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ON BOUNDARY EXTENSION THEOREMS FOR QUASICONFORMAL MAPPINGS IN $\overline{R}^n$

BY

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Abstract. In this article we generalize the theorem 17.15 [5], and give some boundary extension theorems for quasiconformal mappings.

Keywords: quasiconformal mapping, locally connected, m-connected, property P_2^*, locally quasiconformally m -collared.

All the sets considered in this note are assumed to lie in $\overline{R}^n = R^n \cup \{\infty\}$, $n \geq 2$, the one-point compactification of the euclidian n -space R^n. Given a point $x \in R^n$and a number $r > 0$, we let $B^n(x,r)$ denote the n- dimensional ball $\{y \in R^n \ , |y-x| < r\}$ and $S^{n-1}(x,r)$ its $(n-1)$-dimensional boundary sphere $\{y \in R^n, |y-x| = r\}$. We will also employ the notations $B^n(r) = B^n(0,r), S^{n-1}(r) = S^{n-1}(0,r)$, where 0 denotes the origin. In this note for each point $x \in \overline{R}^n$, a neighborhood of x is an open set containing x.

A path in $\overline{R}^n$ is a continuous mapping $\alpha : I \to \overline{R}^n$, where I is an interval in R^1. The path is said to be closed or open according to whether I is closed or open. We denote $|\alpha| = \alpha(I)$.

If Γ is a family of paths in $\overline{R}^n$ we denote by $F(\Gamma)$ the family of all Borel functions $\rho : \overline{R}^n \to [0,\infty]$ for which $\int_\gamma \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma$.

The modulus of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^n dm.$$

If $E, F, G \subset \overline{R}^n$,$E \subset \overline{G}$, $F \subset \overline{G}$, we denote by $\Delta(E, F; G)$ the family of all paths which join E and F in G; where a path $\gamma : [a, b] \to \overline{R}^n$ is an element of $\Delta(E, F; G)$ if and only if one of its end-points $\gamma(a), \gamma(b)$ belongs to E, the other to F and $\gamma(t) \in G$ for $a < t < b$.

Let D, D' be two domains in $\overline{R}^n$.A homeomorphism $f : D \to D'$ is said to be K-quasiconformal, $1 \leq K < \infty$, if it satisfies the double inequality:

$$\frac{1}{K} M(\Gamma) \leq M(f(\Gamma)) \leq K M(\Gamma)$$

for each path family Γ in D. Here $f(\Gamma) = \{f \circ \gamma, \gamma \in \Gamma\}$. A homeomorphism f is said to be quasiconformal, if it is K-quasiconformal for some K. The maximal dilatation of f, denoted by $K(f)$, is then defined as the least K for which f is K -quasiconformal.

Let D be a domain in $\overline{R}^n$ and $b \in \partial D$.

Definition 1. D is locally connected at b if b has arbitrarily small neighborhoods U such that $U \cap D$ is connected.

We remember that property of $b \in \partial D$ in Definition 1, means that "for every neighborhood V of b, there exists a nieghborhood U of b, $U \subset V$, such that $U \cap D$ has specified property".

Definition 2. D is m-connected at b $(m = 1, 2, ..)$ if m is the least integer for which there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components.

Definition 3. D has property P_2^* at b if the following condition is satisfied: $(\forall) b_1 \in \partial D, b_1 \neq b, (\exists) F \subset D, F$ continuum and $\delta > 0$ such that $M(\Delta(E, F; D)) \geq \delta, (\forall) E \subset D, E$ connected and $b, b_1 \in \overline{E}$.

This property P_2^* is different by property P_2 considerate in [5], Definition 17.5. (4).

Definition 4. D is locally quasiconformally collared at b if there is a neighborhood U of b and a homeomorphism $g : U \cap \overline{D} \to B_+^n \cup B^{n-1}$ such that $g/_{U \cap D}$ is quasiconformal, where $B_+^n = \{x \in R^n, |x| < 1, x_n > 0\}$ and $B^{n-1} = \{x \in R^n, |x| < 1, x_n = 0\}$

Definition 5. D is locally quasiconformally m-collared at b, $m = 1, 2, ...$, if there is a neighborhood U of b such that $U \cap D$ consists of m components $E_1, ..., E_m$ and for each $i \in \{1, .., m\}$, there is a homeomorphism $g_i : U \cap \overline{E}_i \to B_+^n \cup B^{n-1}$ such that the restriction $g_i/_{E_i}$ is quasiconformal.

In this note $D, D^{'}$ are supposed to be domains in $\overline{R}^n$.

Let f be a mapping of D into $\overline{R}^n$ and let b be a point in ∂D. The cluster set $C(f,b)$ of f at b is the set of all points $b' \in \overline{R}^n$ for which there exists a sequence (b_k) in D such that $b_k \to b$ and $f(b_k) \to b'$. It is clear that, $C(f,b) = \cap \overline{f(U \cap D)}$ where U runs through all neighborhoods of b. Obviously, $C(f,b)$ is a non-empty compact set, there exists the limit of f at b if and only if $C(f,b)$ reduces to a single point, and $C(f,b) \subset \partial f(D)$ if f is a homeomorphism.

Remark 1. Suppose that $f : D \to D^{'}$ is a homeomorphism, D is m-connected at $b \in \partial D$ and U is a neighborhood of b appearing in Definition 2, $E_1, ..., E_m$ being components of $U \cap D$. Then $\bigcup_{i=1}^{m} C(f/_{E_i}, b) = C(f,b)$.

Theorem 1. *Suppose that $f : D \to D^{'}$ is a quasiconformal mapping and that D is locally connected at $b \in \partial D$. If $D^{'}$ has property P_2 at some point of $C(f,b)$ then there exists the limit of f at b .([5] Theorem 17.15).*

Remark 2. Theorem 1 is true if instead of property P_2 one considers property P_2^*.

Corollar 1. *Suppose that $f : D \to D'$ is a quasiconformal mapping and that D is locally connected at $b \in \partial D$. If D' is locally quasiconformally m -collared at least at a point of $C(f,b)$, then there exists the limit of f at b.*

Proof. By Theorem 2 [1], $D^{'}$ has property P_2^* at least at a point of $C(f,b)$. Using Remark 2, it follows that, there exists the limit of f at b.

Theorem 2. *Suppose that $f : D \to D'$ is a quasiconformal mapping and that D is m -connected at $b \in \partial D$. Then, $C(f,b)$ either contains at most $m-1$ points at which D' has property P_2^*, or consists of m points.*

Proof. Suppose, contrary to the assertion, that $C(f,b)$ contains at least $m+1$ distinct points $b'_1, ..., b'_{m+1}$ and that D' has property P_2^* at $b'_1, ..., b'_m$.

Suppose that both b and each $b'_j, j \in \{1, ..., m+1\}$, are finite points and we use the euclidian distance. If some of them are infinite, we use the spherical (chordal) distance. (see [5], p. 37)

Let A' be a continuum in D' containing at least to points. Since D' has property P_2^* at b'_i for each $i \in \{1, ..., m\}$ it follows, by Theorem 1.[1], that there is $\delta_i > 0$ such that

$$(1) \qquad M(\Delta(F', A'; D')) \geq \delta_i$$

whenever F' is a connected set in D' with $b'_i, b'_j \in \overline{F}'$, $j \neq i, j \in \{1, ..., m+1\}$.

Let $\delta = \min\limits_{1 \leq i \leq m} \delta_i$, $d = d(A, \partial D)$, where $A = f^{-1}(A')$. Then δ and d are both strictly positive.

Since $b'_j \in C(f, b)$, for each $j \in \{1, ..., m+1\}$, choose a sequence (b_{jk}) in D such that $b_{jk} \to b$ and $f(b_{jk}) \to b'_j$.

Fix $\varepsilon, 0 < \varepsilon < d$. Because D is m -connected at b, there exists, by Theorem 1.10.(5)[4], a component F of $B^n(b, \varepsilon) \cap D$ and integers i and j, $1 \leq i < j \leq m+1$, such that F contains subsequences of (b_{ik}) and (b_{jk}). Let $F' = f(F), \Gamma = \Delta(A, F; D), \Gamma' = \Delta(A', F'; D')$. Since F' is connected and $b'_i, b'_j \in \overline{F}'$ it follows, by (1), that

$$(2) \qquad M(\Gamma') \geq \delta.$$

On the other hand, the path family Γ is minorized by the family $\Delta(S^{n-1}(b, \varepsilon), S^{n-1}(b, d); D)$ and by 7.5[5], $M(\Delta(S^{n-1}(b, \varepsilon), S^{n-1}(b, d); D)) = \omega_{n-1}(\log \frac{d}{\varepsilon})^{1-n}$ it follows that

$$(3) \qquad M(\Gamma) \leq \omega_{n-1}(\log \frac{d}{\varepsilon})^{1-n}$$

where ω_{n-1} is $(n-1)$ -dimensional Lebesgue measure of sphere S^{n-1}.

Since the relation (3) holds for all $\varepsilon, 0 < \varepsilon < d$, letting $\varepsilon \to 0$ and using the relation (2), we obtain a contradiction.

Theorem 3. *Let $f : D \to D'$ be a quasiconformal mapping and let D be m-connected at $b \in \partial D$. If D' has property P_2^* at each point of $C(f, b)$ and $C(f, b)$ is a connected set, then there exists the limit of f* at b.

Proof. We give two methods of proof.

Method I. Since D is m-connected at b, by Theorem 1.10(3)[4], there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components $E_1, ..., E_m$ and each E_i is locally connected at b. We show that $C(f/_{E_i}, b)$ has a single point. Suppose that there exist two distinct points $b'_i, b' \in C(f/_{E_i}, b) \subset C(f, b)$ and hence D' has property P_2^* at b'_i and b'. We can assume that $b_i^{'}, b^{'}, b$ are finite points. Let $A^{'}$ be a continuum in $f(E_i)$ containing at least two points. Since $D^{'}$ has property P_2^* at $b_i^{'}$, by Theorem 1[1], there exists $\delta > 0$ such that

(1) $M(\Delta(F^{'}, A^{'}; D^{'})) \geq \delta$ whenever $F^{'}$ is a connected subset of $D^{'}$, $b^{'}, b_i^{'} \in \overline{F}^{'}$.

Let $A = f^{-1}(A^{'}), d = d(A, \partial D)$ and hence $d > 0$. Since $b_i^{'}, b^{'} \in C(f/_{E_i}, b)$ choose two sequences $(b_n), (b_{in})$ in E_i such that $b_n \to b, b_{in} \to b$ and $f(b_n) \to b^{'}, f(b_{in}) \to b_i^{'}$. Fix $\varepsilon, 0 < \varepsilon < d$ and since E_i is 1-connected at b by Theorem 1.10(5)[4], there exists a component F of $B^n(b, \varepsilon) \cap E_i$ which contains two subsequences of (b_n) and (b_{in}). The set $F^{'} = f(F)$ is connected and $b^{'}, b_i^{'} \in \overline{F}^{'}$. By (1) we have:

(2) $M(\Delta(F^{'}, A^{'}; D^{'})) \geq \delta$.

On the other hand, the path family $\Delta(A, F; D)$ is minorized by the family $\Delta(S^{n-1}(b, \varepsilon), S^{n-1}(b, d); D)$, and hence

(3) $M(\Delta(A, F; D)) \leq \omega_{n-1}(\log \frac{d}{\varepsilon})^{1-n}$.

Letting $\varepsilon \to 0$ and using (2), we contradict the fact that f is a quasiconformal mapping. Therefore, $C(f/_{E_i}, b) = \{b_i^{'}\}$.

Consequently, $C(f, b) = \{b_1^{'}, ..., b_m^{'}\}$ and since $C(f, b)$ is a connected set, we obtain $b_1^{'} = ... = b_m^{'} = b^{'}$. Hence, $b' = \lim_{x \to b} f(x)$, and the proof is complete.

Method II. If $m = 1$, then D is 1-connected at b and by Theorem 2, $C(f, b)$ either contains none point at which $D^{'}$ has property P_2^*, or $C(f, b)$ has a single point. Since $D^{'}$ has property P_2^* at each point of $C(f, b)$, it follows that $C(f, b)$ has a single point.

Suppose that $m \geq 2$. By Theorem 2, $C(f, b)$ either contains at most $m - 1$ points at which $D^{'}$ has property P_2^*, or consists of m points. Since $C(f, b)$ is connected, $C(f, b)$ has at most $m - 1$ points at which $D^{'}$ has property P_2^*. If $C(f, b)$ does not have a single point, since $C(f, b)$ is a connected set, there exist even an infinitude of points in $C(f, b)$ and in these points $D^{'}$ has property P_2^*. This contradicts the fact $C(f, b)$ has at most $m - 1$ points at which $D^{'}$ has property P_2^*.

Theorem 4. *Suppose that $f : D \to D'$ is a quasiconformal mapping and D is m-connected at $b \in \partial D$ .If U is a neighborhood of b whith $U \cap D$ consisting of components $E_1, ..., E_m$ and $b^{'}$ is a point belonging to $C(f/_{E_i}, b)$ for $i = 1, ..., m$ and if $D^{'}$ has property P_2^* at $b^{'}$ then $b' = \lim_{x \to b} f(x)$.*

Proof. By Theorem 1.10[4], we can assume that each E_i is locally connected at b. Using similar arguments mentioned at Method I, it follows that $b^{'} = \lim_{x\to b} f/_{E_i}(x)$ and hence $b' = \lim_{x\to b} f(x)$.

Theorem 5. *Suppose that $f : D \to D'$ is a quasiconformal mapping, $b \in \partial D, b^{'} \in C(f,b)$, and D is locally quasiconformally m-collared at each point of $C(f^{-1},b^{'})$. Then f can be extended to a homeomorphism $f^* : D\cup\{b\} \to D'\cup\{b^{'}\}$ if and only if $D^{'}$ is locally quasiconformally m-collared at each point of $C(f,b)$ and $C(f,b), C(f^{-1},b^{'})$ are connected sets .*

Proof. We prove the necessity. Suppose that f can be extended to a homeomorphism $f^* : D \cup \{b\} \to D' \cup \{b^{'}\}$ and hence$f^*(b) = b^{'}$. Obviously $C(f,b) = \{b^{'}\}, C(f^{-1},b^{'}) = \{b\}$ are connected sets. Since D is locally quasiconformally m-collared at b, there exists a neighborhood U of b such that $U\cap D$ consists of the components $E_1,...,E_m$ and for each $i \in \{1,...,m\}$, there exists a homeomorphism $g_i : U \cap \overline{E}_i \to B^n_+ \cup B^{n-1}$ such that the restriction $g_i/_{E_i}$ is quasiconformal. Choose a neighborhood $V^{'}$ of $b^{'}$ such that $(f^*)^{-1}(V^{'}\cap D^{'}) \subset U\cap D$ and for each $i \in \{1,...,m\}$, we set $E^{'}_i = f^*(E_i)$. The set $U^{'} = V^{'} \cup E^{'}_1 \cup ... \cup E^{'}_m$ is a neighborhood of $b^{'}$. On the other hand, $V^{'} \cap D^{'} \subset f^*(U \cap D) = f^*(E_1 \cup ... \cup E_m) = E^{'}_1 \cup ... \cup E^{'}_m$ and hence $U^{'} \cap D^{'}$ consists of the components $E^{'}_1,...,E^{'}_m$. Setting $f_i = f^*/_{U\cap\overline{E}_i}$, we obtain the homeomorphisms $h_i = g_i \circ (f_i)^{-1} : U^{'} \cap \overline{E}^{'}_i \to B^n_+ \cup B^{n-1}$ and since $g_i/_{E_i}$, $(f_i)^{-1}/_{E^{'}_i}$ are quasiconformally, it follows that $h_i/_{E^{'}_i}$ is quasiconformally.

We prove the sufficiency. Since D is locally quasiconformally m-collared at each point of $C(f^{-1},b^{'}), D$ is locally quasiconformally m-collared at b. It follows from Theorem 2.[1] that D is m-connected at b and that $D^{'}$ has property P^*_2 at each point of $C(f,b)$. Hence, by Theorem 3, f has the limit $b^{'}$ at b. Set $f^*(b) = b^{'}$ and $f^*/_D = f$. Since $D^{'}$ is locally quasiconformally m-collared at $b^{'}$, by Theorem 2[1], $D^{'}$ is m-connected at $b^{'}$. On the other hand, D is locally quasiconformally m-collared at each point of $C(f^{-1},b^{'})$ and, by Theorem 2[1], D has property P^*_2 at each point of $C(f^{-1},b^{'})$. It follows from Theorem 3, that f^{-1} has a limit at $b^{'}$.

The proof of the theorem is complete.

Theorem 6. *Let $f : D \to D'$ be a quasiconformal mapping, let D be locally quasiconformally m-collared at $b \in \partial D$ and $b^{'} \in C(f,b)$. Suppose that U is a neighborhood of b, appearing in the Definition 5,$E_1,...,E_m$ being components of $U \cap D$. Then f can be extended to a homeomorphism f^* :*

$D \cup \{b\} \to D' \cup \{b'\}$ if and only if D' is locally quasiconformally m-collared at least at a point belonging to $C(f/_{E_i}, b)$ for each $i \in \{1, ..., m\}$.

Proof. The necessity follows by Theorem 5.

We prove the sufficiency. By Remark 1[1], every E_i is locally quasiconformally collared at b and by Theorem 2[1], E_i is 1-connected at b. Since D' is locally quasiconformally m-collared at least at a point of $C(f/_{E_i}, b)$, by Theorem 2[1], D' has property P_2^* at least a point of $C(f/_{E_i}, b)$. By Theorem 2, $C(f/_{E_i}, b)$ contains a single point, and hence, there exists $b' = \lim_{x \to b} f(x)$.

For $m = 1, D'$ is locally quasiconformally 1-collared at b', by Theorem 2[1], D' is 1-connected at b'. By Theorem 2[1], D has property P_2^* at $b \in C(f^{-1}, b')$ and using Theorem 2, there exists the limit of f^{-1} at b'. Now, assume that $m \geq 2$ and that $b \neq \infty$. If $b = \infty$, then we consider the coresponding spherical distance (Definition 17.1 [5]).

Since D' is locally quasiconformally m-collared at b', there exists a nieghborhood U' of b' such that $U' \cap D'$ consists of components $E'_1, ..., E'_m$ and each E'_i is locally quasiconformally collared at b' (see Remark 1[1]). By $b' = \lim_{x \to b} f(x)$ and by Proposition 1[1], we can find a neighborhood U of b such that $f(U \cap D) \subset U' \cap D', U \cap D$ consists of components $E_1, ..., E_m$ and each E_i is locally quasiconformally collared at b .

Let $d = d(b, \partial U)$. By Theorem 2[1], E_i is 1-connected at b and by theorem 17.7.[5], there exists a connected set $F_i \subset E_i \cap B^n(b, \frac{d}{2})$ with $b \in \overline{F}_i$. Denote by $\Gamma_{i,j} = \Delta(F_i, F_j; D), 1 \leq i < j \leq m$ and hence the path family $\Gamma_{i,j}$ is minorized by the family $\Delta(S^{n-1}(b, \frac{d}{2}), S^{n-1}(b, d); D)$. Therefore, $M(\Gamma_{i,j}) \leq \omega_{n-1}(\log 2)^{1-n} < \infty$, where ω_{n-1} is $(n-1)$-dimensional Lebesgue measure of sphere S^{n-1}.

Since $E'_i, i \in \{1, ..., m\}$, is locally quasiconformally collared at b', by Theorem 2[1], E'_i has property P_1. But $b' \in \overline{f(F_i)} \cap \overline{f(F_j)}$ and if $f(F_i), f(F_j)$, $(i \neq j)$ belong to same component of $U' \cap D'$ it follows that
$M(\Delta(f(F_i), f(F_j); D')) = \infty$. Thus cotradict the fact that f is quasiconformal. Thus, the sets $f(F_i)$ and $f(F_j)$ must belong to different components of $U' \cap D'$ whenever $i \neq j$. Therefore, $b \in C(f^{-1}/_{E'_i}, b')$ for each $i \in \{1, ..., m\}$.

Since D is locally quasiconformally m-collared at b, by Theorem 2[1], D has property P_2^* at $b \in C(f^{-1}/_{E'_i}, b')$ and E'_i is 1-connected at b'. Applying Theorem 2, $C(f^{-1}/_{E'_i}, b') = \{b\}$ and hence $\lim_{y \to b'} f^{-1}(y) = b$.

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