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SOME CLASSES OF BOUNDARY POINTS

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Abstract. In this note we will define a number of concepts allowing us to describe the properties of a domain at a boundary point. We will show that the choice of a continuum is of insignificant consideration in the definition of property P_2^* (this property P_2^* is different by property P_2 considerate in [4], Definition 17.5(4)) and we will generalize the concept of locally quasiconformally collared domain at a boundary point.

Keywords: locally connected, finitely connected, property P_1, property P_2^*, locally quasiconformally collared.

1. Introduction. All the sets considered in this note are assumed to lie in $\overline{R}^n = R^n \cup \{\infty\}, n \geq 2$, the one-point compactification of the euclidian n -space R^n.

Given a point $x \in R^n$ and a number $r > 0$, we let $B^n(x,r)$ denote the n- dimensional ball $\{y \in R^n\ , |y-x| < r\}$ and $S^{n-1}(x,r)$ its $(n-1)$-dimensional boundary sphere $\{y \in R^n, |y-x| = r\}$. We will also employ the notations $B^n(r) = B^n(0,r), S^{n-1}(r) = S^{n-1}(0,r)$, where 0 is the origin. In this note, for each point $x \in \overline{R}^n$, a neighborhood of x is an open set containing x.

A path in $\overline{R}^n$ is a continuous mapping $\alpha : I \to \overline{R}^n$, where I is an interval in R^1. The path is said to be closed or open according as I is closed or open. We denote $|\alpha| = \alpha(I)$.

If Γ is a family of paths in $\overline{R}^n$ we denote by $F(\Gamma)$ the family of all Borel functions $\rho : \overline{R}^n \to [0,\infty]$ for which $\int_\gamma \rho ds \geq 1$, for every rectifiable path $\gamma \in \Gamma$.

The modulus of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^n dm.$$

If $E, F, G \subset \overline{R}^n$, $E \subset \overline{G}$, $F \subset \overline{G}$, we denote by $\Delta(E, F; G)$ the family of all paths which join E and F in G; that is, a path $\gamma : [a, b] \to \overline{R}^n$ is an element of $\Delta(E, F; G)$ if and only if one of its end-points $\gamma(a), \gamma(b)$ belongs to E, the other belongs to F and $\gamma(t) \in G$ for $a < t < b$.

Let D, D' be two domains in $\overline{R}^n$.A homeomorphism $f : D \to D'$ is said to be K-quasiconformal, $1 \leq K < \infty$, if it satisfies the double inequality:

$$\frac{1}{K} M(\Gamma) \leq M(f(\Gamma)) \leq K M(\Gamma)$$

for each path family Γ in D. Here $f(\Gamma) = \{f \circ \gamma, \gamma \in \Gamma\}$. A homeomorphism f is said to be quasiconformal, if it is K-quasiconformal for some K. The maximal dilatation of f, denoted by $K(f)$, is then defined as the least K for which f is K -quasiconformal.

2. Classes of boundary points. Let D be a domain in $\overline{R}^n$ and let $b \in \partial D$.

Definition 1. D is locally connected at b if b has arbitrarily small neighborhoods U such that $U \cap D$ is connected.

We remember that property of $b \in \partial D$ in Definition 1, means that "for every neighborhood V of b, there exists a neighborhood U of b, $U \subset V$, such that $U \cap D$ has specified property".

Definition 2. D is finitely connected at b if b has arbitrarily small neighborhoods U such that $U \cap D$ has a finite number of components.

Definition 3. D is m-connected at b ,$m = 1, 2, ...$, if m is the least integer for which there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components.

Definition 4. D has property P_1 at b if the following condition is satisfied:

$M(\Delta(E, F; D)) = \infty, (\forall)E, F \subset D, E, F$ connected sets, $b \in \overline{E} \cap \overline{F}.$

Definition 5. D has property P_2^* at b if the following condition is satisfied: $(\forall)b_1 \in \partial D, b_1 \neq b, (\exists)F \subset D, F$ continuum and $\delta > 0$ such that $M(\Delta(E, F; D)) \geq \delta, (\forall)E \subset D, E$ connected and $b, b_1 \in \overline{E}$.

Definition 6. D is locally quasiconformally collared at b if there is a neighborhood U of b and a homeomorphism $g : U \cap \overline{D} \to B^n_+ \cup B^{n-1}$ such that $g/_{U\cap D}$ is quasiconformal, where $B^n_+ = \{x \in R^n, |x| < 1, x_n > 0\}$ and $B^{n-1} = \{x \in R^n, |x| < 1, x_n = 0\}$.

Definition 7. D is locally quasiconformally m-collared at b , $m = 1, 2, ...$, if there is a neighborhood U of b such that $U \cap D$ consists of m components $E_1, ..., E_m$ and, for each $i \in \{1, .., m\}$, there is a homeomorphism $g_i : U \cap \overline{E}_i \to B^n_+ \cup B^{n-1}$ such that the restriction $g_i/_{E_i}$ is quasiconformal.

Remark 1. We observe that every E_i in Definition 7 is locally quasiconformally collared at b .

Remark 2. Composing g_i with an auxiliary Möbius transformation, in Definition 7, we can choose g_i such that $g_i(b) = 0$.

Proposition 1. *D is locally quasiconformally m-collared at $b \in \partial D$, $m = 1, 2...$, if and only if there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components, $E_1, ..., E_m$, and for each $i \in \{1, ..., m\}$ there is a homeomorphism $g_i : U \cap \overline{E}_i \to B^n_+ \cup B^{n-1}$ such that the restriction $g_i/_{E_i}$ is quasiconformal.*

Proof. Let D be a locally quasiconformally m-collared domain at b. By Definition 7 and Remark 2 , there is a neighborhood U of b such that $U \cap D$ consists of m components $E_1, ..., E_m$ and for each $i \in \{1, .., m\}$, there is a homeomorphism $g_i : U \cap \overline{E}_i \to B^n_+ \cup B^{n-1}$, such that the restriction $g_i/_{E_i}$ is quasiconformal and $g_i(b) = 0$. Let V be a neighborhood of b. We can find $0 < r < 1$ such that $g_i^{-1}(B^n_+(r) \cup B^{n-1}(r)) \subset V \cap U \cap \overline{E}_i$ and $g_i^{-1}(B^n_+(r)) \subset V \cap U \cap E_i$ for each $i \in \{1, ..., m\}$, where $B^n_+(r) = \{x \in R^n, |x| < r, x_n > 0\}$ and $B^{n-1}(r) = \{x \in R^n, |x| < r, x_n = 0\}$. Then

$$W = (V - \overline{D}) \cup (\bigcup_{i=1}^{m} g_i^{-1}(B^n_+(r) \cup B^{n-1}(r)))$$

is a neighborhood of b. Moreover, $W \subset V$ and $W \cap D$ consists of the components $g_1^{-1}(B^n_+(r)), ..., g_m^{-1}(B^n_+(r))$. Setting $h(y) = \frac{y}{r}$, we obtain the homeomorphisms $h_i = h \circ g_i : W \cap \overline{g_i^{-1}(B^n_+(r))} \to B^n_+ \cup B^{n-1}$such that h_i is quasiconformal in $g_i^{-1}(B^n_+(r)$.

Remark 3. m-connectedness always implies finite connectedness.

Theorem 1. *D has property P_2^* at a boundary point b, if and only if, given any boundary point b_1 of D, $b_1 \neq b$ and any continuum F^* in D containing at least two points, there exists $\delta^* > 0$ such that $M(\Delta(E, F^*; D)) \geq \delta^*$, whenever E is a connected subset of D, with $b, b_1 \in \overline{E}$.*

Proof. It is evidently sufficient to prove the necessity part. Let b_1 be a boundary point of D, $b_1 \neq b$ and let F^* be an arbitrary continuum in D. Since D has property P_2^* at b, there exists a continuum F in D and a positive number δ such that $M(\Delta(E, F; D)) \geq \delta, (\forall)E \subset D, E$ connected and $b, b_1 \in \overline{E}$. We have two cases:

Case I. $F \cap F^* = \Phi$. We will assume, that $\infty \notin F$.

Let $4r = \min\{d(F, F^*), d(F, \partial D)\}$. If $\infty \in F$ then we consider the corresponding spherical distance q. ([4] Definition 12.1). Because F is a continuum and $\infty \notin F$, there exists a finite covering of F by closed balls $A_1, ..., A_p$ with centers $a_i \in F$, $(i = 1, ..., p)$ and radius $r > 0$. Let

$$\Gamma_i^* = \Delta(F^*, A_i; D), M(\Gamma_i^*) = \delta_i. \tag{1}$$

Since F^* and A_i are two continua we have $\delta_i > 0$. Now let E be a connected set in D such that $b, b_1 \in \overline{E}$ and set $\Gamma = \Delta(E, F; D), \Gamma_i = \Delta(A_i, E; D), \Gamma^* = \Delta(F^*, E; D)$. Since the modulus is a monotone and subadditive function, we have:

$$\delta \leq M(\Gamma) \leq \sum_{i=1}^{p} M(\Gamma_i).$$

Thus, there exists $i \in \{1, ..., p\}$ such that $M(\Gamma_i) \geq \frac{\delta}{p}$, and say

$$M(\Gamma_1) \geq \frac{\delta}{p}. \tag{2}$$

Let

$$\delta^* = 3^{-n} \min\{\frac{\delta}{p}, \delta_1, ..., \delta_p, c_n \log 2\}, \tag{3}$$

where $c_n > 0$ is as defined in [4] (theorems 10.9). We will show that:

$$M(\Gamma^*) \geq \delta^*. \tag{4}$$

It is sufficient to consider the case $F^* \cap E = \Phi$, otherwise $M(\Gamma^*) = \infty$ and obviously (4) is satisfied. Choose $\rho \in F(\Gamma^*)$. If

$$\int_{\gamma_1} \rho ds \geq \frac{1}{3}, or, \int_{\gamma_1^*} \rho ds \geq \frac{1}{3} \tag{5}$$

for every rectifiable path $\gamma_1 \in \Gamma_1, \gamma_1^* \in \Gamma_1^*$, then $3\rho \in F(\Gamma_1)$ or $3\rho \in F(\Gamma_1^*)$. This implies

$$\int_{R^n} \rho^n dm \geq 3^{-n} \min\{M(\Gamma_1), M(\Gamma_1^*)\} \tag{6}$$

If (5) does not hold, then, there exist rectifiable paths $\gamma_1 \in \Gamma_1$ and $\gamma_1^* \in \Gamma_1^*$ such that

$$\int_{\gamma_1} \rho ds < \frac{1}{3} \tag{7}$$

and

$$\int_{\gamma_1^*} \rho ds < \frac{1}{3}$$

And now we have two cases:

a) If $E \cap B^n(a_1, 2r) = \Phi$, let R_1 be the ring $r < |x - a_1| < 2r$ and, set $\Delta_1 = \Delta(|\gamma_1|, |\gamma_1^*|; R_1)$. For every rectifiable path $\alpha_1 \in \Delta_1$ set $|\widetilde{\gamma}_1| = |\gamma_1| \cup |\alpha_1| \cup |\gamma_1^*|$. Thus we have

$$1 \leq \int_{\widetilde{\gamma}_1} \rho ds \leq \int_{\gamma_1} \rho ds + \int_{\alpha_1} \rho ds + \int_{\gamma_1^*} \rho ds < \frac{2}{3} + \int_{\alpha_1} \rho ds$$

and hence $\int_{\alpha_1} \rho ds > \frac{1}{3}$.

Thus, $3\rho \in F(\Delta_1)$. Since every $S^{n-1}(a_1, t)$ meets both $|\gamma_1^*|$ and $|\gamma_1|$ for $r < t < 2r$ and since $B^n(a_1, 2r) \subset D$ according to [4] (theorem10.12), we obtain

$$\int_{R^n} \rho^n dm \geq 3^{-n} c_n \log 2. \tag{8}$$

b) If $E \cap B^n(a_1, 2r) \neq \Phi$, let R_1^* be the ring $2r < |a_1 - x| < 4r$ and set $\Delta_1^* = \Delta(|\gamma_1^*|, E; R_1^*)$. For every rectifiable path $\alpha_1^* \in \Delta_1^*$ set
$|\widetilde{\gamma}_1^*| = |\gamma_1^*| \cup |\alpha_1^*|$. We have:

$$1 \leq \int_{\widetilde{\gamma}_1^*} \rho ds \leq \int_{\gamma_1^*} \rho ds + \int_{\alpha_1^*} \rho ds < \frac{1}{3} + \int_{\alpha_1^*} \rho ds$$

and hence $\int_{\alpha_1^*} \rho ds > \frac{2}{3}$.

Thus, $3\rho \in F(\Delta_1^*)$. Since every $S^{n-1}(a_1, t)$ meets both $|\gamma_1^*|$ and E for $2r < t < 4r$ and since $B^n(a_1, 4r) \subset D$ we obtain (8).

Since $\rho \in F(\Gamma^*)$ was arbitrary and since either (6) or (8) is true, we obtain $M(\Gamma^*) \geq \delta^*$ by combining (1),(2),(3).

Case II. $F \cap F^* \neq \Phi$. We may choose a continuum $F_1^* \subset D - (F \cup F^*)$ such that $\infty \notin F_1^*$, and hence $F \cap F_1^* = \Phi$. We are in case I, with the continua F, F_1^*. Let

$$4r_1 = \min\{d(F, F_1^*), d(F, \partial D)\}$$

Because F is a continuum and $\infty \notin F$, there exists a finite covering of F by closed balls $A'_1, ..., A'_{p_1}$ with centers $a'_i \in F$ $(i = 1, ..., p_1)$ and radius r_1. Applying the procedure from the case I, we find $\delta_1^* = 3^{-n} \min\{\frac{\delta}{p_1}, \delta'_1, ..., \delta'_{p_1}, c_n \log 2\}$, where $\delta'_i = M(\Delta(F_1^*, A'_i; D)), i \in \{1, ..., p_1\}$, such that

$$M(\Delta(E, F_1^*; D)) \geq \delta_1^*,$$

whenever E is a connected subset of D with $b, b_1 \in \overline{E}$.

On the other hand, $F_1^* \cap F^* = \Phi$, and hence we are in the case I, with the continua F_1^*, F^*. Let

$$4r_2 = \min\{d(F_1^*, F^*), d(F_1^*, \partial D)\}.$$

Because F_1^* is a continuum and $\infty \notin F_1^*$, there exists a finite covering of F_1^* by closed balls $B_1, ..., B_{p_2}$ with centers $b_i \in F_1^*$ $(i = 1, ..., p_2)$ and radius r_2. Applying the procedure from the case I, we find $\delta_2^* =$

$3^{-n}\min\{\frac{\delta_1^*}{p_2}, \delta_1^{"}, ..., \delta_{p_2}^{"}, c_n \log 2\}$ where $\delta_i^{"} = M(\Delta(F^*, B_i; D)), i \in \{1, ..., p_2\}$, such that

$$M(\Delta(E, F^*; D)) \geq \delta_2^*,$$

whenever E is a connected subset of D with $b, b_1 \in \overline{E}$.

Theorem 2. *Let D be a domain which is locally quasiconformally m-collared at a boundary point b .Then:*

1) D is m-connected at b ;

2) D has property P_1 at b if and only if $m = 1$;

3) D has property P_2^* at b .

Proof. We establish (1). By Proposition 1, there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components, $E_1, ..., E_m$ and for each $i \in \{1, ..., m\}$, there is a homeomorphism $g_i : U \cap \overline{E}_i \to B_+^n \cup B^{n-1}$ such that the restriction $g_i/_{E_i}$ is quasiconformal. Obviously, E_i is locally quasiconformally collared at b, and by Theorem 17.10 [4], E_i is locally connected at b. Hence, there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists m components, $E_1, ..., E_m$, which are locally connected at b. Applying Theorem 1.10 [3], it follows that D is m connected in b.

We establish (2). If $m = 1$ then D is locally quasiconformally collared at b and according to [4] (Theorem 17.10.) , we obtain that D has property P_1 at b. Suppose that D has property P_1 at b. Using (1) we deduce that D is m-connected at b and hence D is finitely connected at b. By theorem 1.18.[3], D is locally connected at b, and hence $m = 1$.

We establish (3). Let b_1 be a boundary point of D, $b_1 \neq b$, and let U be a neighborhood of b such that $b_1 \in \partial U$. Since D is locally quasiconformally m-collared at b, by Proposition 1, we can find a neighborhood $V \subset U$ of b, such that $V \cap D$ consists of m components $E_1, ..., E_m$ and, for each of which, there is a homeomorphism $g_i : V \cap \overline{E_i} \to B_+^n \cup B^{n-1}$ with $g_i(b) = 0$ and $g_i/_{E_i}$ quasiconformal. Because V is a neighborhood of b, there is $0 < r < 1$ such that $g_i^{-1}(B^n(r) \cap B_+^n) \subset E_i \cap V = E_i$. Denote by $I' = \{x \in R^n, x_1 = ... = x_{n-1} = 0, \frac{r}{2} \leq x_n \leq r\}$ and by $I = \overset{m}{\underset{i=1}{\cup}} g_i^{-1}(I')$.

Next, let $F \subset D$ be a continuum which contains I. We will show that the condition from Theorem1 is satisfied by this F and by $\delta = (b_n \log 2)/K$, where b_n is a positive constant depending only on n (see [4] Theorem10.2) and $K = \max_{1\leq i\leq m} K(g_i)$.

Let E be a connected set in D with $b, b_1 \in \overline{E}$. Since $g_i(b) = 0$ and $b \in \overline{E}$ it follows that $0 \in \overline{g_i(E_i \cap E)}$ for at least one i. Fix such i and set
$\Gamma' = \Delta(g_i(E_i \cap E), g_i(E_i \cap F); B^n_+)$.
If $g_i(E_i \cap E) \cap g_i(E_i \cap F) \neq \Phi$ then $M(\Gamma') = \infty$ and hence $M(\Delta(E, F; D)) = \infty$. Thus, $M(\Delta(E, F; D)) > \delta$.

Otherwise choose $\rho \in F(\Gamma')$. From the choice of r and F and from the connectedness of E , we deduce that every hemisphere $S_+(t) = S^{n-1}(t) \cap B^n_+$ meets both $g_i(E_i \cap E)$ and $g_i(E_i \cap F)$ for $\frac{r}{2} < t < r$.

By Fubini's theorem and by Theorem 10.2[4] we have

$$\int_{R^n} \rho^n dm \geq \int_{\frac{r}{2}}^{r} dt \int_{S_+(t)} \rho^n dm_{n-1} \geq \int_{\frac{r}{2}}^{r} \frac{b_n}{t} dt = b_n \log 2$$

where $b_n = 2^{-n-1}\omega_{n-2}(\int_0^\infty t^{-\frac{n-2}{n-1}}(1+t^2)^{-\frac{1}{n-1}}dt)^{1-n}, b_2 = \frac{1}{2\pi}$, ω_{n-2} is $(n-2)$-dimensional Lebesgue measure of sphere S^{n-2}, and hence $M(\Gamma') \geq b_n \log 2$.

Finally, the monotoneity of the modulus and the K -quasiconformality of g_i imply

$$M(\Delta(E, F; D)) \geq M(g_i^{-1}(\Gamma')) \geq \frac{1}{K} M(\Gamma') \geq \frac{1}{K} b_n \log 2.$$

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