nu(E \cap K_0 \cap C) + \\
&+ \sup_{s\in (E\cap K_0)|C} |f_n(s) - f(s)| \cdot \nu((E \cap K_0)|C) \leq \\
&\leq 2M\nu(C) + \frac{\varepsilon}{4\nu(K_0)} \cdot \nu(K_0) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned} \tag{65}$$

Since $E \setminus K_0 \in \mathcal{A}$ and $(E \setminus K_0) \cap K_0 = \phi$, from (61) and (62) we obtain

$$\left| \int_{E \setminus K_0} f_n d\varphi \right| < \frac{\varepsilon}{4}, (\forall) n \geq n_0$$

and

$$\left| \int_{E \setminus K_0} f d\varphi \right| < \frac{\varepsilon}{4}.$$

Now, the last two inequalities lead to

$$\begin{aligned} \mathcal{D}\left(\int_{E|K_0} f_n d\varphi, \int_{E|K_0} f d\varphi \right) \leq & \left| \int_{E|K_0} f_n d\varphi \right| + \\ & + \left| \int_{E|K_0} f d\varphi \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \end{aligned} \tag{66}$$

Applying the relations (63), (65) and (66) we have

$$\mathcal{D}\left(\int_E f_n d\varphi, \int_E f d\varphi \right) < \varepsilon, (\forall) n \geq n_0,$$

which assures that $\lim_{n \to \infty} \int_E f_n d\varphi = \int_E f d\varphi$, uniformly in $E \in \mathcal{A}$. ∎

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