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A SURVEY ON THE VECTOR SUBDIFFERENTIALS

BY

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1. Introduction. In 1972, VALADIER published in [37] some generalizations for two theorems of Moreau about the subdifferentiability of convex, continuous vector functions.

In Valadier's opinion, until his paper, the study of the subdifferentiability of vector valued functions was considered only by RAFFIN [25] for the space $\mathbb{R}^I$.

The vector subdifferential used by Valadier for a function between two linear topological spaces X and Y, $f : X \to Y$ is

$$\partial_{\leq} f(x_0) = \{T \in \mathcal{L}(X,Y) \mid \ T(x - x_0) \leq f(x) - f(x_0), \forall x \in X\}$$

and the main theorem of the paper generalize for a convex function with values in a Banach lattice the known result from the real case that the subdifferential for a convex, continuous function is a nonempty, convex equicontinuous subset of $\mathcal{L}(X,Y)$. Besides, he established the link between this vector subdifferential and a directional derivative $f'(x_0,h)$ (see relation (DV)) as

$$f'(x_0,h) = \max\{Th,\ T \in \partial_{\leq} f(x_0)\}.$$

After that, the study of the vector subdifferentiability for functions which take values in a Banach lattice was continued by many authors; among them, we recall ZOWE [38], KUSRAEV and KUTATELADZE [17], PAPAGEORGIOU [2]. The last of them, developed some calculus rules and a duality theory, similarly to the real one. Also, he proposed a generalization for the Clarke subdifferential for locally o-Lipschitz functions (with values in a

Banach lattice, too); for that, he used a directional derivative $f^{\circ}(x,h)$ (see relation (DP)) and defined the subdifferential like:

$$\partial^P f(x) = \{T \in \mathcal{L}(X,Y),\ T(h) \leq f^{\circ}(x,h), \forall h \in X\}.$$

By using this type of subdifferential, he generalized the known results for real lipschitzian maps for vector valued functions also as some calculus rules and optimality conditions.

In [36], THIBAULT proposed another subdifferential, introduced by a directional derivative $f^{\uparrow}(x,h)$ (see relation (DT)), defined with the aid of the Clarke tangent cone; this subdifferential, given by:

$$\partial^T f(x) = \{T \in \mathcal{L}(X,Y) \mid\ T(h) \leq f^{\uparrow}(x,h), \forall h \in X\}$$

was used for the study of "strictly compactly lipschitzian" functions, a notion which generalize the "strictly differentiable" functions.

Let's remark that this subdifferentials are coincident with the Valadier subdifferential for the convex functions.

ZOWE proposed in 1974, in [38] another approach which doesn't request that Y being lattice but which request some special topological conditions for X and Y. This approach open a new direction for the study of vector subdifferentiability, the scalarization. By this way, J.B. Hiriart-Urruty and L. Thibault give a generalization for the notion of Clarke subdifferential for locally Lipschitz functions when X is a separable Banach space and Y is a reflexive separable space. They proved the existence of a closed, convex set $\Gamma(f,x_0)$, subset of $\mathcal{L}(X,Y)$, which is the maximal set with the property that

$$\partial y^* \circ f(x_0) = y^* \circ \Gamma(f,x_0),\ \text{for all } y^* \in Y^*.$$

In fact, this set is $plen\{\partial_H f(x_0)\}$ where $\partial_H f(x_0) = \overline{co}\{D_H f(x_0)\}$ ($D_H f(x_0) = \{\lim_{n\to\infty} Df(x_n) \mid x_n \to x_0, x_n \in H\}$, where $Df(x_n)$ is the Hadamard's derivative of f at x_n and H is the set where f is Hadamard differentiable.)

This subdifferential can be considered as a generalization for the Clarke real subdifferential expressed like the convex hull of the gradients limits.

For the finite dimensional spaces, Clarke already introduced the vector subdifferential like the convex hull of all the matrices obtained as the limits of the classical jacobian matrices.

It is known that if $Y = \mathbb{R}$, this subdifferential is coincident with the Valadier subdifferential for a convex function but generally, this is not true.

Also, for the locally Lipschitz functions between two Banach spaces, A Ioffe proposed a notion which generalize the vector subdifferential expressed by the correspondence:

$$y^* \rightrightarrows \{x^* \in X^* \mid \langle x^*, h\rangle \leq \langle y^*, f(x+h) - f(x)\rangle,\ \forall h \in X\}.$$

After 1980, another types of vector subdifferential was considered, especially the Pareto subdifferential which make possible the study of an optimization problem, by linking the topological properties for the space and for the ordering cone.

Some results about this subdifferentials was given in finite dimensional spaces by Tanino and Sawaragi [29] and in infinite dimensional spaces by Nemeth [21], Isac and Postolică [14].

After 1990, Mordukhowich introduced in [20] another object for the study of the closed graph multifunctions, the coderivative, which made possible for the introduction of a real nonconvex subdifferential used in nonlinear analysis and in optimal control.

By the scope to express these subdifferentials by a general notion, we introduce in [31], a new type of vector subdifferentials; in what follows we intend to give the links between these subdifferentials and the notions which were studied until now.

2. Notations and preliminaries. Throughout this paper, without other suppositions, X, Z will be locally convex spaces, Z endowed with a partial order induced by a cone Z_+ which is proper, closed, *convex* ($Z_+ + Z_+ \subset Z_+$), *pointed* ($Z_+ \cap -Z_+ = \{0\}$).

We make the convention that $A + \emptyset = \lambda\emptyset = \emptyset$ for all $A \subset Z,\ \lambda \in \mathbb{R}$.

We adjoin to Z an abstract maximal element denoted by $+\infty$ and an abstract minimal element denoted $-\infty$. We set $\overline{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$. Infinity satisfies:

$$\lambda \cdot (\pm\infty) = \pm\infty \quad y + (\pm\infty) = \pm\infty,$$

for all $\lambda > 0$ and any y in Z.

For all $z, y \in Z$, we write

$$z \leq_{Z_+} y \iff y - z \in Z_+$$

$$z \not\leq_{Z_+} y \iff y - z \notin Z_+ \setminus \{0\}.$$

We will renounce to index the order if no confusion can happen. We denote by $\mathcal{V}_Z(x_0)$ a fundamental system of neighborhoods of x_0 for the given topology on Z. If the interior of the ordering cone Z_+ (denoted Int Z_+) is nonempty and $\varepsilon \in \text{Int } Z_+$, then we shall consider for $+\infty$ a fundamental system of neighborhoods consisting of sets $(\varepsilon + Z_+) \cup \{+\infty\}$ and for $-\infty$ a fundamental system of neighborhoods consisting of sets $(-\varepsilon - Z_+) \cup \{-\infty\}$.

We denote by Z^* the topological *dual* of Z; Z_+^* denotes the *dual cone* of Z_+ defined by

$$Z_+^* = \Big\{ z^* \in Z^* \;\Big|\; \forall z \in Z_+,\ \langle z^*, z\rangle \geq 0 \Big\},$$

while $Z_+^{\#}$ stands for the *quasi-interior* of Z_+^* and is given by

$$Z_+^{\#} = \Big\{ z^* \in Z_+^* \mid \forall z \in Z_+ \setminus \{0\}, \langle z^*, z\rangle > 0 \Big\}.$$

For a function $f : X \to \overline{Z}$, dom $f = \{x \in X \mid f(x) \neq \{+\infty\}\}$ and for a multifunction $F : X \rightrightarrows Z$, we denote Dom $F = \{x \in X \mid\ F(x) \neq \emptyset\}$ and Gr $F = \{(x, z) \in X \times Z \mid\ z \in F(x)\}$.

If A is a nonempty subset of Z and $\varepsilon \in Z_+$, the principal sets of efficient points used in this paper are:

$$^{\varepsilon}SUP\ A = \{z \in \overline{Z} \mid z \not< a - \varepsilon,\ \forall a \in A\}$$

ε-supremum points set;

$$^{\varepsilon}SUP_1 A = \{z \in^{\varepsilon} SUP\ A \mid \forall \eta > 0,\ \exists a_\eta \in A,\ a_\eta > z - \eta\}$$

proximal ε-supremum points set;

$$^{\varepsilon}INF\ A = \{z \in \overline{Z} \mid z \not> a + \varepsilon,\ \forall a \in A\}$$

ε-infimum points set;

$$^{\varepsilon}INF_1 A = \{z \in^{\varepsilon} INF\ A \mid \forall \eta > 0,\ \exists a_\eta \in A,\ a_\eta < z + \eta\}$$

proximal ε-infimum points set;

$$IMAX\ A = \{z \in A \mid z \geq a, \forall a \in A\}$$

ideal maximal points set;

$$IMIN\ A = \{z \in A \mid z \leq a, \forall a \in A\}$$

ideal minimal points set;

$$\inf A = IMAX\{z \in Z \mid z \leq a, \forall a \in A\}$$

strong infimum points set;

$$\sup A = IMIN\{z \in Z \mid z \geq a, \forall a \in A\}$$

strong supremum points set.

Let's remark that if the set $IMIN\ A$, or $IMAX\ A$, $\inf A$, $\sup A$ is nonempty then the set contains only one element, respectively.
If the interior of Z_+ (denoted Int Z_+) is nonempty, the efficient points sets corresponding to the cone $K = \text{Int } Z_+ \cup \{0\}$ will be denoted:

$$w^{\varepsilon}SUP\ A = \{z \in \overline{Z} \mid z \not\leq_{\text{Int } Z_+ \cup \{0\}} a - \varepsilon,\ \forall a \in A\};$$
$$w^{\varepsilon}SUP_1 A = \{z \in w^{\varepsilon}SUP\ A \mid \forall \eta >_K 0,\ \exists a_\eta \in A,\ a_\eta >_K z - \eta\};$$
$$w^{\varepsilon}INF\ A = \{z \in \overline{Z} \mid z \not\geq_{\text{Int } Z_+ \cup \{0\}} a + \varepsilon,\ \forall a \in A\};$$
$$w^{\varepsilon}INF_1 A = \{z \in w^{\varepsilon}INF\ A \mid \forall \eta >_K 0,\ \exists a_\eta \in A,\ a_\eta <_K z + \eta\};$$

(for more details see [12]).

Recall that a convex cone Z_+ is normal if there exists a basis of neighborhoods of the origin $\mathcal{V}_Z(0)$ such that

$$(V - Z_+) \cap (V + Z_+) = V,\ \forall V \in \mathcal{V}_Z(0);$$

The cone Z_+ is called *Daniell* if every decreasing net which has strong infimum is convergent to its strong infimum.

An ordered locally convex space Z is *order complete* with respect to the ordering cone Z_+ if every nonempty set bounded from below has an infimum.

A *Banach lattice* is an ordered Banach space in which the following monotonicity relation holds: $\mid x \mid \leq \mid z \mid$ implies $\|x\| \leq \|z\|$.
As usual, $\mathcal{L}(X, Z)$ stands for the set of linear continuous mappings from X to Z. A mapping $f\colon X \to \overline{Z}$ is said to be *convex* (with respect to Z_+) if the epigraph of f given by

$$\text{epi } f = \{(x, z) \in X \times Z \mid z \geq f(x)\}$$

is a convex set in $X \times Z$.
If $\Omega \subset X \times Z$, we denote by $Pr_X \Omega = \{x \in X \mid\ \exists z \in Z, (x, z) \in \Omega\}$.

Definition 2.1. A mapping $f : X \to Z$ is lower semicontinuous at x_0 if for all $V \in \mathcal{V}_Z(f(x_0))$ there exists $U \in \mathcal{U}_X(x_0)$ such that $f(U) \subset V + Z_+$. We say that f is lower semicontinuous if it is lower semicontinuous at each point of X.

If the interior of the cone Z_+ is nonempty, then we can extend this definition to the case of $f : X \to \overline{Z}$.

Definition 2.2. A function $f : X \to \overline{Z}$ is lower semicontinuous at $x_0 \in X$ with $f(x_0) \in Z \cup \{+\infty\}$ if for all $V \in \mathcal{V}_Z(f(x_0))$ there exists $U \in \mathcal{U}_X(x_0)$ such that $f(U) \subset V + Z_+$. If $f(x_0) = -\infty$, then we consider that f is lower semicontinuous at x_0.
We say that f is lower semicontinuous if it is lower semicontinuous at each point of X.

We will say that $f : X \to \overline{Z}$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ with $f(x) \in Z$.

For a proper function $f : X \to \overline{Z}$, $\varepsilon \in Z_+$, the usual Fenchel ε-subdifferential will be denoted

$$\partial^{\varepsilon}_{\leq} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid T(x - x_0) \leq f(x) - f(x_0) + \varepsilon,\ \forall x \in \operatorname{dom} f\}$$

and the Pareto ε-subdifferential will be denoted

$$\partial^{\varepsilon}_{\not\geq} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid\ T(x - x_0) \not\geq f(x) - f(x_0) + \varepsilon,\ \forall x \in \operatorname{dom} f\}$$

If the interior of the cone Z_+ is nonempty, we denote:

$$w\partial^{\varepsilon}_{\not\geq} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid\ T(x - x_0) \not\geq_{\operatorname{Int} Z_+ \cup \{0\}} f(x) - \\ -f(x_0) + \varepsilon,\ \forall x \in \operatorname{dom} f\}$$

If $Z = \mathbb{R}$, we set by $\partial_{\leq} f(x)$ the usual convex subdifferential of f at x and by $\partial^{Cl} f(x)$ the usual Clarke subdifferential of f at x. In [33] we have seen another kinds of vector subdifferentials introduced by using the normal cones to the epigraph of a lower semicontinuous function. Thus, if $N_{\operatorname{epi} f}(x_0, f(x_0))$ denotes a normal cone to the epigraph of a lower semicontinuous function f at $(x_0, f(x_0))$ with $x_0 \in \operatorname{dom} f$, $z^* \in Z^*_+ \setminus \{0\}$, we set

$$(1) \qquad \partial_{z^*} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid\ (z^* \circ T, -z^*) \in N_{\operatorname{epi} f}(x_0, f(x_0))\}$$

$$(2) \qquad \partial'_{z^*} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid\ (z^* \circ T, -1) \in N_{\operatorname{epi} z^* \circ f}(x_0, z^* \circ f(x_0))\}$$

If $Z = \mathbb{R}$, the two sets from definitions are coincident and we get, for different kinds of normal cones, the known subdifferentials for real functions; for example, if the normal cone is the polar of the cone generated by the epigraph, we find the Fenchel subdifferential, the normal cone is the polar of the Clarke tangent cone we find the Clarke subdifferential, if the normal cone is the Mordukhowich normal cone, we get the Mordukhowich subdifferential, etc. If the normal cone is the polar cone for $C_{\text{epi } f}(x, f(x))$-the cone generated by the epigraph of f at $(x, f(x))$, the sets given in (1) and (2) was denoted respectively, $\partial_{z^*} f(x)$ and $\partial'_{z^*} f(x)$; if X and Z are normed spaces and the normal cone is the polar cone for $K_{\text{epi } f}(x, f(x))$-the contingent cone (or Ursescu's tangent cone) to the epigraph of f at $(x, f(x))$, $T_{\text{epi } f}(x, f(x))$-the Clarke tangent cone to the epigraph of f at $(x, f(x))$ or it is $\hat{N}_{\text{epi } f}(x, f(x))$- the Fréchet cone to the epigraph of f at $(x, f(x))$ or $N^M_{\text{epi } f}(x, f(x))$-the Mordukhowich normal cone to the epigraph of f at $(x, f(x))$, the sets given in (1) and (2) was denoted respectively, $\partial^{co}_{z^*} f(x)$, $\partial^{Cl}_{z^*} f(x)$, $\hat{\partial}_{z^*} f(x)$, $\partial^{M}_{z^*} f(x)$, $\partial'^{co}_{z^*} f(x)$, $\partial'^{Cl}_{z^*} f(x)$, $\hat{\partial}'_{z^*} f(x)$ and $\partial'^{M}_{z^*} f(x)$.

3. Directional derivatives and convex subdifferentials. Excepting the Fréchet and Mordukhowich cones which are not polar cones, we were considered in [31] a directional derivative Df defined as

$$\text{Gr } Df(x) = TG_{\text{epi } f}(x, f(x))$$

where $TG_{\text{epi } f}(x, f(x))$ is ones of the tangent cones $C_{\text{epi } f}(x, f(x))$, $T_{\text{epi } f}(x, f(x))$, $K_{\text{epi } f}(x, f(x))$.
Using this directional derivative we can express the subdifferential like:

$$\partial_{z^*} f(x) = \{T \in \mathcal{L}(X, Z) \mid z^* \circ T(u) \leq z^*(v), \\ \forall v \in Df(x)(u),\ u \in Pr_X T_{\text{epi } f}(x, f(x))\}$$

$$\partial'_{z^*} f(x) = \{T \in \mathcal{L}(X, Z) \mid z^* \circ T(u) \leq w, \\ \forall w \in Dz^* \circ f(x)(u),\ u \in Pr_X T_{\text{epi } z^* \circ f}(x, z^* \circ f(x))\}.$$

In [1] we find the notion of directional derivative for a multifunction $F : X \rightrightarrows Z$ at $(x, y) \in \text{Gr } F$ given as $\text{Gr}\tilde{D}\, F(x, y) = TG_{\text{Gr } F}(x, y)$.
For a function $f : X \to \overline{Z}$, if we consider

$$\tilde{f}(x) = \begin{cases} f(x) + Z_+ & if \quad x \in \text{dom f} \\ Z & if \quad f(x) = -\infty \\ \emptyset & if \quad f(x) = +\infty \end{cases} \qquad (F)$$

then $D\ f(x) = \tilde{D}\ \tilde{f}(x, f(x))$.

If $Z = \mathbb{R}$, we find that $\inf Df(x)$ are the known directional derivative. If we denote $D'f(x)$, $C^{\uparrow}f(x)$, $D^{\uparrow}f(x)$ the directional derivative corresponding for the cones $C_{\text{epi } f}(x, f(x))$, $T_{\text{epi } f}(x, f(x))$, respectively $K_{\text{epi } f}(x, f(x))$, we find that

$$\inf D'f(x)(u) = f'(x, u) = \inf_{t>0} \frac{f(x+tu) - f(x)}{t}$$

the convex directional derivative at x in the direction u;

$$\inf C^{\uparrow}f(x)(u) = f^{\uparrow}(x_0, d) = \limsup_{\substack{x \to x_0 \\ f(x) \to f(x_0)}} \inf_{\substack{t \searrow 0 \\ y \to d}} \frac{f(x+ty) - f(x)}{t}.$$

the Clarke directional derivative at x in the direction u;

$$\inf D^{\uparrow}f(x)(u) = f'_D(x, u) = \liminf_{t \downarrow 0,\ y' \to u} \frac{f(x+ty') - f(x)}{t}$$

the inferior Dini derivative at x in the direction u.

In [37], M. Valadier introduced another type of directional derivative for a convex function $f : X \to Z$ where Z is an ordered vectorial space which has the property that every lower bounded decreasing net has an strong infimum.

Thus,

$$f'_V(x_0, h) \overset{def}{=} \inf\{\frac{f(x_0+th) - f(x_0)}{t}, t > 0\} \quad (DV)$$

He proved that if Z_+ is closed and Daniell (in respect to the weak topology), then

$$f'_V(x_0, h) = \lim_{n \to \infty} \Big\{ \frac{f(x_0 + t_n h) - f(x_0)}{t_n}, t_n \downarrow 0 \Big\}.$$

Zowe proved in [38] that if $\text{Int } (Z_+^*) \neq \emptyset$, then

$$f'_V(x_0, h) = \lim_{t \to 0, t>0} \frac{f(x_0+th) - f(x_0)}{t}.$$

In [32] we can find another kind of directional derivative defined by using the efficient points.

Definition 3.1. [32] Let $f : X \to \overline{Z}$ be a proper function, $x_0 \in \text{dom } f$, $h \in X$. The set given by

$$f'_\varepsilon(x_0, h) = INF_1 \left\{ \frac{f(x_0 + th) - f(x_0) + \varepsilon}{t}, t > 0 \right\}$$

will be called the ε-directional derivative of f at (x_0, h).

We denote $\tilde{f}'_\varepsilon(x_0, h) = INF \left\{ \frac{f(x_0+th)-f(x_0)+\varepsilon}{t}, t > 0 \right\}$; if the interior of the cone is nonempty, we denote $wf'_\varepsilon(x_0, h)$ and $w\tilde{f}'_\varepsilon(x_0, h)$ the sets obtained with the aid of $wINF_1$, respectively $wINF$.

Proposition 3.1. [32] *Let $f : X \to \overline{Z}$ be a convex, proper function where Z is ordered by the convex, pointed, normal, closed cone Z_+ with nonempty interior and which is Daniell. Then for all $h_1, h_2 \in X$, $\varepsilon \in Z_+$, $x_0 \in \text{dom } f$ we have*

$$\begin{cases} wf'_\varepsilon(x_0, h_1 + h_2) \subseteq wINF_1(wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2)) - Z_+ \\ wINF_1(wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2)) \subseteq wf'_\varepsilon(x_0, h_1 + h_2) + Z_+ \\ wf'_\varepsilon(x_0, \lambda h) = \lambda wf'_\varepsilon(x_0, h), \ \forall \lambda \in R_+, \ h \in X. \end{cases}$$

Corollary 3.1. *Under the previous hypothesis, for $x_0 \in \text{dom } f$,*

$$T \in w\partial^\varepsilon_{\not\to} f(x_0) \Longleftrightarrow \forall h \in X, \ Th \in wf'_\varepsilon(x_0, h) - \text{Int } Z_+ \cup \{0\}.$$

We are interested to see the link between this types of directional derivatives.

Proposition 3.2. *Let $f : X \to \overline{Z}$ be a convex, proper function where Z is ordered by the convex, pointed, normal, closed cone Z_+ with nonempty interior and which is Daniell. Then,*

$$wf'(x_0, h) = f'_V(x_0, h) + (Z_+ \backslash \text{Int } Z_+). \tag{3}$$

Proof. Let $\alpha \in wf'(x_0, h)$; it follows that for all $\varepsilon \in \text{Int } Z_+$, there exists $t_\varepsilon > 0$ such that

$$\alpha + \varepsilon > \frac{f(x_0 + t_\varepsilon h) - f(x_0)}{t_\varepsilon} \geq f'_V(x_0, h).$$

For $\varepsilon \to 0$, we get $\alpha \geq f'_V(x_0, h)$ and thus

$$wf'(x_0, h) \subseteq f'_V(x_0, h) + Z_+.$$

Suppose that for $\alpha \in wf'(x_0,h)$ there exists $\varepsilon \in \text{Int } Z_+$ such that $\alpha = \varepsilon + f'_V(x_0,h)$. Using the definition of $f'_V(x_0,h)$, we derive the existence of $t_\varepsilon > 0$ such that

$$f'_V(x_0,h) + \frac{\varepsilon}{2} > \frac{f(x_0+t_\varepsilon h)-f(x_0)}{t_\varepsilon}$$

which implies

$$\alpha - \frac{f(x_0+t_\varepsilon h)-f(x_0)}{t_\varepsilon} \in \text{Int } Z_+.$$

This relation is contradicting the fact that $\alpha \in wINF_1\left\{\frac{f(x_0+th)-f(x_0)}{t}, t>0\right\}$ and thus

$$wf'(x_0,h) \subseteq f'_V(x_0,h) + (Z_+ \backslash \text{Int } Z_+). \tag{4}$$

Conversely, for $\mu \in Z_+ \backslash \text{Int } Z_+$, we have

$$f'_V(x_0,h) + \mu \not> \frac{f(x_0+th)-f(x_0)}{t}, \forall t > 0.$$

Indeed, if we suppose the existence of $t_\mu > 0$ such that

$$f'_V(x_0,h) + \mu - \frac{f(x_0+t_\mu h)-f(x_0)}{t_\mu} \in \text{Int } Z_+,$$

since $\frac{f(x_0+t_\mu h)-f(x_0)}{t_\mu} - f'_V(x_0,h) \in Z_+$ we derive $\mu \in \text{Int } Z_+$, false.

Let observe that for $\varepsilon \in \text{Int } Z_+$ there exists $t_\varepsilon > 0$ such that

$$f'_V(x_0,h)+\mu+\varepsilon > \frac{f(x_0+t_\varepsilon h)-f(x_0)}{t_\varepsilon} + \frac{\varepsilon}{2} >_{\text{Int } Z_+ \cup \{0\}} \frac{f(x_0+t_\varepsilon h)-f(x_0)}{t_\varepsilon}$$

and finally

$$f'_V(x_0,h) + \mu \subseteq wf'(x_0,h). \tag{5}$$

From (4) and (5) we obtain (3). □

It is obvious that if $f'_V(x_0,u)$ exists, then

$$f'_V(x_0,u) \leq v, \ \forall v \in D'f(x_0)(u)$$

and for all $z^* \in Z_+^* \setminus \{0\}$ we have,

$$z^* \circ f'_V(x_0,u) \leq (z^* \circ f)'_V(x_0,u).$$

Pennamen and Eckstein proved in [24] that if $f : X \to \overline{Z}$ is a lower semicontinuous, proper, convex function, and if the limit

$$f'(x_0; u) = \lim_{\mu \to 0} \frac{f(x_0 + \mu u) - f(x_0)}{\mu}$$

exists for all $u \in X$, then

$$\text{cl epi } f'(x_0; \cdot) = \text{Gr } D'f(x_0).$$

Thus,

$$D'f(x_0)(u) \supseteq f'(x_0; u) + Z_+, \ \forall u \in X.$$

This yields,

$$\partial_{\leq} f(x_0) \subseteq \{T \in \mathcal{L}(X, Z) \mid \ Tu \leq f'(x_0; u), \ \forall u \in X\}.$$

Conversely, let $T \in \{T \in \mathcal{L}(X, Z) \mid \ Tu \leq f'(x_0; u), \ \forall u \in X\}$ and $v \in D'f(x_0)(u)$. There exist sequences (u_α), (v_α) such that $v_\alpha \geq f'(x_0, u_\alpha)$, $u_\alpha \to u$ and $v_\alpha \to v$. Since $Tu_\alpha \leq f'(x_0, u_\alpha) \leq v_\alpha$, we obtain that $Tu \leq v$, yielding $T \in \partial_{\leq} f(x_0)$.
Finally we get

$$\partial_{\leq} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid \ Tu \leq f'(x_0; u), \ \forall u \in X\}.$$

If f is subdifferentiable at x_0 or f is convex and the ordering is complete then $f'_V(x_0, u) \neq \emptyset$ for all $u \in X$.

Valadier proved in Theorem 6 [22] the following result.

Theorem 3.1. *If Z is an order complete Banach lattice ordered by a normal cone Z_+ and $f : X \to \overline{Z}$ is convex and continuous at x then,*

$$f'_V(x, u) = IMAX \ \{A(u) \mid \ A \in \partial_{\leq} f(x)\}.$$

Proposition 3.3. *Let $f : X \to \overline{Z}$ be a proper, convex, lower semicontinuous function such that Z is order complete and* dom $f - x_0$ *is absorbant. Then,*

$$f'_V(x_0, u) = \inf \ D'f(x_0)(u).$$

If, in addition, the cone Z_+ is Daniell, the limit $f'(x_0; u)$ exists, and

$$f'_V(x_0, u) = f'(x_0; u) = \inf \ D'f(x_0)(u) = IMIN \ D'f(x_0)(u). \tag{6}$$

Proof. If f is convex, the ordering is complete and dom $f - x_0$ is absorbant, then $f'_V(x_0, u)$ exists for all $u \in X$.
Let $v \in D'f(x_0)(u)$. We have that $v \geq f'_V(x_0, u)$ and thus,

$$f'_V(x_0, u) \leq \inf\ D'f(x_0)(u).$$

Let remark that for all $t \in M_u = \{t > 0 \mid x_0 + tu \in \text{dom } f\}$, $\frac{f(x_0+tu)-f(x_0)}{t} \in D'f(x_0)(u)$ and so, since the ordering is complete we have

$$f'_V(x_0, u) \geq \inf\ D'f(x_0)(u)$$

and the equality follows.
Since f is convex, then the directed family $t \to \frac{f(x_0+tu)-f(x_0)}{t} \in D'f(x_0)(u)$ is increasing and bounded from below since dom $f - x_0$ absorbs $-u$. Under the additional condition that Z_+ is Daniell the family $t \to \frac{f(x_0+tu)-f(x_0)}{t} \in D'f(x_0)(u)$ has a limit and therefore, $f'(x_0, u)$ exists and coincides with $f'_V(x_0, u)$.
Since $f'(x_0; u) \in D'f(x_0)(u)$, we obtain that $f'(x_0; u) = IMIN\ D'f(x_0)(u)$ and relation (6) does holds. □

Remark 3.1. Observe that the additional hypothesis of the previous proposition yields

$$D'f(x_0)(u) = f'(x_0; u) + Z_+.$$

Thus, $C_{\text{epi } f}(x_0, f(x_0)) = \text{epi } f'(x_0; \cdot)$ and so, the application

$$u \mapsto f'(x_0; u)$$

is a closed and sublinear function.

Another type of vector subdifferential, the V-subdifferential was introduced by Thibault in [36] and it extends the ε-subdifferentials. In addition, the V-subdifferential has the property of being Lipschitzian if V is a neighborhood of 0, property which ε-subdifferential does not enjoy if Int Z_+ is empty.

Definition 3.2. [36] Let $f : X \to \bar{Z}$ be a proper lower semicontinuous function, $a \in \text{dom } f$ and $V \subset Z$.

$$\partial_V f(a) = \{T \in \mathcal{L}(X, Z),\ T(x - a) \in f(x) - f(a) + V - Z_+, \forall x \in \text{dom } f\}.$$

This subdifferential enjoies the property of equicontinuity in the class of continuous linear operators from X to Z.

Proposition 3.4. *Let $f : X \to \overline{Z}$ be a convex map, continuous at $a \in \operatorname{dom} f$. If Z_+ is normal, then for all bounded set S containing 0 there exists $U \subset X$, a neighborhood of 0 such that $\partial_S f(a+U)$ is equicontinuous in $\mathcal{L}_p(X,Z)$.*

In particular, for $S = [-\varepsilon,\varepsilon]_{Z_+}$, we obtain the result for ε-subdifferentials given in [32].

4. The case of locally Lipschitz functions. Now, we intend to present a comparison with some types of vector subdifferentials for locally Lipschitz functions.

In [13], the authors present a characterization of the maximal, closed convex set $\Gamma(f,x_0)$ from $\mathcal{L}_p(X,Z_w)$ (with the pointwise topology and Z endowed with the weak topology) such that for all $z^* \in Z^*$, we have

$$\partial^{Cl} z^* \circ f(x_0) = z^* \circ \Gamma(f,x_0)$$

where $f : X \to Z$ is a locally Lipschitz function. They proved that if X is a separable Banach space and Z is reflexive and separable,

$$\Gamma(f,x_0) = plen\{\partial_H f(x_0)\}.$$

where $\partial_H f(x_0) = \overline{co}\{\underset{\to_H}{D} f(x_0)\}$ with $\underset{\to_H}{D} f(x_0)$ being the set of all limits of the Hadamard differential. For more details, consult [7], [26], [34].
If $f : X \to Z$, we can consider (2) for $z^* \in Z^* \setminus \{0\}$ and we get $\partial'^{Cl}_{z^*} f(x_0) = \{T \in \mathcal{L}(X,Z) \mid (z^* \circ T, -1) \in T^{\circ}_{epi\ z^* \circ f}(x, z^* \circ f(x))\}$.
Thus, by using our notions, we derive that for $z^* \in Z^* \setminus \{0\}$,

$$\begin{aligned} \partial'^{Cl}_{z^*} f(x_0) &= \Gamma(f,x_0) + \{T \in \mathcal{L}(X,Z) \mid z^* \circ T = 0\} \\ &= plen\{\partial_H f(x_0)\} + \{T \in \mathcal{L}(X,Z) \mid z^* \circ T = 0\}. \end{aligned}$$

We observe that

$$\bigcap_{z^* \in Z^* \setminus \{0\}} \partial'^{Cl}_{z^*} f(x_0) \supseteq plen\{\partial_H f(x_0)\}.$$

From this, we obtain for each $z^* \in Z^* \setminus \{0\}$ the relation:

$$z^* \circ \bigcap_{y^* \in Z^* \setminus \{0\}} \partial'^{Cl}_{y^*} f(x_0) = \partial^{Cl} z^* \circ f(x_0). \tag{7}$$

Obviously relation (7) does hold for $z^* = 0$. Since $\bigcap\limits_{z^* \in Z^* \setminus \{0\}} \partial'^{Cl}_{z^*} f(x_0)$ is a closed convex set from $\mathcal{L}_p(X, Z_w)$, we obtain using the maximality of Γ that

$$\bigcap_{z^* \in Z^* \setminus \{0\}} \partial'^{Cl}_{z^*} f(x_0) = \Gamma(f, x_0).$$

Let's denote by $\partial'^{Cl}_{+} f(x_0) = \bigcap\limits_{z^* \in Z^*_+ \setminus \{0\}} \partial'^{Cl}_{z^*} f(x_0)$.

Remark 4.1. If $Z^*_+ - Z^*_+ = Z^*$, then $z^* \circ \partial^{Cl}_{+} f(x_0) = z^* \circ \Gamma(f, x_0)$ for all $z^* \in Z^* \setminus \{0\}$ which implies that

$$\partial'^{Cl}_{+} f(x_0) = \Gamma(f, x_0).$$

In the following we give a vector result similar to the theorem concerning the equivalence between the monotonicity of the subdifferential and the convexity of a real lower semicontinuous, proper function, due to Correa-Jofré -Thibault [9].

Proposition 4.1. *Suppose that X is a Banach separable space, Z is a reflexive, separable Banach space and $f : X \to Z$ is a locally Lipschitz function. Then f is convex if and only if $\partial'^{Cl}_{+} f$ is a monotone operator.*

Proof. Since f is convex, then $\partial^{Cl} z^* \circ f(x) = \partial_{\leq} z^* \circ f(x)$ for all $x \in X$ and $z^* \in Z^*$ and thus $\partial'^{Cl}_{z^*} f(x) = \partial'_{z^*} f(x)$ for all $z^* \in Z^*_+ \setminus \{0\}$ and $x \in X$. We derive that $\partial'^{Cl}_{+} f(x) = \partial_{\leq} f(x)$ for all $x \in X$ and obviously we get the monotonicity of $\partial'^{Cl}_{+} f$.

Conversely, if $\partial'^{Cl}_{+} f$ is monotone, then from (7) we obtain that

$$z^* \circ \bigcap_{y^* \in Z^* \setminus \{0\}} \partial'^{Cl}_{y^*} f(x) = z^* \circ \bigcap_{y^* \in Z^*_+ \setminus \{0\}} \partial'^{Cl}_{y^*} f(x) = \partial^{Cl} z^* \circ f(x)$$

for all $x \in X$ and so, $\partial^{Cl} z^* \circ f$ is monotone for all $z^* \in Z^*_+ \setminus \{0\}$. From [9], we derive that $z^* \circ f$ is convex for all $z^* \in Z^*_+ \setminus \{0\}$ which implies that f is convex. □

In [19], the authors proved that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz function, f is convex if and only if the generalized Jacobian Jf is monotone.

The generalized Jacobian for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$Jf(x) = \overline{co}\{Df \in \mathbb{R}^n \times \mathbb{R}^m \mid \ Df = \lim_{x_i \to x} Df(x_i),\ x_i \in D\}$$

(D is the set where f is differentiable).
This Jacobian extend the Clarke subdifferential for the $\mathbb{R}^n$-valued functions.

It is easy to prove that in this case, for all $z^* \in Z_+^* \setminus \{0\}$,

$$z^* \circ Jf(x) = Jz^* \circ f(x) = \partial^{Cl} z^* \circ f(x)$$

and thus $Jf(x) \subseteq \partial_+^{'Cl} f(x)$.
In [19] we find an example of a convex function such that the above inclusion is strict.

In fact, $Jf(x) = \partial_H f(x)$ since for a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$\underset{\to H}{D} f(x_0) = \{ \lim_{x_i \to x} Df(x_i),\ x_i \in H\} = \{ \lim_{x_i \to x} Df(x_i),\ x_i \in D\}$$

and $\partial_+^{'Cl} f(x) = \Gamma(f, x)$ thanks to Remark 4.1.

Remark 4.1. In [19], the authors proves also the fact that $\partial_{\leq} f$ is a maximal monotone operator if $Z = \mathbb{R}^n$.

The following result extend to the vector case the known properties from the real case.

Proposition 4.2. *By the same hypothesis like in Proposition 4.1, if* Int Z_+ *is nonempty, then f is convex if and only if $\partial_+^{'Cl} f$ is an maximal monotone operator.*

Proof. Using Proposition 4.1, it rest to prove that if f is convex, then $\partial_{\leq} f$ is a maximal monotone operator. Since f is continuous and convex, using the Zowe's existence result, we derive that$\partial_{\leq} f(x) \neq \emptyset$ for all $x \in X$. We will show that if $A \notin \partial_{\leq} f(x)$, then there are $y \in X$ and $B \in \partial_{\leq} f(x)$ such that $(B - A)(y - x) \notin Z_+$. Since $A \notin \partial_{\leq} f(x)$, there exists $\overline{y}$ such that $f(\overline{y}) - f(x) \not\geq A(\overline{y} - x)$ or $(f - A)(\overline{y}) - (f - A)(x) \notin Z_+$. Using Hahn-Banach theorem we find $z^* \in Z_+^* \setminus \{0\}$ such that $z^*(f - A)(\overline{y}) - z^*(f - A)(x) < 0$. By the mean theorem, there exists $y \in]x, \overline{y}[$, $\gamma \in \partial z^* \circ (f - A)(y)$ with $\gamma(y - x) < 0$; then we get B such that $z^*(B - A)(y - x) < 0$ and finally $(B - A)(y - x) \notin Z_+$. □

Corollary 4.1. *If X, Z are Banach reflexive separable spaces and $f : X \to Z$ is a locally Lipschitz function, then f is convex if and only if $\partial_{\leq} f(x) = \partial_+^{'Cl} f(x)$ for all $x \in X$.*

In [23], is given a study of Clarke's vectorial subdifferentiability in order complete Banach lattices. Recall that a vector function $f : X \to Z$ is *locally o-Lipschitz* if and only if for every bounded open set U of X there is $k \in Z_+$ such that

$$| f(x) - f(z) | \leq k\|x - z\|, \ \ \forall x, z \in U.$$

Thus, for a locally o-Lipschitz function between two normed spaces X, Z, such that Z is an order complete Banach lattice, it was defined a directional derivative at x in the direction d as follows:

$$f^{\circ}(x, d) = \limsup_{h \to 0,\ \lambda \downarrow 0} \frac{f(x + h + \lambda d) - f(x + h)}{\lambda} \quad (DP).$$

When $Z = \mathbb{R}$, it reduces to the usual Clarke's directional derivative of a real Lipschitz function. For more details, the reader is referred to [23].
As it is easily seen, if $z^* \in Z_+^*$, then

$$(z^* \circ f)^{\circ}(x, d) \leq z^* \circ f^{\circ}(x, d).$$

This relation implies that

$$\bigcap_{z^* \in Z_+^* \setminus \{0\}} \partial'^{Cl}_{z^*} f(x) \subseteq \partial^P f(x) \tag{8}$$

where $\partial^P f(x)$ denote the Clarke vectorial subdifferential introduced by Papageorgiou and defined as the set

$$\partial^P f(x) = \{T \in \mathcal{L}(X, Z) \mid \ Tu \leq f^{\circ}(x, u), \ \forall u \in X\}.$$

We remark that in [36] the author introduced the *generalized directional derivative* $f_T^{\circ}(x, \cdot)$ for a *strictly compactly Lipschitzian* function with values in an order complete vector lattice given by:

$$f_T^{\circ}(\overline{x}, v) = \sup D_f(\overline{x}, v)$$

where

$$D_f(\overline{x}, v) = \left\{ \lim_{j \in J} \frac{f(x_j + t_j v) - f(x_j)}{t_j} \mid \ t_i > 0, \ \lim_{j \in J} t_j = 0, \ \lim_{j \in J} x_j = \overline{x} \right\}.$$

Recall that a mapping f from X into Z is said to be *strictly compactly Lipschitzian* at a point $\overline{x} \in X$ if there exist a mapping K from X into the set

Comp(Z) of nonempty compact subsets of Z, a mapping r of $(0,1)\times X\times Z$ into Z and neighborhoods U of $\overline{x}$ and V of 0 in Z verifying

(a) $\lim\limits_{t\downarrow 0,\ x\to\overline{x}} r(t,x;v)=0$ for each $v\in V$ and $\lim\limits_{\substack{x\to\overline{x}\\ t\to 0,\ v\to 0}} r(t,x;v)=0$;

(b) for all $x\in U,\ v\in V$ and $t\in(0,1)$

$$t^{-1}\mid f(x+tv)-f(x)\mid\in K(v)+r(t,x;v);$$

(c) $K(0)=\{0\}$ and the set-valued mapping K is upper semicontinuous at the origin.

Using this directional derivative the author considered the Lipschitzian subdifferential of a strictly compactly Lipschitzian map given by

$$\partial_l f(\overline{x})=\{T\in\mathcal{L}(X,Z)\mid\ Tv\le f_T^{\circ}(\overline{x},v),\ \forall v\in X\}.$$

Let's observe that if f is strictly compactly Lipschitzian then

$$f_T^{\circ}(\overline{x},v)\le f^{\circ}(\overline{x},v)$$

and

$$\partial_l f(\overline{x})\subseteq\partial^P f(\overline{x}).$$

In the same paper, the author define the directional subderivative of f at $\overline{x}$ for a mapping f with values in a complete vector lattice by:

$$f^{\uparrow}(\overline{x},v)=\inf\{l\in Z\mid\ (v,l)\in T_{\text{epi f}}(\overline{x},f(\bar{x}))\}\quad(DT).$$

In our context, $f^{\uparrow}(\overline{x},v)=\inf\ C^{\uparrow}f(\overline{x})(v)$.

The subdifferential defined with the aid of this directional subderivative is

$$\partial^T f(\overline{x})=\{T\in\mathcal{L}(X,Z)\mid\ Tv\le f^{\uparrow}(\overline{x},v),\ \forall v\in X\}$$

and we have

$$\bigcap_{z^*\in Z_+^*\setminus\{0\}}\partial_{z^*}^{Cl}f(\overline{x})=\partial^T f(\overline{x}).$$

Using the coincidence between $f^{\uparrow}(\overline{x},v)$ and $f_T^{\circ}(\overline{x},v)$ for strictly compactly lipschitzian maps given in [36] we derive that if f is a mapping which is strictly compactly Lipschitzian at $\overline{x}$, $f:X\to\overline{Z}$ where Z is an order complete lattice, then

$$\partial_l f(\overline{x})=\partial^T f(\overline{x})=\bigcap_{z^*\in Z_+^*\setminus\{0\}}\partial_{z^*}^{Cl}f(\overline{x}).$$

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